RACSAM
Rev. R. Acad. Cien. Serie A. Mat.
VOL. **95** (1), 2001, pp. 65–83
Geometría y Topología / Geometry and Topology

On the geometric prequantization of brackets

M. de León, J. C. Marrero and E. Padrón

Abstract. In this paper we consider a general setting for geometric prequantization of a manifold endowed with a non-necessarily Jacobi bracket. The existence of a generalized foliation permits to define a notion of prequantization bundle. A second approach is given assuming the existence of a Lie algebroid on the manifold. Both approaches are related, and the results for Poisson and Jacobi manifolds are recovered.

Precuantización geométrica de corchetes

Resumen. En este artículo se considera un marco general para la precuantización geométrica de una variedad provista de un corchete que no es necesariamente de Jacobi. La existencia de una foliación generalizada permite definir una noción de fibrado de precuantización. Se estudia una aproximación alternativa suponiendo la existencia de un algebroide de Lie sobre la variedad. Se relacionan ambos enfoques y se recuperan los resultados conocidos para variedades de Poisson y Jacobi.

1. Introduction

Since the seminal results by Kostant and Souriau [14, 26] a lot of work has been done in order to develop a geometric theory of quantization. The inspiration behind these ideas was to develop a method to quantize a classical system and to obtain the quantum system reproducing the Dirac scheme for canonical quantization.

The procedure starts with a phase space which in the most favourable case is a symplectic manifold (M, ω). Then, we associate to M a Hilbert space, which at the first step is the space of sections $\Gamma(K)$ of a complex line bundle K over M. Thus, to each function $f \in C^\infty(M, \mathbb{R})$ (an observable) we attach an operator $\hat{f} : \Gamma(K) \longrightarrow \Gamma(K)$, $\hat{f}(s) = \nabla_{X_f} s + 2\pi i f s$, where X_f is the Hamiltonian vector field defined by f, and ∇ is a covariant derivative on K. K is said to be a prequantization bundle of M if $\widehat{\{f, g\}} = \hat{f} \circ \hat{g} - \hat{g} \circ \hat{f}$, that is, the commutator of the operators corresponds to the Poisson bracket of the observables. This condition can be translated as the existence of a covariant derivative ∇ such that its curvature is ω. The condition is just fullfilled for integral symplectic manifolds.

In constrained Hamiltonian systems and other physical instances, there appear more general phase spaces, endowed with a non-symplectic Poisson bracket, or even, a Jacobi bracket. An approach to this problem is the use of symplectic and contact groupoids, and there is an extensive list of results due mainly to Karasev, Maslov, Weinstein, Dazord, Hector and others (see [4], [5], [6], [11] and [31]; see also [29] and the references therein).

Presentado por Pedro Luis García Pérez
Recibido: 3 de Noviembre 1999. *Aceptado*: 8 de Marzo 2000.
Palabras clave / Keywords: Jacobi manifolds, Poisson manifolds, Lie algebroids, H-Chevalley-Eilenberg cohomology, Lichnerowicz-Jacobi cohomology, Lichnerowicz-Poisson cohomology, foliated cohomology, foliated covariant derivatives, geometric prequantization.
Mathematics Subject Classifications: 53D50, 53D17, 17B66

On the other hand, in [10] Huebschmann extended the Kostant-Souriau geometric quantization procedure of symplectic manifolds to Poisson algebras and, particularly, to the Lie algebra of functions of a Poisson manifold. In [28] (see also [29]), Vaisman obtains essentially the same quantization of a Poisson manifold M straightforwardly, without resorting to any special algebraic machinery. He introduced contravariant instead of covariant derivatives on M, in order to take account that the Poisson bracket is given by a contravariant 2-vector. This permits to obtain similar results for Poisson manifolds. Here, the traditional de Rham cohomology has to be substitutted by the so-called Lichnerowicz-Poisson cohomology (LP-cohomology). The LP-cohomology has a clear lecture in the Chevalley-Eilenberg cohomology of the Lie algebra of functions on M through the representation $(f, g) \mapsto \{f, g\}$.

The above results were recently extended for a Jacobi manifold M (see [19]). In this case one has to consider the Lie algebroid associated to M and the so-called Lichnerowicz-Jacobi cohomology (LJ-cohomology) of M. The Lie algebroid (respectively, the LJ-cohomology) has been introduced in [12] (respectively, in [18, 19]). The LJ-cohomology is the cohomology of a subcomplex of the H-Chevalley-Eilenberg complex. The cohomology of this last complex is just the cohomology of the Lie algebra of functions on M relative to the representation defined by the Hamiltonian vector fields (for a detailed study of the cohomologies of another subcomplexes of the H-Chevalley-Eilenberg complex, we refer to [15, 16, 17]).

The purpose of the present paper is to consider a more general setting which extends in some sense the precedent ones. We consider a manifold M endowed with a skew-symmetric bracket $\{\ ,\ \}$ of functions which satisfies the Jacobi identity, a rule $\mathcal{H}$ that assigns a vector field $\mathcal{H}(f) = X_f$ to each function f such that $[X_f, X_g] = X_{\{f,g\}}$, and, in addition, we assume that the generalized distribution $\mathcal{F}$ defined by the vector fields X_f is in fact a generalized foliation. Note that $\{\ ,\ \}$ does not satisfy necessarily any local property, so that it is not in principle a Jacobi bracket.

Even in this general setting it is still possible to define the corresponding H-Chevalley-Eilenberg cohomology, and the de Rham and the $\mathcal{F}$-foliated cohomologies are related with it. By introducing a suitable definition of $\mathcal{F}$-foliated derivatives, we give a first definition of prequantization bundle, and obtain a characterization of a quantizable manifold $(M, \{\ ,\ \}, \mathcal{H})$ (Theorem 3.9) in the setting of foliated forms.

If the existence of a Lie algebroid $\tilde{K} \longrightarrow M$ is assumed, then we can define a new cohomology by using the representation of $\Gamma(\tilde{K})$ on $C^\infty(M, \mathbb{R})$ and, under certain conditions, this cohomology is related with the precedent ones in a natural way. A second definition of prequantization bundle is given in this context making use of a convenient notion of $\Gamma(\tilde{K})$-derivative, and the corresponding characterization of quantizable manifold is obtained (Theorem 4.8). It should be remarked that this second notion of prequantization bundle is more general that the precedent one. Finally, we discuss the cases of Poisson and Jacobi manifolds recovering the results previously obtained in [28] and [19], respectively.

2. Jacobi manifolds

All the manifolds considered in this paper are assumed to be connected.

A *Jacobi structure* on an m-dimensional manifold M is a pair (Λ, E) where Λ is a 2-vector and E a vector field on M satisfying the following properties:

$$[\Lambda, \Lambda] = 2E \wedge \Lambda\ , \qquad [E, \Lambda] = 0\ . \tag{1}$$

Here $[\ ,\ ]$ denotes the Schouten-Nijenhuis bracket ([3, 29]). The manifold M endowed with a Jacobi structure is called a *Jacobi manifold*. A bracket of functions (the *Jacobi bracket*) is defined by

$$\{f, g\} = \Lambda(df, dg) + fE(g) - gE(f)\ , \qquad \text{for all} \qquad f, g \in C^\infty(M, \mathbb{R}). \tag{2}$$

The Jacobi bracket $\{\ ,\ \}$ is skew-symmetric, satisfies the Jacobi identity and

$$\text{support}\{f, g\} \subseteq (\text{support} f) \cap (\text{support} g).$$

Thus, the space $C^\infty(M, \mathbb{R})$ of C^∞ real-valued functions on M endowed with the Jacobi bracket is *a local Lie algebra* in the sense of Kirillov (see [13]). Conversely, a structure of local Lie algebra on

$C^\infty(M,\mathbb{R})$ defines a Jacobi structure on M (see [8, 13]). If the vector field E identically vanishes then $\{\ ,\ \}$ is a derivation in each argument and, therefore, $\{\ ,\ \}$ defines a *Poisson bracket* on M and (M,Λ) is a *Poisson manifold.* Jacobi and Poisson manifolds were introduced by Lichnerowicz ([20, 21]; see also [3], [22] and [29]).

Examples of Poisson structures are *symplectic* and *Lie-Poisson* structures (see [20] and [30]).

Other interesting examples of Jacobi manifolds, which are not Poisson manifolds, are contact manifolds and locally conformal symplectic manifolds which we will describe below.

A *contact manifold* is a pair (M,η), where M is a $(2m+1)$-dimensional manifold and η is a 1-form on M such that $\eta\wedge(d\eta)^m\neq 0$ at every point (see, for example, [1], [2], [21] and [22]). If $\flat:\mathfrak{X}(M)\longrightarrow\Omega^1(M)$ is the isomorphism of $C^\infty(M,\mathbb{R})$-modules of the space of vector fields $\mathfrak{X}(M)$ on M onto the space of 1-forms $\Omega^1(M)$ defined by $\flat(X)=i_X d\eta+\eta(X)\eta$, then the vector field $\xi=\flat^{-1}(\eta)$ is called the *Reeb vector field*. A contact manifold (M,η) is a Jacobi manifold. In fact, the vector field E is the Reeb vector field ξ and the 2-vector Λ on M is defined by

$$\Lambda(\alpha,\beta)=d\eta(\flat^{-1}(\alpha),\flat^{-1}(\beta)), \tag{3}$$

for all $\alpha,\beta\in\Omega^1(M)$.

An *almost symplectic manifold* is a pair (M,Φ), where M is an even dimensional manifold and Φ is a non-degenerate 2-form on M. An almost symplectic manifold is said to be *locally conformal symplectic (l.c.s.)* if for each point $x\in M$ there is an open neighborhood U such that $d(e^{-\sigma}\Phi)=0$, for some function $\sigma:U\longrightarrow\mathbb{R}$ (see, for example, [8] and [27]). So, $(U,e^{-\sigma}\Phi)$ is a symplectic manifold. An almost symplectic manifold (M,Φ) is l.c.s. if and only if there exists a closed 1-form ω such that $d\Phi=\omega\wedge\Phi$. The 1-form ω is called the *Lee 1-form* of M. It is obvious that the l.c.s. manifolds with Lee 1-form identically zero are just the symplectic manifolds.

