RACSAM
Rev. R. Acad. Cien. Serie A. Mat.
VOL. **95** (1), 2001, pp. 19–28
Análisis Matemático / Mathematical Analysis

Holomorphic extension maps for spaces of Whitney jets

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Abstract. The key result (Theorem 1) provides the existence of a holomorphic approximation map for some space of C^∞-functions on an open subset of $\mathbb{R}^n$. This leads to results about the existence of a continuous linear extension map from the space of the Whitney jets on a closed subset F of $\mathbb{R}^n$ into a space of holomorphic functions on an open subset D of $\mathbb{C}^n$ such that $D \cap \mathbb{R}^n = \mathbb{R}^n \setminus F$.

Operadores de extensión holomorfa para espacios de jets de Whitney

Resumen. El resultado clave (Teorema 1) prueba que existen operadores de extensión holomorfa para ciertos espacios de funciones de clase C^∞ en un abierto de $\mathbb{R}^n$. Esto conduce a resultados sobre la existencia de un operador de extensión lineal y continuo del espacio de jets de Whitney sobre un subconjunto cerrado F de $\mathbb{R}^n$ en un espacio de funciones holomorfas en D, un subconjunto abierto de $\mathbb{C}^n$ tal que $D \cap \mathbb{R}^n = \mathbb{R}^n \setminus F$.

1. Introduction

Let Ω be a proper open subset of $\mathbb{R}^n$. For a C^∞-function f on Ω, we set

$$\mathrm{D}^\alpha f(x) = \frac{\partial^{|\alpha|} f}{\partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}}(x), \quad \forall \alpha \in \mathbb{N}_0^n, x \in \Omega.$$

By $\mathrm{BC}^\infty(\Omega)$, we designate the Fréchet space of the C^∞-functions on Ω which are bounded on Ω as well as all their derivatives, endowed with the system of norms $\{\|\cdot\|_m : m \in \mathbb{N}\}$ defined by

$$\|f\|_m = \sup_{|\alpha| \le m} 2^{(m+1)|\alpha|} \|\mathrm{D}^\alpha f\|_\Omega .$$

Moreover Ω^* stands for the following open subset of $\mathbb{C}^n$:

$$\Omega^* := \{u + iv : u \in \Omega, v \in \mathbb{R}^n, |v| < \mathrm{d}(u, \partial\Omega)\}.$$

For a closed subset F of $\mathbb{R}^n$, we designate as usual by $\mathcal{E}(F)$ the Fréchet space of the Whitney jets on F (cf. [4]).

In [2], the key result states that *there is a continuous linear map T from* $\mathrm{BC}^\infty(\Omega)$ *into* $\mathrm{BC}^\infty(\mathbb{R}^n)$ *such that, for every* $f \in \mathrm{BC}^\infty(\Omega)$,

Presentado por Manuel López Pellicer
Recibido: 9 de Mayo 2001. *Aceptado*: 6 de Junio 2001.
Palabras clave / Keywords: Whitney jet, extension map, real-analytic extension, holomorphic extension.
Mathematics Subject Classifications: Primary: 46E10, 32D99. Secondary: 26E05.

a) *Tf has a holomorphic extension on Ω^*;*
b) *for every $s \in \mathbb{N}$ and $\varepsilon > 0$, there is a compact subset K of Ω such that*

$$\sup_{|\alpha| \leq s} \|\mathrm{D}^\alpha f - \mathrm{D}^\alpha(Tf)\|_{\Omega \setminus K} \leq \varepsilon.$$

This result is then used to prove that, *if K is a compact subset of $\mathbb{R}^n$, then*
a) *every Whitney jet on K has a $\mathrm{BC}^\infty(\mathbb{R}^n)$-extension which is real-analytic on $\mathbb{R}^n \setminus K$;*
b) *there is a continuous linear extension map from $\mathcal{E}(K)$ into $\mathrm{C}^\infty(\mathbb{R}^n)$ if and only if there is such a map with values real-analytic outside K.*

Since then L. Frerick and D. Vogt have solved in [1] the problem raised in [2] as to how this last property extends for a closed subset. Their result reads as follows: *if there is a continuous linear extension map from $\mathcal{E}(F)$ into $\mathrm{C}^\infty(\mathbb{R}^n)$, then there also is such a map E with values having a holomorphic extension on Ω^* if and only if, for every bounded subset B of $\mathbb{R}^n$, the boundary of the union of the connected components of $\mathbb{R}^n \setminus F$ having non empty intersection with B is compact.* Their proof makes use of the key result mentioned here above.

The purpose of this paper is to generalize all these results.

