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On the uniform limit of quasi-continuous functions

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Abstract. We study when the uniform limit of a net of quasi-continuous functions with values in a locally convex space X is a quasi-continuous function, emphasizing that this fact depends on the least cardinal of a fundamental system of neighbourhoods of 0 in X, and giving necessary and sufficient conditions. The main result of the paper is Theorem 15, where the results of [7] and [10] are improved, in relation with a Theorem of L. Schwartz.

Sobre el límite uniforme de funciones cuasi-continuas

Resumen. Estudiamos cuando el límite uniforme de una red de funciones cuasi-continuas con valores en un espacio localmente convexo X es también una función cuasi-continua, resaltando que esta propiedad depende del menor cardinal de un sistema fundamental de entornos de 0 en X, y estableciendo condiciones necesarias y suficientes. El principal resultado de este trabajo es el Teorema 15, en el que los resultados de [7] y [10] son mejorados, en relación al Teorema de L. Schwartz.

In general, we shall work with a measure space (Ω, Σ, μ) where Ω is a topological space, the σ-algebra Σ contains the Borel sets of Ω and $\mu(\Omega) = 1$. Suppose that X is a locally convex Hausdorff space.

We say that a function $f : \Omega \to X$ is *quasi-continuous* if the set of points D where f is not continuous has outer measure $\mu^*(D) = 0$.

A function $f : \Omega \to X$ is said to be *Lusin measurable* if for any $\varepsilon > 0$, there is a closed set $F \subseteq \Omega$ such that $\mu(\Omega \setminus F) < \varepsilon$ and the restriction $f|_F$ is continuous.

We shall use the following axiom:

Axiom L *The interval $[0, 1]$ cannot be covered by a family $(F_i)_{i\in I}$ of closed subsets of Lebesgue measure zero where the cardinal of I is less than the continuous c.*

Then, according to [11,1-6-4], we have:

Proposition 1 *Let Ω be a compact metrizable space, and let μ be a Radon measure on Ω. Then Axiom L implies that the union of a family $(F_i)_{i\in I}$ consisting of closed sets of measure zero such that $card(I) < c$ does not cover any set of positive measure.*

Theorem 1 *Assume Axiom L in the conditions of Proposition 1. Let X be a locally convex space with a base of neighbourhoods $(V_\alpha)_{\alpha\in A}$ of zero such that $card(A) < c$. Let $(f_i)_{i\in I}$ be a net of quasi-continuous functions $f_i : \Omega \to X$, converging uniformly to f. Then, if C is the set of points where f is continuous, we have $\mu^*(C) = 1$.*

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PROOF. It is obvious that if X is a metrizable space, then f is quasi-continuous. In the general case, we can suppose, using [8,5.4], that X is the product $\Pi_{\alpha\in A}X_\alpha$ of a family of Banach spaces. Let π_α be the projection $X \to X_\alpha$ and ω_α the oscillation function of $\pi_\alpha \circ f$; then $\pi_\alpha \circ f$ is quasi-continuous. Let

$$G_{\alpha,n} = \{x \in \Omega : \omega_\alpha(x) < 1/n\}.$$

Then, since ω_α is upper semicontinuous, $G_{\alpha,n}$ is an open set of measure $\mu(G_{\alpha,n}) = 1$. By Proposition 1, the union $\bigcup_{\alpha\in A,n\in\mathbb{N}}(X \setminus G_{\alpha,n})$ does not cover any set of positive measure. Hence, its inner measure is zero and

$$\mu^*(\bigcap_{\alpha\in A,n\in\mathbb{N}} G_{\alpha,n}) = 1.$$

To finish, it suffices to note that $C = \bigcap_{\alpha\in A,n\in\mathbb{N}} G_{\alpha,n}$. ■

It is also useful to consider the following axiom:

Axiom M *If $(A_i)_{i\in I}$ is a family of subsets of $[0,1]$ with Lebesgue measure zero and such that $card(I) < c$, then the union of the A_i's has measure zero.*

According to [11,1.6.2], we have:

Proposition 2 *Assume Axiom M and suppose that (Ω,Σ,μ) is a probability space, where Σ is countably generated. Then, if $(A_i)_{i\in I}$ is a family of subsets with measure zero and such that $card(I) < c$, we have that the union also has measure zero.*

Theorem 2 *Assume Axiom M in the conditions of Proposition 2. Let X be a locally convex space with a base of neighbourhoods $(V_\alpha)_{\alpha\in A}$ of zero such that $card(A) < c$. Let $(f_i)_{i\in I}$ be a net of quasi-continuous functions $f_i : \Omega \to X$, converging uniformly to f. Then f is quasi-continuous.*

PROOF. We just have to proceed as in Theorem 1 , using Proposition 2 instead of Proposition 1. ■

Remark 1 In Theorems 1 and 2, if Ω is a (T_1)-space and the measure μ is diffuse, we have that $card(\Omega) \geq c > card(A)$. As we shall soon see, these theorems do not hold if $\Omega = [0,1]$, μ is the Lebesgue measure and $card(A) = c$.