In a similar way that for contact manifolds, we define a 2-vector Λ and a vector field E on M which are given by

$$\Lambda(\alpha,\beta)=\Phi(\flat^{-1}(\alpha),\flat^{-1}(\beta)),\qquad E=\flat^{-1}(\omega)\ ,$$

for all $\alpha,\beta\in\Omega^1(M)$, where $\flat:\mathfrak{X}(M)\longrightarrow\Omega^1(M)$ is the isomorphism of $C^\infty(M,\mathbb{R})$-modules defined by $\flat(X)=i_X\Phi$. Then (M,Λ,E) is a Jacobi manifold.

The contact and l.c.s. manifolds are called the transitive Jacobi manifolds (see [7]).

Now, let (M,Λ,E) be a Jacobi manifold. Define a homomorphism of $C^\infty(M,\mathbb{R})$-modules $\#:\Omega^1(M)\longrightarrow\mathfrak{X}(M)$ by

$$(\#(\alpha))(\beta)=\Lambda(\alpha,\beta), \tag{4}$$

for $\alpha,\beta\in\Omega^1(M)$.

This homomorphism can be extended to a homomorphism, which we also denote by $\#$, from the space of k-forms $\Omega^k(M)$ on M onto the space of k-vectors $\mathcal{V}^k(M)$ by putting:

$$\#(f)=f,\qquad \#(\alpha)(\alpha_1,\dots,\alpha_k)=(-1)^k\alpha(\#(\alpha_1),\dots,\#(\alpha_k)), \tag{5}$$

for $f\in C^\infty(M,\mathbb{R})$, $\alpha\in\Omega^k(M)$ and $\alpha_1,\dots,\alpha_k\in\Omega^1(M)$.

We also denote by $\#:\bigcup_{x\in M}(\Lambda^k T_x^*M)\longrightarrow\bigcup_{x\in M}(\Lambda^k T_xM)$ the corresponding vector bundle morphism.

If f is a C^∞ real-valued function on a Jacobi manifold M, the vector field X_f defined by

$$X_f=\#(df)+fE \tag{6}$$

is called the *Hamiltonian vector field* associated with f. It should be noticed that the Hamiltonian vector field associated with the constant function 1 is just E. A direct computation proves that (see [21] and [24])

$$[X_f,X_g]=X_{\{f,g\}}. \tag{7}$$

Now, for every $x\in M$, we consider the subspace $\mathcal{F}_x$ of T_xM generated by all the Hamiltonian vector fields evaluated at the point x. In other words, $\mathcal{F}_x=\#_x(T_x^*M)+\langle E_x\rangle$. Since $\mathcal{F}$ is involutive, one easily follows that $\mathcal{F}$ defines a generalized foliation, which is called the *characteristic foliation* in [7]. Moreover, the Jacobi structure of M induces a Jacobi structure on each leaf. In fact, if L_x is the leaf over a point x of M and $E_x \notin Im\#$ then L_x is a contact manifold with the induced Jacobi structure. If $E_x \in Im\#$, L_x is a l.c.s. manifold (for a more detailed study of the characteristic foliation of a Jacobi manifold we refer to [7] and [8]). If M is a Poisson manifold then the characteristic foliation of M is just the *canonical symplectic foliation* of M (see [20] and [30]).

Next, we will recall the definition of the Lie algebroid associated to a Jacobi manifold (see [12]).

Let M be a differentiable manifold. A *Lie algebroid structure* on a differentiable vector bundle $\tilde{\pi} : \tilde{K} \longrightarrow M$ is a pair that consists of a Lie algebra structure $\{\ ,\ \}\tilde{}$ on the space $\Gamma(\tilde{K})$ of the global cross sections of $\tilde{\pi} : \tilde{K} \longrightarrow M$ and a vector bundle morphism $\varrho : \tilde{K} \longrightarrow TM$ (the *anchor map*) such that:

1. The induced map $\varrho : (\Gamma(\tilde{K}), \{\ ,\ \}\tilde{}) \longrightarrow (\mathfrak{X}(M), [\ ,\ ])$ is a Lie algebra homomorphism.
2. For all $f \in C^\infty(M, \mathbb{R})$ and for all $\tilde{s}_1, \tilde{s}_2 \in \Gamma(\tilde{K})$ one has

$$\{\tilde{s}_1, f\tilde{s}_2\}\tilde{} = f\{\tilde{s}_1, \tilde{s}_2\}\tilde{} + (\varrho(\tilde{s}_1)(f))\tilde{s}_2. \tag{8}$$

A triple $(\tilde{K}, \{\ ,\ \}\tilde{}, \varrho)$ is called *a Lie algebroid over M* (see [23], [25] and [29]).

Now, let (M, Λ, E) be a Jacobi manifold. In [12], the authors obtain a Lie algebroid structure on the jet bundle $J^1(M, \mathbb{R})$ as follows. It is well-known that if T^*M is the cotangent bundle of M, the space $J^1(M, \mathbb{R})$ can be identified with the product manifold $\tilde{K} = T^*M \times \mathbb{R}$ in such a sense that the space $\Gamma(\tilde{K})$ of the global cross sections of the vector bundle $\tilde{K} = T^*M \times \mathbb{R} \longrightarrow M$ can be identified with $\Omega^1(M) \times C^\infty(M, \mathbb{R})$. Now, we consider on $\Omega^1(M) \times C^\infty(M, \mathbb{R})$ the bracket $\{\ ,\ \}\tilde{}$ given by (see [12])

$$\begin{aligned}
\{(\alpha, f), (\beta, g)\}\tilde{} &= j(\mathcal{L}_{(\#(\alpha)+fE)}g - \mathcal{L}_{(\#(\beta)+gE)}f - \Lambda(\alpha, \beta)) \\
&\quad + ((\mathcal{L}_{(\#(\alpha)+fE)} - i_E\alpha)(\beta - dg) - (\mathcal{L}_{(\#(\beta)+gE)} - i_E\beta)(\alpha - df), 0) \\
&= (\mathcal{L}_{\#(\alpha)}\beta - \mathcal{L}_{\#(\beta)}\alpha - d(\Lambda(\alpha, \beta)) + f\mathcal{L}_E\beta - g\mathcal{L}_E\alpha - i_E(\alpha \wedge \beta), \\
&\quad \alpha(\#(\beta)) + \#(\alpha)(g) - \#(\beta)(f) + fE(g) - gE(f)),
\end{aligned} \tag{9}$$

where $j : C^\infty(M, \mathbb{R}) \longrightarrow \Omega^1(M) \times C^\infty(M, \mathbb{R})$ is the *prolongation mapping* defined by

$$j(f) = (df, f). \tag{10}$$

We have (see [12])

Theorem 1 *Let (M, Λ, E) be a Jacobi manifold and $\{\ ,\ \}\tilde{}$ the bracket on $\Omega^1(M)\times C^\infty(M, \mathbb{R})$ defined by (9). Then, the triple $(T^*M \times \mathbb{R}, \{\ ,\ \}\tilde{}, (\#, E))$ is a Lie algebroid over M, where $(\#, E) : T^*M \times \mathbb{R} \longrightarrow TM$ is the vector bundle morphism*

$$(\#, E)(\alpha_x, \lambda) = \#(\alpha_x) + \lambda E_x, \tag{11}$$

for $(\alpha_x, \lambda) \in T^*_x(M)\times\mathbb{R}$. Moreover, if we consider on $C^\infty(M, \mathbb{R})$ the Jacobi bracket then the prolongation mapping

$$j : C^\infty(M, \mathbb{R}) \longrightarrow \Omega^1(M) \times C^\infty(M, \mathbb{R}), \qquad f \mapsto j(f) = (df, f) \tag{12}$$

is a Lie algebra homomorphism.

Remark 1 If $\tilde{F} = T^*M \times \{0\}$ then the canonical projection $\tilde{F} \longrightarrow M$ defines a vector subbundle of the vector bundle $\tilde{K} = T^*M \times \mathbb{R} \longrightarrow M$ in such a sense that

$$\tilde{K}_x = \tilde{F}_x \oplus < j(1)(x) >$$

for $x \in M$, where $\tilde{K}_x$ (respectively, $\tilde{F}_x$) is the fibre of $\tilde{K} \longrightarrow M$ (respectively, $\tilde{F} \longrightarrow M$) over x. Note that $\tilde{F}$ can be identified with the cotangent bundle T^*M and that, under this identification, the restriction of the anchor map $(\#, E)$ to $\tilde{F}$ is just the vector bundle morphism $\# : T^*M \longrightarrow TM$.

3. H-Chevalley-Eilenberg cohomology, foliated covariant derivatives and prequantization

In this section we will assume that M is a differentiable manifold which satisfies the following conditions:

1. There exists a bracket of functions
$$\{\ ,\ \} : C^\infty(M,\mathbb{R}) \times C^\infty(M,\mathbb{R}) \longrightarrow C^\infty(M,\mathbb{R})$$
which is $\mathbb{R}$-bilinear, skew-symmetric and satisfies the Jacobi identity.

2. There exists a $\mathbb{R}$-linear map
$$\mathcal{H} : C^\infty(M,\mathbb{R}) \longrightarrow \mathfrak{X}(M), \qquad f \in C^\infty(M,\mathbb{R}) \mapsto \mathcal{H}(f) = X_f \in \mathfrak{X}(M)$$
and
$$[X_f, X_g] = X_{\{f,g\}} \tag{13}$$
for $f, g \in C^\infty(M,\mathbb{R})$.
$\mathcal{H}(f) = X_f$ is called the Hamiltonian vector field associated with f.
If x is a point of M, we will denote by $\mathcal{F}_x$ the subspace of T_xM defined by
$$\mathcal{F}_x = \{X_f(x)/f \in C^\infty(M,\mathbb{R})\}.$$
A vector field X on M is said to be tangent to $\mathcal{F}$ if $X_x \in \mathcal{F}_x$ for all $x \in M$. The space of the vector fields tangent to $\mathcal{F}$ is denoted by $\mathfrak{X}(\mathcal{F})$.

