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A note on the convolution of inverted-gamma distributions with applications to the Behrens-Fisher distribution

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Abstract. The generalized Behrens-Fisher distribution is defined as the convolution of two Student t distributions and is related to the inverted-gamma distribution by means of a representation theorem as a scale mixture of normals where the mixing distribution is a convolution of two inverted-gamma distributions.

One important result in this paper establishes that for odd degrees of freedom the Behrens-Fisher distribution is distributed as a finite mixture of Student t distributions. This result follows from the main theorem concerning the form of the convolution of inverted-gamma distributions with demi-integer shape parameter.

Una nota sobre la convolución de distribuciones gammas invertidas con aplicaciones a la distribución de Behrens-Fisher

Resumen. La distribución de Behrens-Fisher generalizada se define como convolución de dos distribuciones t de Student y se relaciona con la distribución gamma invertida por medio de un teorema de representación como una mixtura, respecto del parámetro de escala, de distribuciones normales cuando la distribución de mezcla es la convolución de dos distribuciones gamma invertidas.

Un resultado importante de este artículo establece que la distribución de Behrens-Fisher con grados de libertad impares es mixtura finita de distribuciones t de Student. Este resultado se deduce del teorema fundamental sobre la convolución de distribuciones gamma invertidas con parámetros de forma semienteros.

1. Introduction

The generalized Behrens-Fisher distribution is defined as the convolution of two arbitrary Student t distributions. It is a rather complex and ackward distribution which does not even have a simple density function. The main purpose of this note is to establish the relation between the Behrens-Fisher distribution, when certain conditions are imposed on their degrees of freedom, and a finite mixture of t distributions. This result is important from a theoretical as well as from a practical point of view for it may provide exact results when computing, for instance, the density function os some Behrens-Fisher distributions, and may also simplify the computation of their percentiles. A sketch of the proof, namely Theorem 3, is given in section 3, along with other related results. Their proof rests basically on the main theorem of the paper, namely

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theorem 1. In the preceding section 2, we present some facts and results concerning the Behrens-Fisher and inverted-gamma distributions which are needed for the proof of the main theorem and can be found in the existing literature, mainly in Girón et al. [3].

2. Some definitions and needed results

A random variable X follows an inverted-gamma distribution, and will be denoted as $X \sim \mathrm{Ga}^{-1}(\alpha, \beta)$, if its reciprocal $1/X$ follows a gamma distribution, $Ga(\alpha, \beta)$, with shape parameter α and scale parameter β.

Its density function is

$$f(x\,|\,\alpha,\beta) = \begin{cases} \dfrac{\beta^\alpha}{\Gamma(\alpha)} x^{-(\alpha+1)} e^{-\beta/x} & \text{if} \quad x > 0, \\ 0 \quad \text{otherwise,} \end{cases}$$

and its characteristic function is

$$\varphi_x(t) = \frac{2\beta^{\alpha/2}}{\Gamma(\alpha)}(-it)^{\alpha/2} K_\alpha\left(2\sqrt{\beta}\sqrt{-it}\right)$$

where $K_m(\cdot)$ denotes the modified Bessel function of the second kind of order m.

The inverted-gamma distribution appears in Bayesian inference, in a natural way, as the posterior distribution of the variance in normal sampling when reference or conjugate distributions on the parameters are used.

It also appears, within the context of the Bayesian analysis of the normal model, in the following well known representation of the Student t distribution as a scale mixture of normals when the mixing distribution is inverted-gamma.

In fact, if t is a standard Student distribution with ν degrees of freedom, denoted as $t \sim t(0, 1; \nu)$, then

$$t \sim \int_o^\infty N(t\,|\,0,\lambda) d\mathrm{Ga}^{-1}\left(\lambda \,\middle|\, \frac{\nu}{2}, \frac{\nu}{2}\right) \tag{1}$$

On the other hand, the Behrens-Fisher distribution appears in Bayesian statistics as the posterior distribution of the difference of two independent normal means when the variances of the two populations are not assumed to be equal, and reference or conjugate distributions over the parameters are assumed.

The standard Behrens-Fisher distribution with degrees of freedom $\nu_1 > 0, \nu_2 > 0$ and angle $\phi \in [0, \frac{\pi}{2}]$ is defined as $b_0 = x_1 \operatorname{sen}\phi - x_2 \cos\phi$, where x_1 and x_2 are independent random variables and $x_i \sim t(0, 1; \nu_i)$, for $i = 1, 2$, and it is denoted as $b_0 \sim \text{Be-Fi}(b_0\,|\,\phi, \nu_1, \nu_2)$.

The density and the distribution function of the Behrens-Fisher distribution cannot be given in a simple explicit form (except for the case $\nu_1 = \nu_2 = 1$).