Let U be a proper open subset of $\mathbb{C}^n$. For a holomorphic function f on U, we set

$$\mathrm{D}^\alpha f(z) = \frac{\partial^{|\alpha|} f}{\partial_{z_1}^{\alpha_1} \ldots \partial_{z_n}^{\alpha_n}}(z), \quad \forall \alpha \in \mathbb{N}_0^n, z \in U.$$

By $\mathcal{H}_\infty(U)$, we designate the Fréchet space of the holomorphic functions on U which are bounded on U as well as all their derivatives, endowed with the system of norms $\{\|\cdot\|_m : m \in \mathbb{N}\}$ defined by

$$\|f\|_m = \sup_{|\alpha| \leq m} \|\mathrm{D}^\alpha f\|_U .$$

Firstly we are going to construct in Paragraph 2. an open subset D_Ω of $\mathbb{C}^n$ such that $D_\Omega \cap \mathbb{R}^n = \Omega$ and $u \in \Omega$ for every $u + iv \in D_\Omega$. Then we prove in Paragraph 4. the following extension of the previous key result.

Theorem 1 *There is a continuous linear map T_Ω from $\mathrm{BC}^\infty(\Omega)$ into $\mathcal{H}_\infty(D_\Omega)$ such that for every $f \in \mathrm{BC}^\infty(\Omega)$, $\varepsilon > 0$ and $s \in \mathbb{N}$, there is a compact subset K of Ω such that*

$$|D^\alpha(T_\Omega f)(u + iv) - D^\alpha f(u)| \leq \varepsilon$$

for every $u + iv \in D_\Omega$ and $\alpha \in \mathbb{N}_0^n$ verifying $u \in \Omega \setminus K$ and $|\alpha| \leq s$.

Finally we use this new key result to enhance the previous extension theorems (cf. Section 5.).

In some way, this paper is a continuation of [2]. The idea is to use the fact that the functions $G(\cdot, f)$ have a holomorphic extension on $\Omega^* \supset D_\Omega$.

2. Construction of the open subset D_Ω of $\mathbb{C}^n$

Given a proper open subset Ω of $\mathbb{R}^n$, the construction of the open subset D_Ω of $\mathbb{C}^n$ comes from a refinement of the construction of the sequence $(\lambda_r)_{r \in \mathbb{N}}$ made in the Paragraph 2 of [2].

For the sake of clarity and completeness, we give the construction explicitely. The reader may skip it at first reading and come back to it as needed. The point is that we want to be able to use the inequalities of [2] involving the numbers λ_r in order to obtain supplementary results about the space $\mathrm{BC}^\infty(\Omega)$.

We first fix a compact cover $\{K_r : r \in \mathbb{N}\}$ of Ω subject to the following requirements: $(K_1)^\circ \neq \emptyset$, $\mathrm{d}(K_1, \partial\Omega) < 1$ and for every $r \in \mathbb{N}$, $(K_r)^{\circ,-} = K_r \subset (K_{r+1})^\circ$ as well as

$$\eta_r := \mathrm{d}(K_r, \mathbb{R}^n \setminus K_{r+1}) > \tfrac{1}{2}\mathrm{d}(K_r, \partial\Omega).$$

Of course the sequence $(\eta_r)_{r\in\mathbb{N}}$ strictly decreases to 0 and $\eta_1 < 1$.

Next for every $r \in \mathbb{N}$, a_r designates an element of $\mathrm{C}^\infty(\mathbb{R}^n)$, identically 1 on a neighbourhood of $K_{r+2} \setminus (K_{r+1})^\circ$ and with support contained in $(K_{r+3})^\circ \setminus K_r$.

Then for every r, $m \in \mathbb{N}$, we choose $d_{r,m} > 1$ such that

$$\begin{aligned}(r+1)d_{r,m}^2 &< d_{r+1,m},\\ d_{r,m} &< d_{r,m+1},\\ \sup_{|\alpha|\le m} 2^{(m+r+1)|\alpha|}\,\|\mathrm{D}^\alpha a_r\|_{\mathbb{R}^n} &\le d_{r,m}.\end{aligned}$$

Finally we remark that, by the Poisson formula,

$$\Phi(\rho) := \pi^{-n/2}\int_{|y|\le\rho} \mathrm{e}^{-|y|^2}\,dy \uparrow 1$$

as $\rho > 0$ increases to $+\infty$.

So if we introduce the numbers $p_r = d_{r,r}$, $\varepsilon_r = 2^{-rp_{r+2}}$ and $\delta_r = \varepsilon_r(3np_r^2p_{r+1}2^{2r+2})^{-1}$ for every $r \in \mathbb{N}$, we can fix a strictly increasing sequence $(\lambda_r)_{r\in\mathbb{N}}$ of positive integers by the following procedure.