3. The involutive generalized distribution
$$x \in M \longrightarrow \mathcal{F}_x \subseteq T_xM$$
is completely integrable. Thus, $\mathcal{F}$ defines a generalized foliation on M, which is called the characteristic foliation.

3.1. H-Chevalley-Eilenberg cohomology

We consider the cohomology of the Lie algebra $(C^\infty(M,\mathbb{R}), \{\ ,\ \})$ relative to the representation defined by the Hamiltonian vector fields, that is, to the representation given by
$$C^\infty(M,\mathbb{R}) \times C^\infty(M,\mathbb{R}) \longrightarrow C^\infty(M,\mathbb{R}), \qquad (f,g) \mapsto X_f(g).$$
This cohomology is denoted by $H^*_{HCE}(M)$ and it is called the *H-Chevalley-Eilenberg cohomology* associated to M (see [16, 17, 18, 19] for the case of a Jacobi manifold). In fact, if $C^k_{HCE}(M)$ is the real vector space of the $\mathbb{R}$-multilinear skew-symmetric mappings $c^k : C^\infty(M,\mathbb{R}) \times \ldots^{(k} \cdots \times C^\infty(M,\mathbb{R}) \longrightarrow C^\infty(M,\mathbb{R})$ then
$$H^k_{HCE}(M) = \frac{\ker\{\partial_H : C^k_{HCE}(M) \longrightarrow C^{k+1}_{HCE}(M)\}}{\mathrm{Im}\{\partial_H : C^{k-1}_{HCE}(M) \longrightarrow C^k_{HCE}(M)\}},$$
where $\partial_H : C^r_{HCE}(M) \longrightarrow C^{r+1}_{HCE}(M)$ is the linear differential operator defined by
$$\begin{aligned}(\partial_H c^r)(f_0, \cdots, f_r) &= \sum_{i=0}^{r} (-1)^i X_{f_i}(c^r(f_0, \cdots, \widehat{f_i}, \cdots, f_r)) \\ &+ \sum_{i<j} (-1)^{i+j} c^r(\{f_i, f_j\}, f_0, \cdots, \widehat{f_i}, \cdots, \widehat{f_j}, \cdots, f_r)\end{aligned} \tag{14}$$

for $c^r \in C^r_{HCE}(M)$ and $f_0, \ldots, f_r \in C^\infty(M, \mathbb{R})$.

Let $I : C^\infty(M, \mathbb{R}) \longrightarrow C^\infty(M, \mathbb{R})$ be the identity map. We will denote by $\bar{\Lambda}$ to the 2-coboundary

$$\bar{\Lambda} = \partial_H I.$$

From (14), it follows that

$$\bar{\Lambda}(f, g) = X_f(g) - X_g(f) - \{f, g\}, \tag{15}$$

for $f, g \in C^\infty(M, \mathbb{R})$.

Using (13) and (14), we obtain the following relation between the de Rham cohomology and the H-Chevalley-Eilenberg cohomology.

Theorem 2 *Let $\tilde{\mathcal{H}} : \Omega^k(M) \longrightarrow C^k_{HCE}(M)$ be the homomorphism of $C^\infty(M, \mathbb{R})$-modules given by*

$$\tilde{\mathcal{H}}(\alpha)(f_1, \ldots, f_k) = \alpha(\mathcal{H}(f_1), \ldots, \mathcal{H}(f_k)) = \alpha(X_{f_1}, \ldots, X_{f_k}) \tag{16}$$

for $\alpha \in \Omega^k(M)$ and $f_1, \ldots, f_k \in C^\infty(M, \mathbb{R})$. Then $\tilde{\mathcal{H}}$ induces a homomorphism of complexes $\tilde{\mathcal{H}} : (\Omega^*(M), d) \longrightarrow (C^*_{HCE}(M), \partial_H)$. Thus, if $H^*_{dR}(M)$ is the de Rham cohomology of M, we have the corresponding homomorphism in cohomology $\tilde{\mathcal{H}} : H^*_{dR}(M) \longrightarrow H^*_{HCE}(M)$.

Next, we will show the relation between the $\mathcal{F}$-foliated cohomology of M and the H-Chevalley - Eilenberg cohomology.

First, we will introduce the $\mathcal{F}$-foliated cohomology of M (for the definition of the $\mathcal{F}$-foliated cohomology associated to a regular foliation $\mathcal{F}$ on a differentiable manifold, we refer to [9]).

Let $\Omega^k(\mathcal{F})$ be the space of the *$\mathcal{F}$-foliated k-forms* on M. An element α of $\Omega^k(\mathcal{F})$ is a mapping

$$x \in M \longrightarrow \alpha(x) \in \Lambda^k \mathcal{F}^*_x$$

such that:

1. If x is a point of M, the restriction of α to the leaf L_x of $\mathcal{F}$ over x is a k-form on L_x.
2. If $X_1, \ldots, X_k$ are C^∞-differentiable local vector fields defined on an open subset U of M and $X_1, \ldots, X_k$ are tangent to $\mathcal{F}$ in U then the function $\alpha(X_1, \ldots, X_k) : U \longrightarrow \mathbb{R}$ is C^∞-differentiable, where $\alpha(X_1, \ldots, X_k)$ is given by

   $$\alpha(X_1, \ldots, X_k)(x) = \alpha(x)(X_1(x), \ldots, X_k(x)),$$

   for $x \in U$.

We can consider the linear differential operator $d : \Omega^k(\mathcal{F}) \longrightarrow \Omega^{k+1}(\mathcal{F})$ given by

$$(d(\alpha))_x = (d(\alpha_{|L_x}))_x \tag{17}$$

for $\alpha \in \Omega^k(\mathcal{F})$ and $x \in M$.

It is clear that $d^2 = 0$. This fact allows us to introduce the differential complex

$$\cdots \longrightarrow \Omega^{k-1}(\mathcal{F}) \xrightarrow{d} \Omega^k(\mathcal{F}) \xrightarrow{d} \Omega^{k+1}(\mathcal{F}) \longrightarrow \cdots$$

The cohomology of this complex is denoted by $H^*(\mathcal{F})$ and it is called the *$\mathcal{F}$-foliated cohomology* of M.

Now, using (13), (14) and (17), we prove the following result which relates the cohomologies $H^*(\mathcal{F})$ and $H^*_{HCE}(M)$.

Theorem 3 *Let $\tilde{\mathcal{H}}(\mathcal{F}) : \Omega^k(\mathcal{F}) \longrightarrow C^k_{HCE}(M)$ be the homomorphism of $C^\infty(M, \mathbb{R})$-modules defined by*

$$\tilde{\mathcal{H}}(\mathcal{F})(\alpha)(f_1, \ldots, f_k) = \alpha(X_{f_1}, \ldots, X_{f_k}) \tag{18}$$

for $\alpha \in \Omega^k(\mathcal{F})$ and $f_1, \ldots, f_k \in C^\infty(M, \mathbb{R})$. Then $\tilde{\mathcal{H}}(\mathcal{F})$ induces a homomorphism of complexes $\tilde{\mathcal{H}}(\mathcal{F}) : (\Omega^*(\mathcal{F}), d) \longrightarrow (C^*_{HCE}(M), \partial_H)$ and we have the corresponding homomorphism in cohomology $\tilde{\mathcal{H}}(\mathcal{F}) : H^*(\mathcal{F}) \longrightarrow H^*_{HCE}(M)$. Moreover, the following diagram is commutative

$$\begin{tikzcd} H^*(\mathcal{F}) \arrow[r, "\tilde{\mathcal{H}}(\mathcal{F})"] & H^*_{HCE}(M) \\ H^*_{dR}(M) \arrow[u, "r(\mathcal{F})"] \arrow[ur, "\tilde{\mathcal{H}}"'] & \end{tikzcd}$$

where $r(\mathcal{F}) : H^*_{dR}(M) \longrightarrow H^*(\mathcal{F})$ is the canonical homomorphism between the de Rham cohomology of M and the cohomology $H^*(\mathcal{F})$.

3.2. $\mathcal{F}$-foliated covariant derivatives

Let $\pi : K \longrightarrow M$ be a complex line bundle over M. Denote by $\Gamma(K)$ the space of the cross sections of $\pi : K \longrightarrow M$ and by $K_x = \pi^{-1}(x)$ the fibre over $x \in M$.

If L_x is the leaf of $\mathcal{F}$ over $x \in M$ then it is clear that the projection $\pi_{|\pi^{-1}(L_x)} : \pi^{-1}(L_x) \longrightarrow L_x$ defines a complex line bundle over L_x. If y is a point of L_x, we will denote by $\mathrm{Lin}_{\mathbb{C}}(\Gamma(\pi^{-1}(L_x)), K_y)$ the space of the $\mathbb{C}$-linear maps of $\Gamma(\pi^{-1}(L_x))$ onto K_y.

Definition 1 *A $\mathcal{F}$-foliated covariant derivative ∇ on $\pi : K \longrightarrow M$ is a map*

$$\nabla : \bigcup_{x \in M} \mathcal{F}_x = \bigcup_{x \in M} T_x(L_x) \longrightarrow \bigcup_{x \in M} Lin_{\mathbb{C}}(\Gamma(\pi^{-1}(L_x)), K_x)$$

which satisfies the following conditions:

1. *If $v \in \mathcal{F}_x = T_x(L_x)$ then $\nabla_v \in Lin_{\mathbb{C}}(\Gamma(\pi^{-1}(L_x)), K_x)$ and the map $\nabla_{|T(L_x)} : T(L_x) \longrightarrow \bigcup_{y \in L_x} Lin_{\mathbb{C}}(\Gamma(\pi^{-1}(L_x)), K_y)$ is a covariant derivative on $\pi_{|\pi^{-1}(L_x)} : \pi^{-1}(L_x) \longrightarrow L_x$.*
2. *If U is an open subset of M, $s : U \longrightarrow K$ is a C^∞-differentiable local section of $\pi : K \longrightarrow M$ and X is a C^∞-differentiable local vector field defined in U which is tangent to $\mathcal{F}$, then the map $\nabla_X s : U \longrightarrow K$ given by*

   $$(\nabla_X s)(x) = \nabla_{X_x}(s_{|U \cap L_x})$$

   for $x \in U$, is a C^∞-differentiable local section of $\pi : K \longrightarrow M$.