The following espression for the density can be found in Box and Tiao [1].

$$\begin{aligned} p(b_0\,|\,\phi,\nu_1,\nu_2) = k \int_{-\infty}^{+\infty} &\left[1 + \frac{(\tau \operatorname{sen}\phi + z\cos\phi)^2}{\nu_1}\right]^{-\frac{1}{2}(\nu_1+1)} \\ &\times \left[1 + \frac{(z \operatorname{sen}\phi - \tau\cos\phi)^2}{\nu_2}\right]^{-\frac{1}{2}(\nu_2+1)} dz, \quad -\infty < b_0 < +\infty \end{aligned}$$

where

$$k^{-1} = \frac{\Gamma\left(\frac{\nu_1+1}{2}\right)\Gamma\left(\frac{\nu_2+1}{2}\right)}{\pi\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)}(\nu_1\nu_2)^{-1/2}.$$

For an extension and generalization of the Behrens-Fisher distribution, see Girón et al. [3], where it is proven that the convolution of any two arbitrary t distributons is a generalized Behrens-Fisher distribution.

The following mixture representation of the standard Behrens-Fisher distribution which follows from (1) —the proof of which can be found in Girón et al. [3]—, relates the Behrens-Fisher distribution to the convolution of inverted-gamma distributions.

$$\begin{aligned} b_0 &\sim \int_0^\infty \int_0^\infty N(b_0\,|\,0, \lambda_1 \mathrm{sen}^2\phi + \lambda_2 \cos^2\phi) \prod_{i=1}^{2} d\mathrm{Ga}^{-1}\left(\lambda_i \,\middle|\, \frac{\nu_i}{2}, \frac{\nu_i}{2}\right) \\ &\sim \int_0^\infty \int_0^\infty N(b_0\,|\,0, x_1 + x_2) d\mathrm{Ga}^{-1}\left(x_1 \,\middle|\, \frac{\nu_1}{2}, \frac{\nu_1}{2}\mathrm{sen}^2\phi\right) d\mathrm{Ga}^{-1}\left(x_2 \,\middle|\, \frac{\nu_2}{2}, \frac{\nu_2}{2}\cos^2\phi\right) \\ &\sim \int_0^\infty N(b_0\,|\,0, z) d\mathrm{CGa}^{-1}\left(z \,\middle|\, \frac{\nu_1}{2}, \frac{\nu_1}{2}\mathrm{sen}^2\phi, \frac{\nu_2}{2}, \frac{\nu_2}{2}\cos^2\phi\right) \end{aligned}$$

where $\mathrm{CGa}^{-1}(z\,|\,\alpha_1, \beta_1, \alpha_2, \beta_2)$ denotes the convolution of two independent inverted-gamma distributions with parameters (α_1, β_1) y (α_2, β_2), respectively.

Last formula provides the connection or link between the Behrens-Fisher distribution and the convolution of two inverted-gamma distributions. It essentialy shows that the Behrens-Fisher distribution is a scale mixture of normals when the mixing distribution is a convolution of two inverted-gamma's. Thus, it becomes apparent that properties and results on the convolution of inverted-gamma distributions may result on interesting properties of the Behrens-Fisher distribution.

But, in general, the convolution of two inverted-gamma densities does not have an explicit or simple form except for some very especific values of their parameters. Next section presents, however, some results in this direction.

3. Main results

Next theorem shows that under some restrictions on the shape parameters —namely, that they be demi-integers—, the convolution of inverted-gamma distributions is distributed as a finite mixture of inverted-gamma distributions all having the same scale parameter.

Theorem 1 *If $x \sim Ga^{-1}(n+\frac{1}{2}, \beta_1), y \sim Ga^{-1}(m+\frac{1}{2}, \beta_2)$, $n, m \in \mathbb{N}$, $m \geq n$ and x and y are independent, then the convolution of x and y is distributed as the following mixture*

$$x + y \sim \sum_{i=1}^{m+1} p_i\, Ga^{-1}\left(n - \frac{1}{2} + i, (\sqrt{\beta}_1 + \sqrt{\beta}_2)^2,\right)$$

where the weights $p_i \geq 0$, $i = 1, \ldots, m+1$, $\sum_{i=1}^{m+1} p_1 = 1$ are computed in a recursive manner from the formulae

$$\begin{aligned} p_{m+1} &= \frac{\sqrt{\pi}\Gamma\left(n+m+\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)} \frac{(\sqrt{\beta_1})^n (\sqrt{\beta_2})^m}{(\sqrt{\beta_1}+\sqrt{\beta_2})^{n+m}} \\ p_{j+1} &= 2^{2j+2}\Gamma\left(n+\frac{1}{2}+j\right)\left(\frac{c\gamma_{n+j}}{2^{2+2j}\sqrt{\pi}(\sqrt{\beta_1}+\sqrt{\beta_2})^{n+j}}\right. \\ &\quad \left. - \sum_{i=j+2}^{m+1} \frac{p_i}{2^{2i}\Gamma\left(n-\frac{1}{2}+i\right)} \cdot \frac{(n+2i-2-j)!}{(n+j)!(i-1-j)!}\right), \quad j = 0, \ldots, m-1 \end{aligned} \tag{2}$$