We choose $\lambda_1 > 1$ verifying the conditions hereunder if they apply to λ_1 only and then the numbers λ_2, $\lambda_3, \ldots$ successively, submitted to the following requirements:

(1) $p_r^2(1-\Phi(\lambda_r\delta_r)) < \delta_r$;
(2) $\pi^{-n/2}\lambda_r^n\mathrm{e}^{-\lambda_r^2r^{-2}}p_r^2\mu(K_{r+3}) < 2^{-r}$, where μ is the Lebesgue measure;
(3) $\lambda_r^{-1}2^{n+2}\pi^{-n/2}p_r^2(1+\mu(K_{r+3})) \le 2^{-r}$;
(4) $\lambda_{r+1}^{-1} < \mathrm{d}(K_r, \mathbb{R}^n \setminus \Omega)$;
(5) $\mathrm{e}^{-\frac{1}{2}\lambda_r} \le \lambda_r^{-(n+1)}$;
(6) $\lambda_r(\eta_p^2 - \lambda_{p+1}^{-2}) \ge \frac{1}{2}$ for every $p \in \{1, \ldots, r-1\}$;
(7) $\lambda_{r+1}^{-n} \le \lambda_r^{-(n+1)}$;
(8) $\mathrm{e}^{\lambda_r^2\lambda_{r+1}^{-2}} - 1 \le \lambda_r^{-(n+1)}$;
(9) for every $p \in \mathbb{N}$, we set $R_p = \sup\{|u| : u \in K_p\}$ and, if $\lambda_1, \ldots, \lambda_p$ are fixed, we first choose $\Theta_p > 0$ such that $\left|\mathrm{e}^{i\theta} - 1\right| \le \lambda_p^{-(n+1)}$ for every $\theta \in [-\Theta_p, \Theta_p]$ and next impose $4\lambda_p^2\lambda_r^{-1}R_{r+2} \le \Theta_p$ for every $r > p$.

Let us remark that the requirements (1) and (2) are exactly the conditions imposed in [2] for the definition of the sequence $(\lambda_r)_{r\in\mathbb{N}}$. So all the inequalities established in [2] are available.

Definition 1 *Now we have at our disposal all we need to introduce the open subset D_Ω of $\mathbb{C}^n$ as the interior of*

$$\bigcup_{r=0}^{\infty}\{u+iv\colon u \in K_{r+1}\setminus K_r, v \in \mathbb{R}^n, |v| < \lambda_{r+2}^{-1}\}$$

where $K_0 := \emptyset$.

The requirement (4) has been introduced in order to have $D_\Omega \subset \Omega^*$.

3. Auxiliary result about $\mathrm{BC}^\infty(\Omega)$

As in [2], given $f \in \mathrm{BC}^\infty(\Omega)$, we define the sequence $(G_r(\cdot, f))_{r\in\mathbb{N}_0}$ of functions on $\mathbb{C}^n$ by the following recursion: we set $G_0(w, f) = 0$ and

$$G_r(w,f) = \pi^{-n/2}\lambda_r^n\int_{\mathbb{R}^n} a_r(y)(f(y) - \sum_{j=1}^{r-1}G_j(y,f))\mathrm{e}^{-\lambda_r^2\sum_{j=1}^n(w_j-y_j)^2}\,dy$$

for every $r \in \mathbb{N}$ and $w \in \mathbb{C}^n$.

As the functions $a_r \in \mathrm{C}^\infty(\mathbb{R}^n)$ have compact support contained in Ω, this makes sense and the functions $G_r(\cdot, f)$ are holomorphic on $\mathbb{C}^n$. Moreover we have

$$\begin{aligned}&\mathrm{D}^\alpha G_r(w,f)\\&= \pi^{-n/2}\lambda_r^n \int_{\mathbb{R}^n} \mathrm{D}^\alpha \left(a_r(y)(f(y) - \sum_{j=1}^{r-1} G_j(y,f)) \right) \cdot \mathrm{e}^{-\lambda_r^2 \sum_{j=1}^n (w_j - y_j)^2}\, dy\end{aligned}$$

for every $\alpha \in \mathbb{N}_0^n$ and $r \in \mathbb{N}$.

We are going to estimate $|\mathrm{D}^\alpha G_r(u+iv,f) - \mathrm{D}^\alpha G_r(u,f)|$ for every $u+iv \in D_\Omega$, $\alpha \in \mathbb{N}_0^n$ and $r \in \mathbb{N}$. By use of the inequality (1) of the Proposition 3.1 of [2], we certainly have

$$|\mathrm{D}^\alpha G_r(u+iv,f) - \mathrm{D}^\alpha G_r(u,f)| \leq \pi^{-n/2}\lambda_r^n \cdot d_{r,m}^2 2^{-m|\alpha|} \left\| f \right\|_m \cdot I_r$$

for every $m \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^n$ verifying $|\alpha| \leq m$, with

$$I_r := \sup_{u+iv \in D} \int_{K_{r+3}\setminus K_r} \left| \mathrm{e}^{-\lambda_r^2 \sum_{j=1}^n (u_j + iv_j - y_j)^2} - \mathrm{e}^{-\lambda_r^2 \sum_{j=1}^n (u_j - y_j)^2} \right| dy.$$