Let h be a Hermitian metric on $\pi : K \longrightarrow M$. A $\mathcal{F}$-foliated covariant derivative ∇ on $\pi : K \longrightarrow M$ is said to be Hermitian if

$$v(h(s_1, s_2)) = h_x(\nabla_v(s_1)_{|L_x}, s_2(x)) + h_x(s_1(x), \nabla_v(s_2)_{|L_x})$$

for $x \in M$, $v \in \mathcal{F}_x$ and $s_1, s_2 \in \Gamma(K)$.

It is clear that a (Hermitian) covariant derivative on $\pi : K \longrightarrow M$ induces a (Hermitian) $\mathcal{F}$-foliated covariant derivative.

Definition 2 *Let $\pi : K \longrightarrow M$ be a complex line bundle over M and ∇ a $\mathcal{F}$-foliated covariant derivative on $\pi : K \longrightarrow M$. The curvature of ∇ is the mapping $C_\nabla : \mathfrak{X}(\mathcal{F}) \times \mathfrak{X}(\mathcal{F}) \times \Gamma(K) \longrightarrow \Gamma(K)$ given by*

$$C_\nabla(X,Y)(s) = (\nabla_X \circ \nabla_Y - \nabla_Y \circ \nabla_X - \nabla_{[X,Y]})s. \tag{19}$$

If x is a point of M, we have that

$$(C_\nabla(X,Y)(s))_{|L_x} = (C_{\nabla_{|T(L_x)}})(X_{|L_x}, Y_{|L_x})(s_{|L_x}),$$

where $C_{\nabla_{|T(L_x)}}$ is the curvature of the covariant derivative $\nabla_{|T(L_x)}$ on $\pi_{|\pi^{-1}(L_x)} : \pi^{-1}(L_x) \longrightarrow L_x$.

Thus, there exists a globally defined complex $\mathcal{F}$-foliated 2-form Ω_∇ such that

$$C_\nabla(X,Y)(s) = \Omega_\nabla(X,Y)s. \tag{20}$$

Since the $\mathcal{F}$-foliated 2-form Ω_∇ completely determines to the map C_∇, we will say also that Ω_∇ is the curvature of ∇.

Proceeding as in the case of a usual covariant derivative (see, for instance, [14]), we prove

Theorem 4 *Let $\pi : K \longrightarrow M$ be a complex line bundle over M. Suppose that ∇ is a $\mathcal{F}$-foliated covariant derivative on $\pi : K \longrightarrow M$ with curvature Ω_∇ and that $H^*_{\mathbb{C}}(\mathcal{F})$ is the complex $\mathcal{F}$-foliated cohomology of M. Then:*

1. *The complex $\mathcal{F}$-foliated 2-form Ω_∇ defines a cohomology class in $H^2_{\mathbb{C}}(\mathcal{F})$.*
2. *The cohomology class $[\Omega_\nabla]$ does not depend of the $\mathcal{F}$-foliated covariant derivative.*
3. *If h is a Hermitian metric on $\pi : K \longrightarrow M$ and ∇ is a Hermitian $\mathcal{F}$-foliated covariant derivative then Ω_∇ is purely imaginary.*

3.3. Prequantization

For a complex line bundle $\pi : K \longrightarrow M$ over M we will denote by $\mathrm{End}_{\mathbb{C}}(\Gamma(K))$ the space of the $\mathbb{C}$-linear endomorphisms of $\Gamma(K)$. Then, we introduce the following definition.

Definition 3 *A complex line bundle $\pi : K \longrightarrow M$ over M is said to be a prequantization bundle if*

$$\widehat{\{f,g\}} = \widehat{f} \circ \widehat{g} - \widehat{g} \circ \widehat{f} \qquad f, g \in C^\infty(M, \mathbb{R}) \tag{21}$$

with $\widehat{f} \in End_{\mathbb{C}}(\Gamma(K))$ defined by

$$s \in \Gamma(K) \longmapsto \widehat{f}(s) = \nabla_{X_f} s + 2\pi i f s, \tag{22}$$

where ∇ is a $\mathcal{F}$-foliated covariant derivative on $\pi : K \longrightarrow M$. The manifold M is said to be quantizable if there exists a prequantization bundle $\pi : K \longrightarrow M$ over M.

Let $\bar{\Lambda}$ be the 2-coboundary in the H-Chevalley-Eilenberg complex given by (15) and let $\tilde{\mathcal{H}}(\mathcal{F}) : \Omega^2(\mathcal{F}) \longrightarrow C^2_{HCE}(M)$ be the homomorphism defined by (18).

From (13), (19), (20), (21), (22) and Definition 1, we deduce

Lemma 1 *The manifold M is quantizable if and only if there exist a complex line bundle $\pi : K \longrightarrow M$ over M and a $\mathcal{F}$-foliated covariant derivative ∇ on $\pi : K \longrightarrow M$ such that the curvature Ω_∇ of ∇ is purely imaginary and*

$$\tilde{\mathcal{H}}(\mathcal{F})(\frac{i}{2\pi}\Omega_\nabla) = \bar{\Lambda}.$$

Next, we will obtain another characterization. For this purpose, we recall the following result.

Theorem 5 *[14] (i) If $\pi : K \longrightarrow M$ is a complex line bundle over M, h is a Hermitian metric on $\pi : K \longrightarrow M$ and $\tilde{\nabla}$ is a Hermitian covariant derivative then the curvature $\Omega_{\tilde{\nabla}}$ of $\tilde{\nabla}$ is purely imaginary and $\Omega = \dfrac{i}{2\pi}\Omega_{\tilde{\nabla}}$ is an integral closed 2-form.*

(ii) If Ω is an integral closed 2-form then there exist a complex line bundle $\pi : K \longrightarrow M$ over M, a Hermitian metric on $\pi : K \longrightarrow M$ and a Hermitian covariant derivative $\tilde{\nabla}$ with curvature $\Omega_{\tilde{\nabla}}$ such that $\Omega = \dfrac{i}{2\pi}\Omega_{\tilde{\nabla}}$.

Now, if $\tilde{\mathcal{H}} : \Omega^2(M) \longrightarrow C^2_{HCE}(M)$ is the homomorphism given by (16) and ∂_H is the H-Chevalley-Eilenberg cohomology operator (see (14)) then, using Lemma 1 and Theorems 3, 4 and 5, we prove

Theorem 6 *The manifold M is quantizable if and only if there exist an integral closed 2-form Ω on M and a $\mathcal{F}$-foliated 1-form α such that*

$$\tilde{\mathcal{H}}(\Omega) = \bar{\Lambda} + \partial_H(\tilde{\mathcal{H}}(\mathcal{F})(\alpha)) = \partial_H(I + \tilde{\mathcal{H}}(\mathcal{F})(\alpha)).$$

4. H-Chevalley-Eilenberg cohomology, Lie algebroids and prequantization

Let M be a differentiable manifold as in Section 3. Moreover, we will assume that there is a Lie algebroid $\tilde{\pi} : \tilde{K} \longrightarrow M$ over M with anchor map $\varrho : \tilde{K} \longrightarrow TM$ which satisfies the following conditions:

1. If $\Gamma(\tilde{K})$ is the space of the cross sections of $\tilde{\pi} : \tilde{K} \longrightarrow M$ then there exists a Lie algebra homomorphism $j : C^\infty(M, \mathbb{R}) \longrightarrow \Gamma(\tilde{K})$ such that $\varrho \circ j = \mathcal{H}$.
2. For all $x \in M$
$$\varrho(\tilde{K}_x) = \mathcal{F}_x,$$
where $\tilde{K}_x = \tilde{\pi}^{-1}(x)$ is the fibre over $x \in M$.
3. For all $x \in M$ we have that $j(1)(x) \neq 0$. Furthermore, there is a vector subbundle $\tilde{\pi} : \tilde{F} \longrightarrow M$ of $\tilde{\pi} : \tilde{K} \longrightarrow M$ such that
$$\tilde{K}_x = \tilde{F}_x \oplus < j(1)(x) > .$$

We will denote by $\{\ ,\ \}\tilde{}$ the Lie bracket on $\Gamma(\tilde{K})$ and by

$$\# : \tilde{F} \longrightarrow TM$$

the restriction to $\tilde{F}$ of the anchor map $\varrho : \tilde{K} \longrightarrow TM$. We also will denote by $\# : \Gamma(\tilde{F}) \longrightarrow \mathfrak{X}(M)$ the induced homomorphism between the cross sections of $\tilde{\pi} : \tilde{F} \longrightarrow M$ and the vector fields on M.

4.1. Lie algebroids and H-Chevalley-Eilenberg cohomology

In this section, we introduce a cohomology associated to the Lie algebroid and we study the relation between this cohomology and the H-Chevalley-Eilenberg cohomology (for a detailed study of the cohomologies associated to a Lie algebroid we refer to [23]).