where

$$
\begin{aligned}
c &= \frac{\pi}{2^{2(n+m)}\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)} \\
\gamma_k &= 2^{2k}\sum_{i=0}^{k}\frac{(2n-i)!}{i!(n-i)!}\frac{(2m-k+i)!}{(k-i)!(m-k+i)!}(\sqrt{\beta_1})^i(\sqrt{\beta_2})^{k-i}, \quad k=0,\ldots,n. \\
\gamma_{n+k} &= 2^{2(n+k)}\sum_{i=0}^{n}\frac{(2n-i)!}{i!(n-i)!}\frac{(2m-n-k+i)!}{(n+k-i)!(m-n-k+i)!}(\sqrt{\beta_1})^i(\sqrt{\beta_2})^{n+k-i} \\
k &= 0,\ldots,m-n. \\
\gamma_{m+k} &= 2^{2(m+k)}\sum_{i=k}^{n}\frac{(2n-i)!}{i!(n-i)!}\frac{(m-k+i)!}{(m+k-i)!(i-k)!}(\sqrt{\beta_1})^i(\sqrt{\beta_2})^{m+k-i}, \quad k=0,\ldots,n.
\end{aligned}
\tag{3}
$$

PROOF. The characteristic function of the convolution of x and y is

$$
\varphi_{x+y}(t) = \frac{4\beta_1^{\frac{2n+1}{4}}\beta_2^{\frac{2m+1}{4}}}{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)}(-it)^{\frac{n+m+1}{2}}K_{n+\frac{1}{2}}\left(2\sqrt{\beta_1}\sqrt{-it}\right)K_{m+\frac{1}{2}}\left(2\sqrt{\beta_2}\sqrt{-it}\right)
$$

Note that if the order of the modified Bessel function of the second kind is $n+1/2$, where $n \in \mathbb{N}$, then, using Gradshteyn and Ryzhik [2], we have that

$$
K_{n+1/2}(x) = \frac{\sqrt{\pi}}{(2x)^{n+\frac{1}{2}}}e^{-x}\sum_{p=0}^{n}\frac{(2n-p)!}{(n-p)!p!}(2x)^p
$$

Consequently, the function $\varphi_{x+y}(t)$ can be expressed in the following form

$$
\begin{aligned}
\varphi_{x+y}(t) =& \frac{\pi}{2^{2(n+m)}\Gamma(n+\frac{1}{2})\Gamma(m+\frac{1}{2})}\exp\{-2(\sqrt{\beta}_1+\sqrt{\beta}_2)\sqrt{-it}\} \\
&\times\sum_{p=0}^{n}\frac{(2n-p)!}{p!(n-p)!}(4\sqrt{\beta}_1\sqrt{-it})^p\sum_{q=0}^{m}\frac{(2m-q)!}{q!(m-q)!}(4\sqrt{\beta}_2\sqrt{-it})^q \\
&= c\cdot\exp\{-2(\sqrt{\beta}_1+\sqrt{\beta}_2)\sqrt{-it}\}\sum_{i=0}^{n+m}\gamma_i(\sqrt{-it})^i
\end{aligned}
$$

where c and γ_i, $i=1,\ldots,m+1$ are given in (3).

On the other hand, the characteristic function of the mixture

$$
\sum_{i=1}^{m+1}p_i\,\mathrm{Ga}^{-1}\left(n-\frac{1}{2}+i,(\sqrt{\beta}_1+\sqrt{\beta}_2)^2\right)
$$

is the same mixture of the characteristic functions of the corresponding inverted-gamma distributions. Thus we have