Theorem 3 *Let $\Omega = [0,1]$, μ the Lebesgue measure on Ω and $X = \mathbb{R}^A$ with $card(A) = c$. Then, there exists a function $f : \Omega \to X$ which is the uniform limit of a net of quasi-continuous functions, which is not continuous at any point and having the property that any restriction $f|_H$ $(H \subseteq \Omega)$ is continuous only on the countable set consisting of the isolated points of H. Hence, f is not Lusin measurable. Moreover, for any set $H \subseteq \Omega$, there exists an open set $G \subseteq X$ such that $f^{-1}(G) = H$ and f is not Borel measurable, but it is weakly measurable and its Pettis integral $\int f d\mu$ is zero.*

PROOF. We can assume that $A = \Omega$. Let $f = (\mathcal{X}_\alpha)_{\alpha\in A}$, where $\mathcal{X}_\alpha(x) = 1$ for $x = \alpha$ and $\mathcal{X}_\alpha(x) = 0$ for $x \neq \alpha$. The function $f : \Omega \longrightarrow X$ is nowhere continuous but it is the uniform limit of a net $(f_i)_{i\in I}$, where every $i \in I$ is a finite subset of Ω, I is ordered by inclusion and $\pi_\alpha \circ f_i = \mathcal{X}_\alpha$ for any $\alpha \in i$ and $\pi_\alpha \circ f_i = 0$ for any $\alpha \notin i$, which implies that every function f_i is quasi-continuous. Moreover, let H be a subset of Ω, then the restriction $f|_H$ is continuous only on the countable set that contains the isolated points of H.

Finally, if $U = (0,2)$ and $U_\alpha = \pi_\alpha^{-1}(U)$, we have

$$f^{-1}(U_\alpha) = \mathcal{X}_\alpha^{-1}(U) = \alpha,$$

and hence $G = \bigcup_{\alpha\in H} U_\alpha$ is an open subset of X such that

$$f^{-1}(G) = \bigcup\nolimits_{\alpha\in H} f^{-1}(U_\alpha) = H.$$

In addition, for any $x^* \in X^*$, since by [3,Proposition 3.14.1] $x^* \circ f$ vanishes outside a finite set, it follows that $x^* o f$ is zero almost everywhere in Ω and hence, the Pettis integral $\int f d\mu$ is zero. ■

Remark 2 We can choose I to be the lattice consisting of the countable subsets of Ω. In this case, for any sequence (i_n) in I, there exists $i = \bigcup_{k\in\mathbb{N}} i_k \geq i_n$ for all $n \in \mathbb{N}$.

We can prove in a similar fashion the following theorem:

Theorem 4 *Let $\Omega = A$ be a (T_1)-topological space, μ a diffuse probability measure on Ω and $X = \mathbb{R}^A$. Then there exists a function $f : \Omega \to X$ which is the uniform limit of a net of quasi-continuous functions, and having the property that any restriction $f|_H$ is continuous only on the set of isolated points of H, and that for every subset $H \subseteq \Omega$, there exists an open subset $G \subseteq X$ such that $f^{-1}(G) = H$. Therefore, if the set of isolated points of Ω has measure less that 1, f is not quasi-continuous and if there exists a non-measurable set H, f is neither Borel measurable nor Lusin measurable.*

Remark 3 If $card(\Omega)$ is of measure zero, the measure of the set consisting of the isolated points of Ω has μ-measure zero and there is a non-measurable set in Ω. According to [5,2.5], the same happens if $card(\Omega)$ is not measurable and μ is a perfect measure. We also have that if μ is a τ-additive measure, the set of isolated points of Ω has measure zero. Since the same holds for every subset H of Ω and the induced measure μ_H, f is not Lusin measurable.

A set is said to be a (D)-set if it is the set $D(f)$ of the discontinuity points of a function $f : \Omega \to \mathbb{R}$. It is clear that any (D)-set is an F_σ-set.