We consider the cohomology of the Lie algebra $(\Gamma(\tilde{K}), \{\ ,\ \}\tilde{})$ relative to the representation of $\Gamma(\tilde{K})$ on $C^\infty(M, \mathbb{R})$ defined by

$$\Gamma(\tilde{K}) \times C^\infty(M, \mathbb{R}) \longrightarrow C^\infty(M, \mathbb{R}), \qquad (\tilde{s}, f) \mapsto \varrho(\tilde{s})(f).$$

This cohomology is denoted by $H^*(\Gamma(\tilde{K}); C^\infty(M,\mathbb{R}))$. Therefore, if $C^k(\Gamma(\tilde{K}); C^\infty(M,\mathbb{R}))$ is the real vector space of the $\mathbb{R}$-multilinear skew-symmetric mappings $R^k : \Gamma(\tilde{K})\times\ldots^{(k}\cdots\times\Gamma(\tilde{K}) \longrightarrow C^\infty(M,\mathbb{R})$ then

$$H^k(\Gamma(\tilde{K}); C^\infty(M,\mathbb{R})) = \frac{\ker\{\partial : C^k(\Gamma(\tilde{K}); C^\infty(M,\mathbb{R})) \longrightarrow C^{k+1}(\Gamma(\tilde{K}); C^\infty(M,\mathbb{R}))\}}{\mathrm{Im}\{\partial : C^{k-1}(\Gamma(\tilde{K}); C^\infty(M,\mathbb{R})) \longrightarrow C^k(\Gamma(\tilde{K}); C^\infty(M,\mathbb{R}))\}},$$

where $\partial : C^r(\Gamma(\tilde{K}); C^\infty(M,\mathbb{R})) \longrightarrow C^{r+1}(\Gamma(\tilde{K}); C^\infty(M,\mathbb{R}))$ is the linear differential operator defined by

$$\begin{aligned}(\partial R^r)(\tilde{s}_0,\cdots,\tilde{s}_r) \;=\; & \sum_{i=0}^{r}(-1)^i \varrho(\tilde{s}_i)(R^r(\tilde{s}_0,\cdots,\widehat{\tilde{s}_i},\cdots,\tilde{s}_r)) \\ & +\sum_{i<j}(-1)^{i+j}R^r(\{\tilde{s}_i,\tilde{s}_j\}\tilde{}, \tilde{s}_0,\cdots,\widehat{\tilde{s}_i},\cdots,\widehat{\tilde{s}_j},\cdots,\tilde{s}_r)\end{aligned} \tag{23}$$

for $\tilde{s}_0,\ldots,\tilde{s}_r \in \Gamma(\tilde{K})$.

Next, we will obtain some relations between the cohomology $H^*(\Gamma(\tilde{K}); C^\infty(M,\mathbb{R}))$, the $\mathcal{F}$-foliated cohomology and the de Rham cohomology.

We will denote by $r(\mathcal{F}) : \Omega^k(M) \longrightarrow \Omega^k(\mathcal{F})$ the canonical homomorphism between $\Omega^k(M)$ and the space of the $\mathcal{F}$-foliated k-forms on M.

Using (17), (23) and the fact that $\varrho : \Gamma(\tilde{K}) \longrightarrow \mathfrak{X}(M)$ is a Lie algebra homomorphism, we deduce

Theorem 7 *1. Let $\tilde{\varrho}(\mathcal{F}) : \Omega^k(\mathcal{F}) \longrightarrow C^k(\Gamma(\tilde{K}); C^\infty(M,\mathbb{R}))$ be the homomorphism of $C^\infty(M,\mathbb{R})$-modules defined by*

$$(\tilde{\varrho}(\mathcal{F})(\alpha))(\tilde{s}_1,\ldots,\tilde{s}_k) = \alpha(\varrho\tilde{s}_1,\ldots,\varrho\tilde{s}_k), \tag{24}$$

for $\alpha \in \Omega^k(\mathcal{F})$ and $\tilde{s}_1,\ldots,\tilde{s}_k \in \Gamma(\tilde{K})$. Then, $\tilde{\varrho}(\mathcal{F})$ induces a homomorphism of complexes $\tilde{\varrho}(\mathcal{F}) : (\Omega^*(\mathcal{F}), d) \longrightarrow (C^*(\Gamma(\tilde{K}); C^\infty(M,\mathbb{R})), \partial)$. Thus, we have the corresponding homomorphism in cohomology $\tilde{\varrho}(\mathcal{F}) : H^*(\mathcal{F}) \longrightarrow H^*(\Gamma(\tilde{K}); C^\infty(M,\mathbb{R}))$.

2. Let $\tilde{\varrho} : \Omega^k(M) \longrightarrow C^k(\Gamma(\tilde{K}); C^\infty(M,\mathbb{R}))$ be the homomorphism of $C^\infty(M,\mathbb{R})$-modules defined by

$$\tilde{\varrho} = \tilde{\varrho}(\mathcal{F}) \circ r(\mathcal{F}). \tag{25}$$

Then $\tilde{\varrho}$ induces a homomorphism of complexes $\tilde{\varrho} : (\Omega^*(M), d) \to (C^*(\Gamma(\tilde{K}); C^*(M,\mathbb{R})), \partial)$ and thus we have the corresponding homomorphism in cohomology

$$\tilde{\varrho} : H^*_{dR}(M) \longrightarrow H^*(\Gamma(\tilde{K}); C^\infty(M,\mathbb{R})).$$

Now, if $\tilde{\mathcal{H}}(\mathcal{F}) : H^*(\mathcal{F}) \longrightarrow H^*_{HCE}(M)$ (respectively, $\tilde{\mathcal{H}} : H^*_{dR}(M) \longrightarrow H^*_{HCE}(M)$) is the canonical homomorphism between the $\mathcal{F}$-foliated cohomology (respectively, the de Rham cohomology) and the H-Chevalley-Eilenberg cohomology (see Theorems 2 and 3) then, using (14), (23), (24), (25) and the fact that $j : C^\infty(M,\mathbb{R}) \longrightarrow \Gamma(\tilde{K})$ is a Lie algebra homomorphism, we prove

Theorem 8 *Let $\tilde{j} : C^k(\Gamma(\tilde{K}); C^\infty(M,\mathbb{R})) \to C^k_{HCE}(M)$ be the homomorphism of $C^\infty(M,\mathbb{R})$-modules given by*

$$\tilde{j}(R^k)(f_1,\ldots,f_k) = R^k(j(f_1),\ldots,j(f_k)) \tag{26}$$

for $R^k \in C^k(\Gamma(K); C^\infty(M,\mathbb{R}))$ and $f_1,\ldots,f_k \in C^\infty(M,\mathbb{R})$. Then, $\tilde{j}$ induces a homomorphism of complexes $\tilde{j} : (C^*(\Gamma(\tilde{K}); C^\infty(M,\mathbb{R})), \partial) \longrightarrow (C^*_{HCE}(M), \partial_H)$ and thus we have the corresponding homomorphism in cohomology $\tilde{j} : H^*(\Gamma(\tilde{K}); C^\infty(M,\mathbb{R})) \longrightarrow H^*_{HCE}(M)$. Moreover, the following diagram is commutative

$$\begin{tikzcd}
H^*(\Gamma(\tilde{K}); C^\infty(M,\mathbb{R})) \arrow[dr, "\tilde{j}"] & \\
H^*(\mathcal{F}) \arrow[u, "\tilde{\rho}(\mathcal{F})"] \arrow[r, "\tilde{\mathcal{H}}(\mathcal{F})"] & H^*_{HCE}(M) \\
H^*_{dR}(M) \arrow[u, "r(\mathcal{F})"] \arrow[ur, "\tilde{\mathcal{H}}"'] \arrow[uu, bend left=60, "\tilde{\varrho}"]
\end{tikzcd}$$

4.2. Lie algebroids and derivatives on complex line bundles

Let $\pi : K \longrightarrow M$ be a complex line bundle over M. Denote by $\Gamma(K)$ the space of cross sections of $\pi : K \longrightarrow M$, by $K_x = \pi^{-1}(x)$ the fibre over $x \in M$ and by $\mathrm{Lin}_{\mathbb{C}}(\Gamma(K), K_x)$ the space of the $\mathbb{C}$-linear maps of $\Gamma(K)$ onto K_x.

Definition 4 *A $\Gamma(\tilde{K})$-derivative D on $\pi : K \longrightarrow M$ is a map $D : \tilde{K} \to \bigcup_{x\in M} Lin_{\mathbb{C}}(\Gamma(K), K_x)$ which satisfies the following conditions:*

1. *If $\tilde{s}_x \in \tilde{K}_x$ then $D_{\tilde{s_x}} \in Lin_{\mathbb{C}}(\Gamma(K), K_x)$.*
2. *The map $D_{|\tilde{K}_x} : \tilde{K}_x \longrightarrow Lin_{\mathbb{C}}(\Gamma(K), K_x)$ is $\mathbb{R}$-linear.*
3. *For $\tilde{s}_x \in \tilde{K}_x$, $f \in C^\infty(M,\mathbb{R})$ and $s \in \Gamma(K)$, we have*
$$D_{\tilde{s}_x}(fs) = \varrho(\tilde{s}_x)(f)s(x) + f(x)D_{\tilde{s}_x}s.$$
4. *If $s \in \Gamma(K)$, U is an open subset of M and $\tilde{s} : U \longrightarrow \tilde{K}$ is a C^∞-differentiable local section of $\tilde{\pi} : \tilde{K} \longrightarrow M$ then the map $D_{\tilde{s}}s : U \longrightarrow K$ given by*
$$(D_{\tilde{s}}s)(x) = D_{\tilde{s}(x)}s,$$
for $x \in U$, is a C^∞-differentiable local section of $\pi : K \longrightarrow M$.

Let h be a Hermitian metric on $\pi : K \longrightarrow M$. A $\Gamma(\tilde{K})$-derivative D on $\pi : K \longrightarrow M$ is said to be Hermitian if
$$\varrho(\tilde{s}_x)(h(s_1,s_2)) = h_x(D_{\tilde{s}_x}s_1, s_2(x)) + h_x(s_1(x), D_{\tilde{s}_x}s_2) \tag{27}$$
for $x \in M$, $\tilde{s}_x \in \tilde{K}_x$ and $s_1, s_2 \in \Gamma(K)$.

If ∇ is a (Hermitian) $\mathcal{F}$-foliated covariant derivative on $\pi : K \longrightarrow M$ and we put
$$D_{\tilde{s}_x}s = \nabla_{\varrho(\tilde{s}_x)}(s_{|L_x})$$
for $\tilde{s}_x \in \tilde{K}_x$ and $s \in \Gamma(K)$, we obtain a (Hermitian) $\Gamma(\tilde{K})$-derivative.

Definition 5 *Let $\pi : K \longrightarrow M$ be a complex line bundle over M and D a $\Gamma(\tilde{K})$-derivative on $\pi : K \longrightarrow M$. The curvature of D is the mapping $C_D : \Gamma(\tilde{K}) \times \Gamma(\tilde{K}) \times \Gamma(K) \longrightarrow \Gamma(K)$ given by*
$$C_D(\tilde{s}_1,\tilde{s}_2)(s) = (D_{\tilde{s}_1} \circ D_{\tilde{s}_2} - D_{\tilde{s}_2} \circ D_{\tilde{s}_1} - D_{\{\tilde{s}_1,\tilde{s}_2\}^{\sim}})(s) \tag{28}$$
for $\tilde{s}_1, \tilde{s}_2 \in \Gamma(\tilde{K})$ and $s \in \Gamma(K)$.