$$\varphi_{\text{mix}}(t) = \sum_{i=1}^{m+1} p_i \frac{\sqrt{\pi}}{2^{2(n+i-1)}\Gamma\left(n+\frac{1}{2}+i-1\right)} \exp\{-2(\sqrt{\beta}_1+\sqrt{\beta}_2)\sqrt{-it}\}$$
$$\times \sum_{p=0}^{n+i-1} \frac{(2n+2i-2-p)!}{p!(n+i-1-p)!}(4(\sqrt{\beta}_1+\sqrt{\beta}_2)\sqrt{-it})^p$$
$$= \frac{\sqrt{\pi}}{2^{2n-2}} \exp\{-2(\sqrt{\beta}_1+\sqrt{\beta}_2)\sqrt{-it}\}$$
$$\times \left[\sum_{j=0}^{n-1}\left(\sum_{i=1}^{m+1} p_i \frac{1}{2^{2i}\Gamma\left(n-\frac{1}{2}+i\right)} \cdot \frac{(2n+2i-2-j)!}{j!(n+i-1-j)!}\right)\left(4(\sqrt{\beta_1}+\sqrt{\beta_2})\sqrt{-it}\right)^j\right.$$
$$\left.+\sum_{j=0}^{m}\left(\sum_{i=j+1}^{m+1} p_i \frac{1}{2^{2i}\Gamma\left(n-\frac{1}{2}+i\right)} \cdot \frac{(n+2i-2-j)!}{(n+j)!(i-1-j)!}\right)\left(4(\sqrt{\beta_1}+\sqrt{\beta_2})\sqrt{-it}\right)^{n+j}\right].$$

It can be shown that the functions $\varphi_{x+y}(t)$ and $\varphi_{\text{mix}}(t)$ are equal if and only if the following equalities hold

$$c\gamma_j = \frac{\sqrt{\pi}}{2^{2n-2}} \sum_{i=1}^{m+1} p_i \frac{1}{2^{2i}\Gamma\left(n-\frac{1}{2}+i\right)} \cdot \frac{(2n+2i-2-j)!}{j!(n+i-1-j)!}\left(4(\sqrt{\beta_1}+\sqrt{\beta_2})\right)^j, \quad j=0,\ldots,n-1;$$
$$c\gamma_{n+j} = \frac{\sqrt{\pi}}{2^{2n-2}} \sum_{i=1}^{m+1} p_i \frac{1}{2^{2i}\Gamma\left(n-\frac{1}{2}+i\right)} \cdot \frac{(n+2i-2-j)!}{(n+j)!(i-1-j)!}\left(4(\sqrt{\beta_1}+\sqrt{\beta_2})\right)^{n+j}, \quad j=0,\ldots,m.$$

The proof that these equalities hold is simple butsomewhat cumbersome. ■

Using induction and the commuting property of mixtures and convolutions, we can generalize the results of theorem 1 to the convolution of any finite number of inverted-gamma distributions with demi-integer shape parameters.

Theorem 2 *If the random variables $x_1, \ldots, x_k$ are independent and distributed as*

$$x_i \sim Ga^{-1}\left(n_i + \frac{1}{2}, \beta_i\right), \quad i = 1, \ldots, k$$

where $n_i \in \mathbb{N}$, and $n_1 \leq \ldots \leq n_k$, then

$$\sum_{i=1}^{k} x_i \sim \sum_{j=1}^{n_2+\ldots+n_k+1} q_j\, Ga^{-1}\left(n_1 - \frac{1}{2} + j, \left(\sum_{j=1}^{k}\sqrt{\beta}_j\right)^2\right)$$

where q_j are weights, i.e., $q_j \geq 0$ and $\sum_{j=1}^{n_2+\ldots+n_k+1} q_j = 1$.

From theorem 2, the following result follows easily.

Corollary 1 *If the random variables $x_1, \ldots, x_k$ are independent and distributed as $x_i \sim Ga^{-1}\left(\frac{1}{2}, \beta_i\right)$, $i = 1, \ldots, k$, then*

$$\sum_{i=1}^{k} x_i \sim Ga^{-1}\left(\frac{1}{2}, \left(\sum_{j=1}^{k}\sqrt{\beta}_j\right)^2\right)$$

As a consequence of theorem 1 we also obtain the following result, which establishes that the Behrens-Fisher distribution with odd degrees of freedom is a finite mixture of t distributions.

Theorem 3 *If $x \sim$ Be-Fi$(x|\phi, 2n+1, 2m+1)$, $n, m \in \mathbb{N}$, $m \geq n$, then*

$$x \sim \sum_{i=1}^{m+1} p_i \, t\left(0, \frac{\left(\sqrt{2n+1}\,\mathrm{sen}\,\phi + \sqrt{2m+1}\cos\phi\right)^2}{2n+2i-1}, 2n+2i-1\right)$$

where the values of p_i,$i = 1, \ldots, m+1$ are computed from (2).

Another consequence of theorem 2 is the well known result that establishes that the convolution of Cauchy distributions is also a Cauchy distribution.

Corollary 2 *If the random variables $x_1, \ldots, x_k$ are independent and distributed as $x_i \sim$ Ca(μ_i, σ_i), for $i = 1, \ldots, k$, where μ_i and σ_i denote the location and scale parameters, respectively, then*

$$\sum_{i=1}^{k} x_i \sim Ca\left(\sum_{i=1}^{k} \mu_i, \sum_{i=1}^{k} \sigma_i\right)$$

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