It is obvious that the set which contains all the discontinuity points of a function $f = (f_\alpha)_{\alpha\in A} : \Omega \to X = \mathbb{R}^A$ is $\bigcup_{\alpha\in A} D(f_\alpha)$. Then we have:

Theorem 5 *If $X = \mathbb{R}^A$, then for any quasi-continuous function $f_\alpha : \Omega \to \mathbb{R}$, the function $f = (f_\alpha)_{\alpha\in A} : \Omega \to X$ is quasi-continuous if and only if the union of any family $(D_\alpha)_{\alpha\in A}$ of (D)-sets of measure zero, has outer measure zero.*

Theorem 6 *Suppose that the support of μ is Ω and that X is an arbitrary locally convex space having a base $(V_\alpha)_{\alpha\in A}$ of neighbourhoods of zero. Let $f : \Omega \to X$ be the uniform limit of a net of quasi-continuous functions. Then f is quasi-continuous if and only if any union $\bigcup_{\alpha\in A} F_\alpha$ of closed sets of Ω having measure zero, has outer measure zero.*

PROOF. Sufficiency follows as in Theorem 1. Necessity follows from Theorem 5 taking into consideration that if F is a closed set with measure zero, then $int(F) = \emptyset$, and hence, $D(\mathcal{X}_F) = F$. In this last step we have used the fact that $supp(\mu) = \Omega$. ■

Remark 4 Theorem 6 can be extended to any τ-additive measure μ.

If $(F_\alpha)_{\alpha\in A}$ is a family of closed sets having μ-measure zero, such that

$$\mu^*(\bigcup_{\alpha\in A} F_\alpha) > 0,$$

and $card(A)$ is of measure zero, then if we suppose that A is well ordered and we set $A_\alpha = F_\alpha \setminus \bigcup_{\beta<\alpha} F_\beta$, we can easily see that $f = (\mathcal{X}_{A_\alpha})_{\alpha\in A} : \Omega \to X = \mathbb{R}^A$ is a non-measurable Borel function which is the uniform limit of quasi-continuous functions. This result holds as well, according to [5,2.5], if μ is a perfect measure and $card(A)$ is not measurable. On the other hand, if Ω is endowed with the discrete topology, every function $f : \Omega \to X$ is continuous. But if μ is a diffuse measure, the latter is equivalent to the fact that $card(\Omega)$ does not have measure zero.

It is easily checked that if $\kappa = \kappa(\mu)$ is the least cardinal of the sets A having the property that $\mu^*(\bigcup_{\alpha\in A} F_\alpha) > 0$ for a family $(F_\alpha)_{\alpha\in A}$ of closed sets of measure zero, then κ is not the supremum of a sequence of cardinals less than κ, which is obviously less than or equal to $card(\Omega)$ if Ω is a (T_1)- space and μ is a diffuse measure. Axiom M implies that $\kappa(\mu) = c$ for the Lebesgue measure μ on $\Omega = [0, 1]$, and this in turn implies Axiom L.

We say that a cardinal is *primary* if it is not the supremun of a sequence of cardinals less than itself.

Corollary 1 *There exists a Radon measure μ and a function $f : \Omega \to X = \mathbb{R}^{\aleph_1}$ that is not quasi-continuous nor Borel measurable and yet it is the uniform limit of a net of quasi-continuous functions.*

PROOF. It turns out from Theorem 6 by taking into consideration that, according to Haydon [2,15.1], there exists a Radon measure μ for which the union of $\aleph_1$ closed sets of measure zero may be not measurable. ■

In the same way as in Theorem 6 but in the same direction as Theorem 1, we can give a necessary and sufficient condition in order that $\mu^*(C) = 1$ for the set C consisting of the continuity points of every function $f : \Omega \longrightarrow X$ which is the uniform limit of a net of quasi-continuous functions.

Theorem 7 *If (Ω, Σ, μ) verifies $\mu_*(\bigcup_{\alpha\in A} F_\alpha) = 0$ for every family $(F_\alpha)_{\alpha\in A}$ of closed subsets of measure zero with $card(A) \le \kappa$, then there exists a probability space (Ω, Σ', ν) such that ν is an extension of μ and $\kappa(\nu) > \kappa$.*

PROOF. We can assume that $\kappa \ge \aleph_0$. Let $\mathcal{H}$ be the set consisting of the unions $\bigcup_{\alpha\in A} F_\alpha$ and

$$\nu^*(E) = \inf_{H\in\mathcal{H}} \mu^*(E \setminus H) \ \ (E \subseteq \Omega).$$