Using (8), (28), Definition 4 and the fact that $\varrho : \Gamma(\tilde{K}) \longrightarrow \mathfrak{X}(M)$ is a Lie algebra homomorphism, we deduce that C_D is trilinear over $C^\infty(M, \mathbb{R})$ and that

$$C_D(\tilde{s}_1, \tilde{s}_2)(s) = -C_D(\tilde{s}_2, \tilde{s}_1)(s).$$

Thus, we have that there exist two $C^\infty(M, \mathbb{R})$-bilinear skew-symmetric mappings $(R_D)_j : \Gamma(\tilde{K}) \times \Gamma(\tilde{K}) \longrightarrow C^\infty(M, \mathbb{R})$, $j = 1, 2$, such that

$$C_D(\tilde{s}_1, \tilde{s}_2)(s) = ((R_D)_1(\tilde{s}_1, \tilde{s}_2) + i(R_D)_2(\tilde{s}_1, \tilde{s}_2))s. \tag{29}$$

We remark that $(R_D)_1$ and $(R_D)_2$ induce two cross sections of the vector bundle $\Lambda^2 \tilde{K}^* \longrightarrow M$.

We will denote by $R_D : \Gamma(\tilde{K}) \times \Gamma(\tilde{K}) \longrightarrow C^\infty(M, \mathbb{C})$ the map defined by

$$R_D(\tilde{s}_1, \tilde{s}_2) = (R_D)_1(\tilde{s}_1, \tilde{s}_2) + i(R_D)_2(\tilde{s}_1, \tilde{s}_2).$$

Since R_D completely determines to C_D, we will say also that R_D is the curvature of D.

Now, we can extend by linearity the operator ∂ to the space $C^k(\Gamma(\tilde{K}); C^\infty(M, \mathbb{C}))$ given by

$$\begin{aligned} C^k(\Gamma(\tilde{K}); C^\infty(M, \mathbb{C})) \quad = \quad & \{R^k : \Gamma(\tilde{K}) \times \ldots^{(k} \cdots \times \Gamma(\tilde{K}) \longrightarrow C^\infty(M, \mathbb{C}) / \ R^k \text{ is} \\ & \mathbb{R}\text{-multilinear and skew-symmetric}\}. \end{aligned}$$

In fact, if $R^k = R_1^k + iR_2^k \in C^k(\Gamma(\tilde{K}); C^\infty(M, \mathbb{C}))$ we define

$$\partial R^k = \partial R_1^k + i\partial R_2^k.$$

It is clear that $\partial^2 = 0$ and, therefore, we obtain the corresponding cohomology which will be denoted by $H^*(\Gamma(\tilde{K}); C^\infty(M, \mathbb{C}))$.

Moreover, using (23), (27), (28), Definition 4 and proceeding as in the proof of Theorem IV.3 in [19], we conclude

Theorem 9 *Let $\pi : K \longrightarrow M$ be a complex line bundle over M. Suppose that D is a $\Gamma(\tilde{K})$-derivative on $\pi : K \longrightarrow M$ with curvature R_D. Then:*

1. *R_D defines a cohomology class in $H^2(\Gamma(\tilde{K}); C^\infty(M, \mathbb{C}))$.*
2. *If $\bar{D}$ is another $\Gamma(\tilde{K})$-derivative on $\pi : K \longrightarrow M$, there exists a $C^\infty(M, \mathbb{R})$-linear mapping $\tilde{P}_{(\bar{D}-D)} : \Gamma(\tilde{K}) \longrightarrow C^\infty(M, \mathbb{C})$ such that*

   $$R_{\bar{D}} - R_D = \partial(\tilde{P}_{(\bar{D}-D)}).$$

   In particular, $[R_{\bar{D}}] = [R_D]$.
3. *If h is a Hermitian metric on $\pi : K \longrightarrow M$ and D is a Hermitian $\Gamma(\tilde{K})$-derivative on $\pi : K \longrightarrow M$ then R_D is purely imaginary.*

4.3. Prequantization

Let $\pi : K \longrightarrow M$ be an arbitrary complex line bundle over M.

In this section, we will assume that a $\Gamma(\tilde{K})$-derivative D on $\pi : K \longrightarrow M$ always satisfies the following conditions:

(C1) If $X_1(x) = 0$ then $D_{j(1)(x)}s = 0$, for all $s \in \Gamma(K)$.

(C2) If $X_1(x) \neq 0$ and there exists $\tilde{s}_x \in \tilde{F}_x$ such that $\#(\tilde{s}_x) = X_1(x)$ then

$$D_{\tilde{s}_x} s = D_{j(1)(x)} s$$

for all $s \in \Gamma(K)$.

Note that if ∇ is a $\mathcal{F}$-foliated covariant derivative on $\pi : K \longrightarrow M$ and D is the $\Gamma(\tilde{K})$-derivative defined by

$$D_{\tilde{s}_x} s = \nabla_{\varrho(\tilde{s}_x)}(s_{|L_x})$$

for $\tilde{s}_x \in \tilde{K}_x$ and $s \in \Gamma(K)$, then D satisfies the above conditions.

Next, we will introduce a new definition of prequantization bundle for the manifold M.

Definition 6 *A complex line bundle $\pi : K \longrightarrow M$ over M is said to be prequantization bundle if*

$$\widehat{\{f, g\}} = \widehat{f} \circ \widehat{g} - \widehat{g} \circ \widehat{f} \qquad f, g \in C^\infty(M, \mathbb{R}) \tag{30}$$

with $\widehat{f} \in End_{\mathbb{C}}(\Gamma(K))$ given by

$$s \in \Gamma(K) \longmapsto \widehat{f}(s) = D_{j(f)} s + 2\pi i f s, \tag{31}$$

where D is a $\Gamma(\tilde{K})$-derivative on $\pi : K \longrightarrow M$. The manifold M is said to be quantizable if there exits a prequantization bundle $\pi : K \longrightarrow M$ over M.

It is clear that if M is quantizable in the sense of Section 3.3 (see Definition 3) then M is also quantizable in the above sense. However, in general, the converse is not true.

Let $\bar{\Lambda}$ be the 2-coboundary in the H-Chevalley-Eilenberg complex given by (15) and let

$$\tilde{j} : C^2(\Gamma(\tilde{K}); C^\infty(M, \mathbb{R})) \longrightarrow C^2_{HCE}(M)$$

be the homomorphism defined by (26).

Using (28), (29), (30), (31), Definition 4 and the fact that j is a Lie algebra homomorphism, we deduce

Lemma 2 *The manifold M is quantizable if and only if there exist a complex line bundle $\pi : K \longrightarrow M$ over M and a $\Gamma(\tilde{K})$-derivative D on $\pi : K \longrightarrow M$ with curvature $R_D = (R_D)_1 + i(R_D)_2$ such that*

$$\tilde{j}((R_D)_1) = 0, \qquad \tilde{j}(\frac{1}{2\pi}(R_D)_2) = -\bar{\Lambda}.$$

Now, if $\tilde{\mathcal{H}} : \Omega^2(M) \longrightarrow C^2_{HCE}(M)$ is the homomorphism given by (16) then, proceeding as in the proof of Theorem V.2 of [19] and using (23), Lemma 2, Theorems 5, 8 and 9 and the fact that ϱ is a Lie algebra homomorphism, we conclude

Theorem 10 *The manifold M is quantizable if and only if there exist an integral closed 2-form Ω on M and a cross section $\tilde{s}^*$ of the dual bundle $\tilde{\pi}^* : \tilde{K}^* \longrightarrow M$ such that:*

1. *If $\tilde{P} : \Gamma(\tilde{K}) \longrightarrow C^\infty(M, \mathbb{R})$ is the $C^\infty(M, \mathbb{R})$-linear map induced by $\tilde{s}^*$ then*

$$\tilde{\mathcal{H}}(\Omega) = \bar{\Lambda} + \partial_H(\tilde{j}(\tilde{P})) = \partial_H(I + \tilde{j}(\tilde{P})).$$

2. *If x is a point of M and $(X_1)(x) = 0$ then $\tilde{s}^*(x)(j(1)(x)) = 0$.*
3. *If x is a point of M such that $(X_1)(x) \neq 0$ and $\#(\tilde{s}_x) = X_1(x)$, with $\tilde{s}_x \in \tilde{F}_x$, then $\tilde{s}^*(x)(\tilde{s}_x) = \tilde{s}^*(x)(j(1)(x))$.*

5. The particular cases: Jacobi and Poisson manifolds

Let (M, Λ, E) be a Jacobi manifold and $\{\ ,\ \}$ the Jacobi bracket.

Suposse that $(C^*_{HCE}(M), \partial_H)$ is the H-Chevalley-Eilenberg complex associated to M. Using (2), (4), (6) and (15), we deduce that

$$\bar{\Lambda}(f,g) = (\partial_H I)(f,g) = \Lambda(df, dg), \qquad f, g \in C^\infty(M, \mathbb{R}). \tag{32}$$

Next, we will recall the definition of the Lichnerowicz-Jacobi cohomology (see [18] and [19]).

A k-cochain $c^k \in C^k_{HCE}(M)$ is said to be 1-*differentiable* if it is defined by a linear differential operator of order 1. If $\mathcal{V}^r(M)$ is the space of r-vectors on M then we can identify the space $\mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M)$ with the space of all 1-differentiable k-cochains $C^k_{HCE1-diff}(M)$ as follows: define $j_k : \mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M) \longrightarrow C^k_{HCE}(M)$ the monomorphism given by

$$j_k(P,Q)(f_1, \ldots, f_k) = P(df_1, \ldots, df_k) + \sum_{q=1}^{k} (-1)^{q+1} f_q Q(df_1, \ldots, \widehat{df_q}, \ldots, df_k). \tag{33}$$

Then $j_k(\mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M)) = C^k_{HCE1-diff}(M)$ which implies that the spaces $\mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M)$ and $C^k_{HCE1-diff}(M)$ are isomorphic. Note that (see (32) and (33))

$$\bar{\Lambda} = \partial_H I = j_2(\Lambda, 0). \tag{34}$$

On the other hand, using that $\{\ ,\ \}$ is a linear differential operator of order 1, we deduce that $\partial_H \tilde{P} \in C^{k+1}_{HCE1-diff}(M)$, for $\tilde{P} \in C^k_{HCE1-diff}(M)$. Thus, we have the corresponding subcomplex

$$(C^*_{HCE1-diff}(M), \partial_{H|C^*_{HCE1-diff}(M)})$$

of the H-Chevalley-Eilenberg complex whose cohomology $H^*_{HCE1-diff}(M)$ will be called the 1-*differentiable H-Chevalley-Eilenberg cohomology* of M. Moreover, we obtain that (see [18, 19])

$$\partial_H(j_k(P,Q)) = j_{k+1}(\sigma_{LJ}(P,Q)) \tag{35}$$

where

$$\sigma_{LJ}(P,Q) = (-[\Lambda, P] + kE \wedge P + \Lambda \wedge Q, [\Lambda, Q] - (k-1)E \wedge Q + [E, P]). \tag{36}$$

This last equation defines a mapping $\sigma_{LJ} : \mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M) \longrightarrow \mathcal{V}^{k+1}(M) \oplus \mathcal{V}^k(M)$ which is in fact a differential operator that verifies $\sigma^2_{LJ} = 0$. Therefore, we have a complex $(\mathcal{V}^*(M) \oplus \mathcal{V}^{*-1}(M), \sigma_{LJ})$ whose cohomology will be called the *Lichnerowicz-Jacobi cohomology* (*LJ-cohomology*) of M and denoted by $H^*_{LJ}(M)$ (see [18, 19]).

Note that the mappings $j_k : \mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M) \longrightarrow C^k_{HCE}(M)$ given by (33) induce an isomorphism between the complexes $(\mathcal{V}^*(M) \oplus \mathcal{V}^{*-1}(M), \sigma_{LJ})$ and $(C^*_{HCE1-diff}(M), \partial_{H|C^*_{HCE1-diff}(M)})$ and consequently the corresponding cohomologies are isomorphic. Furthermore, from (36), we obtain that

$$\sigma_{LJ}(0,1) = (\Lambda, 0). \tag{37}$$

Now, if M is a Poisson manifold ($E = 0$), it follows that (see (36))

$$\sigma_{LJ}(P, 0) = (-[\Lambda, P], 0) \tag{38}$$

for $P \in \mathcal{V}^k(M)$. Thus, we deduce that the linear differential operator $\sigma_{LP} : \mathcal{V}^k(M) \longrightarrow \mathcal{V}^{k+1}(M)$ defined by

$$\sigma_{LP}(P) = -[\Lambda, P] \tag{39}$$

satisfies the condition $\sigma_{LP}^2 = 0$. This fact allows us to consider the differential complex $(\mathcal{V}^*(M), \sigma_{LP})$. The cohomology of this complex is the *Lichnerowicz-Poisson cohomology* (*LP-cohomology*) associated to M and it is denoted by $H^*_{LP}(M)$ (see [20] and [29]).

Next, we will obtain the relation between the $\mathcal{F}$-foliated cohomology of a Jacobi manifold (M, Λ, E) and the LJ-cohomology.

Denote by $\# : \Omega^k(M) \longrightarrow \mathcal{V}^k(M)$ the map given by (4) and (5) and let $\tilde{\#} : \Omega^k(M) \longrightarrow \mathcal{V}^k(M) \oplus \mathcal{V}^{k+1}(M)$ be the homomorphism of $C^\infty(M, \mathbb{R})$-modules defined by

$$\tilde{\#}(\alpha) = (\#(\alpha), -\#(i_E\alpha)) \tag{40}$$

for $\alpha \in \Omega^k(M)$. In [19] (see also [18]), we prove that

$$\tilde{\#} \circ d = -\sigma_{LJ} \circ \tilde{\#}. \tag{41}$$

This implies that the mappings $\tilde{\#} : \Omega^k(M) \to \mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M)$ induce a homomorphism of complexes $\tilde{\#} : (\Omega^*(M), d) \longrightarrow (\mathcal{V}^*(M) \oplus \mathcal{V}^{*-1}(M), -\sigma_{LJ})$ and, thus, a homomorphism in cohomology $\tilde{\#} : H^*_{dR}(M) \longrightarrow H^*_{LJ}(M)$ (see [18, 19]). In fact, using (4), (5), (6), (33) and (40), we have that

$$j_k \circ \tilde{\#} = (-1)^k \tilde{\mathcal{H}}, \tag{42}$$

$\tilde{\mathcal{H}} : \Omega^k(M) \longrightarrow C^k_{HCE}(M)$ being the map given by (16).

On the other hand, if $\Omega^k(\mathcal{F})$ is the space of the $\mathcal{F}$-foliated k-forms on M and $r(\mathcal{F}) : \Omega^k(M) \longrightarrow \Omega^k(\mathcal{F})$ is the canonical homomorphism, it is clear that there exists a homomorphism of complexes $\tilde{\#}(\mathcal{F}) : (\Omega^*(\mathcal{F}), d) \longrightarrow (\mathcal{V}^*(M) \oplus \mathcal{V}^{*-1}(M), -\sigma_{LJ})$ such that the following diagram is commutative

$$\begin{tikzcd}
(\Omega^*(\mathcal{F}), d) \arrow[rr, "\tilde{\#}(\mathcal{F})"] & & (\mathcal{V}^*(M) \oplus \mathcal{V}^{*-1}(M), -\sigma_{LJ}) \\
& (\Omega^*(M), d) \arrow[ul, "r(\mathcal{F})"] \arrow[ur, "\tilde{\#}"'] &
\end{tikzcd}$$

In fact, if $\#(\mathcal{F}) : \Omega^k(\mathcal{F}) \to \mathcal{V}^k(M)$ is the homomorphism of $C^\infty(M, \mathbb{R})$-modules defined by

$$\#(\mathcal{F})(f) = f, \qquad (\#(\mathcal{F})(\alpha))(\alpha_1, \ldots, \alpha_k) = (-1)^k \alpha(\#(\alpha_1), \ldots, \#(\alpha_k)) \tag{43}$$

for $f \in C^\infty(M, \mathbb{R})$, $\alpha \in \Omega^k(\mathcal{F})$ and $\alpha_1, \ldots, \alpha_k \in \Omega^1(M)$, then

$$\tilde{\#}(\mathcal{F})(\alpha) = (\#(\mathcal{F})(\alpha), -\#(\mathcal{F})(i_E\alpha)). \tag{44}$$

Thus, from (6), (33), (43) and (44), we deduce that

$$j_k \circ \tilde{\#}(\mathcal{F}) = (-1)^k \tilde{\mathcal{H}}(\mathcal{F}), \tag{45}$$

where $\tilde{\mathcal{H}}(\mathcal{F}) : \Omega^k(\mathcal{F}) \longrightarrow C^k_{HCE}(M)$ is defined by (18).

Therefore, using (35) and Theorem 3, we conclude that

$$\tilde{\#}(\mathcal{F}) \circ d = -\sigma_{LJ} \circ \tilde{\#}(\mathcal{F}).$$

Consequently, for the corresponding homomorphisms in cohomology, we obtain the following commutative diagram

$$\begin{tikzcd}
H^k(\mathcal{F}) \arrow[dr, "\tilde{\#}(\mathcal{F})"'] \arrow[drr, "(-1)^k\tilde{\mathcal{H}}(\mathcal{F})"] & & \\
& H^k_{LJ}(M) \arrow[r, "j_k"] & H^k_{HCE}(M) \\
H^k_{dR}(M) \arrow[uu, "r(\mathcal{F})"] \arrow[ur, "\tilde{\#}"] \arrow[urr, "(-1)^k\tilde{\mathcal{H}}"'] & &
\end{tikzcd}$$

In the particular case when M is a Poisson manifold, using the above results, we prove that the mappings $\#(\mathcal{F}) : \Omega^k(\mathcal{F}) \longrightarrow \mathcal{V}^k(M)$ induce two homomorphisms of complexes

$$\# : (\Omega^*(M), d) \longrightarrow (\mathcal{V}^*(M), -\sigma_{LP}), \qquad \#(\mathcal{F}) : (\Omega^*(\mathcal{F}), d) \longrightarrow (\mathcal{V}^*(M), -\sigma_{LP})$$

which satisfy the conditions

$$\#(\mathcal{F}) \circ r(\mathcal{F}) = \#, \quad j_k \circ \# = (-1)^k \tilde{\mathcal{H}}, \quad j_k \circ \#(\mathcal{F}) = (-1)^k \tilde{\mathcal{H}}(\mathcal{F}).$$

Next, we will give necessary and sufficient conditions for a Jacobi manifold M to be quantizable in the sense of Section 3.3 and in the sense of Section 4.3.

5.1. Prequantization bundles in the sense of Section 3.3

From (34), (35), (37), (42), (43), (44), (45) and Theorem 6, it follows

Theorem 11 *Let (M, Λ, E) be a Jacobi manifold. Then, M is quantizable if and only if there exist an integral closed 2-form Ω on M and a $\mathcal{F}$-foliated 1-form α such that*

$$\tilde{\#}(\Omega) = \sigma_{LJ}(\#(\mathcal{F})(\alpha), 1 - \alpha(E)).$$

Using Theorem 11, we deduce

Corollary 1 *Let (M, Λ) be a Poisson manifold. Then, M is quantizable if and only if there exist an integral closed 2-form Ω on M and a $\mathcal{F}$-foliated 1-form α such that*

$$\#(\Omega) = \Lambda + \sigma_{LP}(\#(\mathcal{F})(\alpha)).$$

Now, we will study the case of a transitive Jacobi manifold M ($\mathcal{F}_x = T_xM$, for all $x \in M$).
We distinguish the following cases:

1. *Symplectic manifolds:* Let (M, Ω) be a symplectic manifold and Λ the Poisson bivector. Then the mapping $\# : \Omega^k(M) \longrightarrow \mathcal{V}^k(M)$ is an isomorphism of $C^\infty(M, \mathbb{R})$-modules and $\#(\Omega) = \Lambda$. Using these facts and Corollary 1 we recover the result of Kostant and Souriau (see [14, 26]), that is, M is quantizable if and only if Ω is integral.