First of all, we are going to prove that ν^* is an outer measure. Indeed, for every $\varepsilon > 0$ and $E_n \subseteq \Omega$ there exists an $H_n \in \mathcal{H}$ $(n \in \mathbb{N})$ such that

$$\nu^*(E_n) + \varepsilon 2^{-n} > \mu^*(E_n \setminus H_n),$$

and therefore

$$\sum_n \nu^*(E_n) + \varepsilon > \sum_n \mu^*(E_n \setminus H_n) \ge \mu^*(\bigcup_n E_n \setminus \bigcup_n H_n) \ge \nu^*(\bigcup_n E_n),$$

from which it follows that

$$\nu^*(\bigcup_n E_n) \le \sum_n \nu^*(E_n).$$

Let (E_n) be a disjoint sequence in Σ and take $M \subseteq \Omega$. Then, for every $\varepsilon > 0$, there exists an $H \in \mathcal{H}$ such that

$$\begin{aligned}\nu^*(M \cap \bigcup_n E_n) + \varepsilon &> \mu^*(M \cap \bigcup_n E_n \setminus H)\\ &= \sum_n \mu^*(M \cap E_n \setminus H)\\ &\ge \sum_n \nu^*(M \cap E_n),\end{aligned}$$

and therefore

$$\nu^*(M \cap \bigcup_n E_n) = \Sigma_n \nu^*(M \cap E_n).$$

From the latter, it turns out that the restriction of ν^* to the σ-algebra $\Sigma' \supseteq \Sigma$ consisting of the ν^*-measurable sets, is a measure ν, which is an extension of μ since $\nu(B) = \mu(B)$ for every set $B \in \Sigma$. Then, since $\nu^*(H) = 0$ for every $H \in \mathcal{H}$, it follows that $\kappa(\nu) > \kappa$. ■

A slight change in the previous proof allows us to prove the following theorem:

Theorem 8 *If (Ω, Σ, μ) has the property that $\mu_*(\bigcup_{\alpha\in A} F_\alpha) = 0$ for every family $(F_\alpha)_{\alpha\in A}$ of closed sets having measure zero with $card(A) < \kappa$ and κ being a primary cardinal, then there exists a probability space (Ω, Σ', ν) such that ν is an extension of μ and $\kappa(\nu) \ge \kappa$.*

Corollary 2 *From axiom L it follows that for every cardinal $\kappa < c$ there exists a probability measure μ on $\Omega = [0,1]$ which is the extension of the Lebesgue measure and such that $\kappa(\mu) > \kappa$. If c is not primary, $\kappa(\mu) < c$ holds for such measures, and if c is primary, we can say that $\kappa(\mu) = c$ for one of them.*

Theorem 9 *If μ is a regular measure and there exists a measurable function $f : \Omega \to \mathbb{R}$ such that $\mu \circ f^{-1}$ is a diffuse measure, we have that $\kappa(\mu) \leq c$.*

PROOF. Since μ is a regular measure, by Lusin's theorem there exists a closed set $F \subseteq \Omega$ of positive measure such that the restriction $f|_F$ is continuous. Then, if $F_\alpha = f^{-1}(\alpha) \bigcap F$ for $\alpha \in \mathbb{R}$, we have that every F_α is a closed set of measure zero and $\mu(\bigcup_{\alpha \in \mathbb{R}} F_\alpha) = \mu(F) > 0$. ■

Remark 5 According to a theorem of Zink [5,2.2], if μ is a separable measure, there is a measurable function $f : \Omega \to [0,1]$ such that $\mu \circ f^{-1}$ is the Lebesgue measure. Similarly, by [5,2.1], if μ is a non-atomic measure, there exists a measurable function $f : \Omega \to [0,1]$ such that $\mu \circ f^{-1}$ is the Lebesgue measure.

Corollary 3 *If μ is a regular non-atomic measure, then $\kappa(\mu) \leq c$.*

Remark 6 If μ is an atomic measure then it can trivially happen that $\mu(\cup_{\alpha \in A} F_\alpha) = 0$ for every family $(F_\alpha)_{\alpha \in A}$ consisting of null-measure sets.

If μ is a diffuse probability measure on a metric space Ω whose density character is non-measurable then, according to [6], μ is a non-atomic measure, and hence, $\kappa(\mu) \leq c$.

Theorem 10 *If Ω is a completely regular Hausdorff space and μ is a weakly τ-additive measure such that for every $x \in \Omega$ and every $\varepsilon > 0$ there exists an open neighbourhood V of x with $\mu(V) < \varepsilon$, then $\kappa(\mu) \leq c$.*

PROOF. Let λ be the restriction of μ to the σ-algebra $Ba(\Omega)$ of the Baire subsets of Ω. If there were an atom B of λ then there would also exist a closed atom $F \subseteq B$ of λ. Hence, by the assumption above, for each $x \in F$ there exists an open neighbourhood $V_x \in Ba(\Omega)$ such that $\lambda(V_x \cap F) = 0$. Let $F_x = F \setminus V_x$ ($\in Ba(\Omega)$), then $\cap_{x \in F} F_x = \emptyset$, from which it follows (taking into account the fact that μ is a weakly τ-additive) that there is a sequence (F_{x_n}) such that $\lambda(\cap_n F_{x_n}) = 0$, and this contradicts the fact that $\lambda(F_{x_n}) = \lambda(F) > 0$. Therefore λ is a non-atomic measure and, according to [5, 2.1], there exists a λ-measurable function $f : \Omega \longrightarrow [0,1]$ such that $\lambda \circ f^{-1}$ is the Lebesgue measure. From this it immediately follows, as in Theorem 9, that $\kappa(\mu) \leq \kappa(\lambda) \leq c$. ■

Remark 7 The property that for every $x \in \Omega$ and $\varepsilon > 0$ there exists an open neighbourhood V of x such that $\mu(V) < \varepsilon$ is equivalent to saying $\lambda^*(\{x\}) = 0$ for every $x \in \Omega$. In general, being Ω a Hausdorff space, $\sum_{x \in \Omega} \lambda^*(\{x\}) \leq \mu(\Omega)$ and, if $\sum_{x \in \Omega} \lambda^*(\{x\}) < \mu(\Omega)$, then it follows from the remaining conditions of Theorem 10 that $\kappa(\mu) \leq c$. From this it turns out that $\kappa(\mu) \leq c$ whenever Ω is a compact infinite Hausdorff group and μ is invariant under left translations.

If μ is a diffuse measure then the function $\mathcal{X}_\omega$ ($\omega \in \Omega$) is Lusin measurable if and only if $\lambda^*(\{\omega\}) = 0$.

Theorem 11 *If Ω is a completely regular Hausdorff space and μ is a weakly τ-additive measure such that its support S is not separable then $\kappa(\mu) \leq c$.*

PROOF. Let λ be the restriction of μ to $Ba(\Omega)$ and H the closure of the countable set $\{x \in \Omega : \lambda^*(\{x\}) > 0\}$. Since S is non-separable, we have $S \setminus H \neq \emptyset$ and $\mu(\Omega \setminus H) > 0$ and there exists a closed set $F \in Ba(\Omega)$ with positive measure $\lambda(F) > 0$ which is disjoint from H. Then, by applying Theorem 10 to the induced measure μ_F it turns out that $\kappa(\mu) \leq \kappa(\mu_F) \leq c$. ■

Remark 8 If Ω is a compact Hausdorff space and μ is a diffuse measure with separable support, the question comes down to the case when the support is a singleton. Indeed, if (x_n) are the points of $\{x \in \Omega : \lambda^*(x) > 0\}$ and $\sum_n \lambda^*(x_n) = \mu(\Omega)$, there exists a sequence (F_n) of pairwise disjoint closed Baire sets such that $x_n \in F_n$, and hence the probability measures μ_n defined by

$$\mu_n(A) = \frac{\mu(A \cap F_n)}{\lambda^*(x_n)} \qquad (A \in \Sigma)$$

have the sets $\{x_n\}$ as supports, and $\kappa(\mu) = \min_n \kappa(\mu_n)$.

Theorem 12 *If Ω is a separable Hausdorff space and μ is a diffuse measure, then $\kappa(\mu) \leq 2^c$.*

PROOF. Let D be a dense sequence in Ω. Then for all $x \in \Omega$ there exists an ultrafilter $\mathcal{U}_x$ in D which converges to x, and the mapping $x \mapsto \mathcal{U}_x$ is one-to-one. Since, according to [1], the cardinal of such ultrafilters is less than or equal to 2^c, and μ is a diffuse measure, it follows that $\kappa(\mu) \leq card(\Omega) \leq 2^c$. ■

Remark 9 If μ is a diffuse measure and the σ-algebra of the measurable sets is countably generated, then, as in [11, 1.6.2], one can deduce that $\kappa(\mu) \leq c$.