2. *Locally conformal symplectic manifolds:* If a Jacobi manifold is quantizable in the sense of Section 3.3 then it is also quantizable in the sense of Section 4.3. Using this fact and the results of [19], we conclude that a l.c.s. manifold is quantizable if and only if it is a quantizable symplectic manifold.

3. *Contact manifolds:* Let (M, η) be a contact manifold. Then, from Theorem 11, we obtain that M is quantizable (it is sufficient to take $\Omega = 0$ and $\alpha = \eta$).

5.2. Prequantization bundles in the sense of Section 4.3

Let $(\tilde{K}, \{\ ,\ \}\tilde{}, \varrho)$ be the Lie algebroid associated to a Jacobi manifold M. Then, $\tilde{K} = T^*M \times \mathbb{R}$ (see Section 2) and thus a cross section $\tilde{s}^*$ of the dual bundle $\tilde{\pi}^* : \tilde{K}^* \longrightarrow M$ can be identified with a pair $(\tilde{X}, \tilde{f}) \in \mathfrak{X}(M) \times C^\infty(M, \mathbb{R})$ in such a way that the corresponding $C^\infty(M, \mathbb{R})$-linear map $\tilde{P} : \Gamma(\tilde{K}) \longrightarrow C^\infty(M, \mathbb{R})$ is given by

$$\tilde{P}(\tilde{\alpha}, \tilde{g}) = \tilde{\alpha}(\tilde{X}) + \tilde{f}\tilde{g}, \tag{46}$$

for $(\tilde{\alpha}, \tilde{g}) \in \Gamma(\tilde{K}) = \Omega^1(M) \times C^\infty(M, \mathbb{R})$.

Therefore, if $\tilde{j} : C^1(\Gamma(\tilde{K}); C^\infty(M, \mathbb{R})) \longrightarrow C^1_{HCE}(M)$ is the homomorphism defined by (26) then, from (10), (33) and (46), we obtain that $\tilde{j}(\tilde{P})$ is the linear differential operator of order 1, $j_1(\tilde{X}, \tilde{f})$. Using these facts, (34), (35), (37), (42) and Theorem 10, we deduce a result which has been proved in [19].

Theorem 12 *[19] Let (M, Λ, E) be a Jacobi manifold. Then, M is quantizable if and only if there exist an integral closed 2-form Ω, a vector field A and a real differentiable function f such that:*

1. $\tilde{\#}(\Omega) = \sigma_{LJ}(A, f)$.
2. *If x is a point of M and $E_x = 0$ then $f(x) = 1$.*
3. *If x is a point of M and α_x is a 1-form at x such that $E_x \neq 0$ and $\#(\alpha_x) = E_x$ then $f(x) = \alpha_x(A_x) + 1$.*

From Theorem 12, it follows a result of Vaisman [28].

Corollary 2 *[28] Let (M, Λ) be a Poisson manifold. Then, M is quantizable if and only if there exist an integral closed 2-form Ω and a vector field A such that*

$$\#(\Omega) = \Lambda + \sigma_{LP}(A) = \Lambda - \mathcal{L}_A\Lambda.$$

Finally, for the transitive Jacobi manifolds we obtain the same results that in Section 5.1 (see [19] for more details).

Acknowledgement. This work has been partially supported through grants DGICYT (Spain) (Project PB94-0106) and University of La Laguna (Spain). ML wishes to express his gratitude for the hospitality offered to him in the Departamento de Matemática Fundamental (University of La Laguna) where part of this work was conceived.

References

[1] Albert, C. (1989). Le théorème de réduction de Marsden-Weinstein en géométrie cosymplectique et de contact, *J. Geom. Phys.*, **6** (4), 627-649.

[2] Blair, D. E. (1976). *Contact manifolds in Riemannian geometry*, Lecture Notes in Math., 509, Springer-Verlag, Berlin.

[3] Bhaskara, K. H., Viswanath, K. (1988). *Poisson algebras and Poisson manifolds*, Research Notes in Mathematics, 174, Pitman, London.

[4] Coste, A., Dazord, P., Weinstein, A. (1987). Groupoïdes symplectiques, *Pub. Dép. Math. Lyon,* **2/A**, 1-62.

[5] Dazord, P. (1995). Intégration d'algèbres de Lie locales et groupoïdes de contact, *C. R. Acad. Sci. Paris Sér. I*, **320**, 959-964.

[6] Dazord, P., Hector, G. (1991). Intégration symplectique des variétés de Poisson totalement asphériques, *Symplectic Geometry, Grupoids and Integrable Systems, Séminaire Sud Rhodanien de Géométrie a Berkeley* (1989) (P. Dazord and A. Weinstein, eds.). *MSRI Publ.* **20**, Springer-Verlag, Berlin-Heidelberg-New York, 37-72.

[7] Dazord, P., Lichnerowicz, A., Marle, Ch. M. (1991). Structure locale des variétés de Jacobi, *J. Math. Pures Appl.*, **70**, 101-152.

[8] Guédira, F., Lichnerowicz, A. (1984). Géométrie des algèbres de Lie locales de Kirillov, *J. Math. Pures Appl.*, **63**, 407-484.

[9] Hector, G., Macias, E., Saralegi, M. (1989). Lemme de Moser feuilleté et classification des variétés de Poisson régulières, *Publicacions Matemàtiques*, **33**, 423-430.

[10] Huebschmann, J. (1990). Poisson cohomology and quantization, *J. Reine Angew. Math.* **408**, 57-113.

[11] Karasev, M. V., Maslov, V. P. (1993). *Nonlinear Poisson Brackets. Geometry and Quantization*, Translations of Mathematical Monographs, **119**, A. M. S., Providence. R. I.

[12] Kerbrat, Y., Souici-Benhammadi, Z. (1993). Variétés de Jacobi et groupoïdes de contact, *C. R. Acad. Sci. Paris Sér. I*, **317**, 81-86.

[13] Kirillov. A. (1976). Local Lie algebras, *Russian Math. Surveys*, **31** No. 4, 55-75.

[14] Kostant, B. (1970). Quantization and unitary representations, *Lectures in modern analysis and applications III* (C. T. Taam, ed.). *Lecture Notes in Math.*, **170**, Springer-Verlag, Berlin-Heidelberg-New York, 87-207.

[15] de León, M., Marrero, J. C., Padrón, E. (1997). Lichnerowicz-Jacobi cohomology of Jacobi manifolds, *C. R. Acad. Sci. Paris Sér. I*, **324**, 71-76.

[16] de León, M., Marrero, J. C., Padrón, E. (1998). A generalization for Jacobi manifolds of the Lichnerowicz-Poisson cohomology, *Proceedings of the V Fall Workshop: Differential Geometry and its Aplications, Jaca, September 23-25, 1996.* Memorias de la Real Academia de Ciencias, Serie de Ciencias Exactas, Tomo XXXII, pp. 131-149.

[17] de León, M., Marrero, J. C., Padrón, E. (1997). Lichnerowicz-Jacobi cohomology, *J. Phys. A: Math. Gen.* **30**, 6029-6055.

[18] de León, M., Marrero, J. C., Padrón, E. (1997). H-Chevalley-Eilenberg cohomology of a Jacobi manifold and Jacobi-Chern class, *C. R. Acad. Sci. Paris Sér. I*, **325**, 405-410.

[19] de León, M., Marrero, J. C., Padrón, E. (1997). On the geometric quantization of Jacobi manifolds, *J. Math. Phys.*, **38** (12), 6185-6213.

[20] Lichnerowicz, A. (1977). Les variétés de Poisson et leurs algèbres de Lie associées, *J. Differential Geometry*, **12**, 253-300.

[21] Lichnerowicz, A. (1978). Les variétés de Jacobi et leurs algèbres de Lie associées, *J. Math. Pures Appl.*, **57**, 453-488.

[22] Libermann, P., Marle, Ch. M. (1987). *Symplectic Geometry and Analytical Mechanics*, Kluwer, Dordrecht.

[23] Mackenzie, K. (1987). *Lie groupoids and Lie algebroids in differential geometry*, London Math. Soc. Lecture Notes Series 124, Cambridge Univ. Press, Cambridge.

[24] Marle, Ch. M. (1985). Quelques propriétés des variétés de Jacobi, *Géométrie symplectique et mécanique (Seminaire sud-rhodanien de géométrie).* J-P. DUFOUR éd. pp. 125-139, Travaux en Cours, Hermann, Paris.

[25] Pradines, J. (1966). Théorie de Lie pour les groupoïdes différentiables, *C. R. Acad. Sci. Paris Sér. A*, **263**, 907-910.

[26] Souriau, J. M. (1969). *Structures des systèmes dynamiques,* Dunod, Paris.

[27] Vaisman, I. (1985). Locally conformal symplectic manifolds, *Internat. J. Math. & Math. Sci.*, **8** (3), 521-536.

[28] Vaisman, I. (1991). On the geometric quantization of Poisson manifolds, *J. Math. Phys.*, **32**, 3339-3345.

[29] Vaisman, I. (1994). *Lectures on the Geometry of Poisson Manifolds*, Progress in Math. 118, Birkhäuser, Basel.

[30] Weinstein, A. (1983). The local structure of Poisson manifolds, *J. Differential Geometry*, **18**, 523-557. Errata et addenda: *J. Differential Geometry*, **22** (1985). 255.

[31] Weinstein, A. (1987). Symplectic groupoids and Poisson manifolds, *Bull. Amer. Math. Soc.* **16**, 101-104.

M. de León
Instituto de Matemáticas y Física Fundamental
Consejo Superior de Investigaciones Científicas
28006 Madrid, Spain
mdeleon@imaff.cfmac.csic.es

J. C. Marrero, E. Padrón
Departamento de Matemática Fundamental
Universidad de La Laguna
La Laguna, Tenerife, Canary Islands, Spain
jcmarrer@ull.es, mepadron@ull.es