Theorem 13 *If Ω is a completely regular Hausdorff space and the cardinal of the support S of the measure μ is greater than 2^c, then $\kappa(\mu) \leq c$.*

PROOF.
By using the usual extension ν of μ on the Stone-Cech compactification $\beta\Omega$ of Ω, which has the property that the induced measure ν_Ω coincides with μ, we can assume that Ω is a compact space. Now, since $card(S) > 2^c$, from the proof of Theorem 12 it follows that S is not separable, and from Theorem 11 it turns out that $\kappa(\mu) \leq c$. ■

Corollary 4 *if Ω is a completely regular Hausdorff space and μ is a diffuse measure such that its support has positive measure, then $\kappa(\mu) \leq 2^c$.*

Theorem 14 *For every cardinal κ there exists a diffuse measure μ on a (T_1)-space Ω such that $\kappa(\mu) > \kappa$ and $supp\mu = \Omega$.*

PROOF. We may assume that κ is infinite. Let Ω be a set whose cardinal is greater than κ, and let us endow Ω with the topology whose closed sets are Ω and all the sets with cardinal less than or equal to κ. Let Σ be the corresponding Borel σ-algebra on Ω, and let us define the measure μ by putting, for $A \in \Sigma$, either $\mu(A) = 0$ or $\mu(A) = 1$ depending on whether $card(A) \leq \kappa$ or $card(A) > \kappa$. Then, as $\kappa^2 = \kappa$, it follows that $\kappa < \kappa(\mu) \leq card(\Omega)$. ■

Theorem 15 *For every cardinal κ there exists a completely regular space $\Omega = (C(K), weak)$ and a probability measure μ with empty support on Ω such that $\kappa(\mu) > \kappa$.*

PROOF. We shall proceed as in [10] and [7]. We may assume that $\kappa > \aleph_0$ is not a limit cardinal. Let ω be the first ordinal with cardinal κ and let $T_0 = \{\alpha : \alpha \leq \omega\}$. Let us endow T_0 with the topology consisting of all the subsets of $T = T_0 \setminus \{\omega\}$ and such that the neighbourhoods of ω are the complements of the subsets of T whose cardinal are less than κ. With this topology T_0 is a space (T_{3a}). Let K be the Stone-Cech compactification of T_0 and put $\Omega = (C(K), weak)$. The set $\mathcal{V}_0$ of all the neighbourhoods of ω is stable with respect intersections of families with cardinal less than κ, because κ is not a limit cardinal, and it admits a fundamental system $\mathcal{V}$ consisting of open-closed neighbourhoods. Let F denote the set of all continuous functions from K to $\{0, 1\}$ which vanish at ω. It is clear that F is a weakly closed set.

We shall construct two classes $\mathcal{C}$ and $\mathcal{D}$ of Borel sets in $(F, weak)$ such that

(i) The smallest σ-algebra containing $\mathcal{C}$ is the class $\mathcal{B}_F$ of the Borel sets of $(F, weak)$.

(ii) If $C \in \mathcal{C}$ then either $C \in \mathcal{D}$ or $F \setminus C \in \mathcal{D}$.

(iii) The intersection of any family of elements in $\mathcal{D}$ with cardinal less than κ is not empty.

(iv) For all $t \in K \setminus \{\omega\}$, the set $\{f \in F : f(t) = 1\}$ belongs to $\mathcal{D}$.

Then we can define a Borel measure λ on $(F, weak)$ by putting $\lambda(B) = 1$ whenever $B \in \mathcal{B}_F$ and B contains the intersection of a family of elements of $\mathcal{D}$ with cardinal less than κ, and $\lambda(B) = 0$ otherwise. Hence for every family $(F_\alpha)_{\alpha \in A}$ of null-λ-measure subsets $F_\alpha \in \mathcal{B}_F$ with $cardA < \kappa$ we have $\lambda_*(\cup_{\alpha \in A} F_\alpha) = 0$. And λ is a non-weakly-τ-additive measure with empty support, because F is the union of the open sets $G_t = \{f : f(t) = 0\}$ when $t \in K \setminus \{\omega\}$, $\lambda(F) = 1$, and $\lambda(G_t) = 0$. From this it follows that there exists a measure with similar properties on Ω, which we shall keep denoting λ.

Let k be an integer, J a set, $(\mu_j^p)_{p \le k, j \in J}$ Radon measures on K with $\mu_j^p(K) = 1$, and $(a^p)_{p \le k}$, $(b^p)_{p \le k}$ rational numbers such that $a^p < b^p$. Now we define $\mathcal{C}$ to be the class of all sets $C = \cup_{j \in J} U_j$, where

$$U_j = \{f \in F : \forall p \le k, \mu_j^p(f) \in (a^p, b^p)\},$$

and k, J, the measures μ_j^p, a^p, and b^p vary.

The class $\mathcal{D}$ consists of all the sets $C \in \mathcal{C}$ such that for all $V \in \mathcal{V}$ there exist $j \in J$ and $f \in U_j$ with $f = 1$ on $K \setminus V$, and also of all the complements $F \setminus C$ of the sets which do not satisfy this condition. Only (iii) needs to be proved. To this end, it is enough to show that for every family $(C_\alpha)_{\alpha \in A} \subseteq \mathcal{C} \cap \mathcal{D}$, where $A = \{\alpha : \alpha < \alpha_0\}$ and $\alpha_0 < \omega$, and for every $V_0 \in \mathcal{V}$, there exists $f \in \cap_\alpha C_\alpha$ such that $f = 1$ on $K \setminus V_0$. It is easy to prove (with the obvious notation) that for all $\alpha \in A$ there exists $\varepsilon_\alpha > 0$ such that if $W \in \mathcal{V}$ then there exist $j \in J_\alpha$ and $f \in U'_j$ with $f = 1$ on $K \setminus W$, where

$$U'_j = \{f \in F : \forall p \le k_\alpha, \mu_{j,\alpha}^p(f) \in (a_\alpha^p + \varepsilon_\alpha, b_\alpha^p - \varepsilon_\alpha)\}.$$

Let $\mathcal{U}_\alpha$ be an ultrafilter on J_α containing all the sets

$$\{j \in J_\alpha : \exists f \in U'_j, f = 1 \text{ on } K \setminus V\}$$

whenever $V \in \mathcal{V}$. Let $p \le k_\alpha$ be given, and let us put $\nu_\alpha^p = \lim_{\mathcal{U}_\alpha} \mu_{j,\alpha}^p$. Then, there exists $V \in \mathcal{V}$, $V \subseteq V_0$, such that $\nu_\alpha^p(V \setminus W) = 0$ for all $W \in \mathcal{V}$, $p \le k_\alpha$ and $\alpha \in A$. In the same way as in [7], but performing a transfinite induction in $\alpha \in A$, it can be proved that there exist f_α, $j_\alpha \in J_\alpha$, $V_\alpha, V'_\alpha \in \mathcal{V}$, and open-closed sets $H_{\alpha,0}$, $H_{\alpha,1}$ in K such that

(i) The sets V_α satisfy $V_\alpha \subseteq \cap_{\beta < \alpha} V'_\beta$ and $V_\alpha \cap (H_{\beta 0} \cup H_{\beta 1}) = \emptyset$ for all $\beta < \alpha \in A$, with $V_1 = V$.

(ii) $\mu_{j_\alpha,\alpha}^p(V \setminus V_\alpha) < \varepsilon_\alpha/2$ for all $\alpha \in A$ and $p \le k_\alpha$.

(iii) $f_\alpha \in U'_{j_\alpha}$ and $f_\alpha = 1$ on $K \setminus V_\alpha$.

(iv) $f_\alpha = 0$ on $V'_\alpha \subseteq V_\alpha$ and $\mu_{j_\alpha,\alpha}^p(V'_\alpha \setminus \{\omega\}) = 0$ for all $\alpha \in A$ and $p \le k_\alpha$.

(v) $A_{00} = \emptyset$, $A_{01} = K \setminus V$, $A_{\alpha 0} = \{t \in V_\alpha \setminus V'_\alpha : f_\alpha(t) = 0\}$, and $A_{\alpha 1} = \{t \in V_\alpha \setminus V'_\alpha : f_\alpha(t) = 1\}$.

(vi) $\{\omega\}$, $H_{\alpha 0}$ and $H_{\alpha 1}$ are disjoint sets such that $H_{\alpha i} \supseteq A_{\alpha i} \cup H_{\alpha - 1 i}$ if α has a predecessor $\alpha - 1$, and $H_{\alpha i} \supseteq A_{\alpha i} \cup \overline{\cup_{\beta < \alpha} H_{\beta i}}$ whenever α is a limit ordinal ($H_{0i} = A_{0i}$) for $i = 0, 1$.

It is obvious that for $\alpha = \alpha_0$ (or $\alpha < \alpha_0$), the open sets $G_0 = \cup_{\beta < \alpha} H_{\beta 0}$ and $G_1 = \cup_{\beta < \alpha} H_{\beta 1}$ are disjoint, and moreover

$$\overline{G_0} \cap \overline{G_1} = \overline{(G_0 \cap T)} \cap \overline{(G_1 \cap T)} = \emptyset,$$

because the sets $G_i \cap T$ are disjoint and open-closed in T_0. Then $H = \overline{G_1}$ is an open-closed set in K such that $\overline{G_0} \cap H = \emptyset$ and $\omega \notin H$.

The function $f = \mathcal{X}_H$ satisfies $f = f_\alpha$ on $A_\alpha = A_{\alpha 0} \cup A_{\alpha 1}$. Moreover, for all $\alpha \in A$ and all $p \leq k_\alpha$ we have $\mu^p_{j_\alpha}(f \neq f_\alpha) < \varepsilon_\alpha$. Hence it follows that $f \in U_{j_\alpha}$ for all $\alpha \in A$. Therefore $f \in \cap_{\alpha \in A} C_\alpha$ and $f = 1$ on $K \setminus V_0$.

Since for every family $(F_\alpha)_{\alpha \in A}$ of null λ-measure sets $F_\alpha \in \mathcal{B}_\Omega$ with $cardA < \kappa$ we have $\lambda_*(\cup_{\alpha \in A} F_\alpha) = 0$ then, according to Theorem 8, it turns out that there exists an extension μ of λ such that $\kappa(\mu) \geq \kappa$.

Given a cardinal κ, a σ-algebra Σ is said to be a κ-algebra provided Σ is stable under unions and intersections of cardinal less than κ. A measure μ on a κ-algebra Σ is said to be κ-additive provided that for every disjoint family $(H_\alpha)_{\alpha \in A}$ of sets $H_\alpha \in \Sigma$ with $cardA < \kappa$ one has $\mu(\cup_{\alpha \in A} H_\alpha) = \sum_{\alpha \in A} \mu(H_\alpha)$. ■

Remark 10 Given a cardinal κ, a slight modification of the above proof allows to show, without using Theorem 11, the existence of a κ-additive measure $\mu \neq 0$ taking values in $\{0, 1\}$, with empty support on a κ-algebra Σ of subsets of a space $\Omega = (C(K), weak)$. Then $\kappa(\mu) \geq \kappa$ holds too.

For such measures μ taking values in $\{0, 1\}$, in a similar way and with the same notations as in the remark following Theorem 6, it turns out that if $cardA$ is non-measurable, then $f = (\mathcal{X}_{A_\alpha})_{\alpha \in A} : \Omega \longrightarrow X = \mathbb{R}^A$ is a non-Borel-measurable function which is the uniform limit of a net of quasi-continuous functions.

Going more deeply into this matter, we shall prove the following Theorem without the above hypothesis.

Theorem 16 *Let Ω, $\mathcal{D}$ and $\kappa = \aleph_{\xi+1}$ be as in Theorem 15, let Σ be the κ-algebra generated by the Borel sets of Ω, and let μ be the κ-additive measure defined on Σ by setting $\mu(A) = 1$ if $A \in \Sigma$ and A contains an intersection of $\aleph_\xi$ subsets of $\mathcal{D}$, and $\mu(A) = 0$ otherwise. Then there exists a non-measurable union of a disjoint family of κ closed null measure sets F_α, and therefore $f = (\mathcal{X}_{F_\alpha})_\alpha : \Omega \longrightarrow X = \mathbb{R}^\kappa$ is a non-Borel-measurable function which is the uniform limit of a net of quasi-continuous functions, and $\kappa(\mu) = \kappa$.*

PROOF. Using the same notations as in Theorem 15, let $G_t = \{f \in F : f(t) = 0\}$ for $t \in T = \{t : t < \omega\}$. Then, in the usual way, we can obtain a disjoint family $(F_t)_{t \in T}$ of closed sets such that $F_t \subseteq G_t$ for every t and $\bigcup_{t \in T} F_t = \bigcup_{t \in T} G_t = F$. Assume that every union of sets F_t is μ-measurable. Then we an define a measure ν on all the subsets of T by setting $\nu(H) = \mu(\bigcup_{t \in H} F_t)$ for every subset $H \subseteq T$. Now, proceeding as in [12], we can construct a matrix (A^s_t) of $\aleph_\xi$ rows and $\aleph_{\xi+1}$ columns whose entries are subsets of T with the following properties:

(i) For each row s, $(A^s_t) \cap (A^s_{t'}) = \emptyset$ for $t \neq t'$.

(ii) For each column t, $T \setminus \bigcup_s A^s_t$ is a set of cardinal less than κ.

Being $\nu(T \setminus \bigcup_s A^s_t) = 0$, it follows that $\nu(\bigcup_s A^s_t) = 1$, and therefore for each t there exists $s = s_t$ such that $\nu(A^s_t) > 0$, since the union of $\aleph_\xi$ ν-null measure sets has measure zero. But then there exists a row s with $\kappa > \aleph_0$ pairwise disjoint sets of positive ν measure, which contradicts the fact that the measure ν is finite. ■

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