Lemma 1 *We have* $I_1 \leq (1+\mathrm{e})\mu(K_4)$ *and*

$$I_r \leq 2^{n+2}(1+\mu(K_{r+3}))\lambda_r^{-(n+1)}, \quad \forall r \in \{2,3,\ldots\}.$$

PROOF. The estimation of I_1 is a direct consequence of

$$\left| \mathrm{e}^{-\lambda_1^2 \sum_{j=1}^n (u_j + iv_j - y_j)^2} \right| \leq \mathrm{e}^{\lambda_1^2 |v|^2} \leq \mathrm{e}^{\lambda_1^2 \lambda_2^{-2}} \leq \mathrm{e} \text{ and } \mathrm{e}^{-\lambda_1^2 \sum_{j=1}^n (u_j - v_j)^2} \leq 1.$$

The case $r \geq 2$ needs more care. Let us first make some evaluations of

$$X := \lambda_r^2 \sum_{j=1}^n ((u_j - y_j)^2 - v_j^2) = \lambda_r^2(|u-y|^2 - |v|^2).$$

a) If u belongs to K_1, the requirement (6) of the definition of the numbers λ_r leads to

$$X \geq \lambda_r^2(\mathrm{d}^2(K_1, \mathbb{R}^n \setminus K_r) - |v|^2) \geq \lambda_r^2(\eta_1^2 - \lambda_2^{-2}) \geq \tfrac{1}{2}\lambda_r.$$

b) If u does not belong to K_1, there is a unique $p \in \mathbb{N}$ such that u belongs to $K_{p+1} \setminus K_p$ and we distinguish the following two possibilities:
b.1) if $p+1 \leq r-1$, the requirement (6) provides

$$X \geq \lambda_r^2(\mathrm{d}^2(K_{p+1}, \mathbb{R}^n \setminus K_r) - \lambda_{p+2}^{-2}) \geq \lambda_r^2(\eta_{p+1}^2 - \lambda_{p+2}^{-2}) \geq \tfrac{1}{2}\lambda_r;$$

b.2) if $p+1 \geq r$, then we set $J_r = \prod_{j=1}^n [u_j - \lambda_{r+1}^{-1}, u_j + \lambda_{r+1}^{-1}]$ and successively get
b.2.i) if $y \in J_r$: $X \geq -\lambda_r^2 \lambda_{p+2}^{-2} \geq -1$;
b.2.ii) if $y \notin J_r$: as $y \in K_{r+3}$, the requirements (8) and (9) give

$$\begin{aligned}&\left| \mathrm{e}^{-\lambda_r^2 \sum_{j=1}^n (u_j + iv_j - y_j)^2} - \mathrm{e}^{-\lambda_r^2 \sum_{j=1}^n (u_j - y_j)^2} \right|\\&\leq \left| \mathrm{e}^{\lambda_r^2 \sum_{j=1}^n (v_j^2 - 2iv_j(u_j - y_j))} - 1 \right| \leq \mathrm{e}^{\lambda_r^2 |v|^2} \left| \mathrm{e}^{-2i\lambda_r^2 \sum_{j=1}^n v_j(u_j - y_j)} - 1 \right| + (\mathrm{e}^{\lambda_r^2 |v|^2} - 1)\\&\leq \mathrm{e}\lambda_r^{-(n+1)} + \lambda_r^{-(n+1)} \leq 2^2 \lambda_r^{-(n+1)}\end{aligned}$$

since $\left| 2i\lambda_r^2 \sum_{j=1}^n v_j(u_j - y_j) \right| \leq 2\lambda_r^2 |v|\,|u-y| \leq 4\lambda_r^2 \lambda_{p+2}^{-1} R_{p+4} \leq \Theta_r$.

Consequently
a) if $u \in K_1$ or if $u \in K_{p+1} \setminus K_p$ with $p+1 \le r-1$, the requirement (5) provides

$$I_r \le 2 \int_{K_{r+3} \setminus K_r} \mathrm{e}^{-\lambda_r^2 \sum_{j=1}^n ((u_j - y_j)^2 - v_j^2)} \, dy$$
$$\le 2\mathrm{e}^{-\frac{1}{2}\lambda_r} \mu(K_{r+3}) \le 2\lambda_r^{-(n+1)} \mu(K_{r+3});$$

b) if $u \in K_{p+1} \setminus K_p$ with $p+1 \ge r$, the requirement (7) leads to

$$I_r \le \int_{(K_{r+3} \setminus K_r) \setminus J_r} + \int_{(K_{r+3} \setminus K_r) \cap J_r} \cdots$$
$$\le 2^2 \lambda_r^{-(n+1)} \mu(K_{r+3}) + (\mathrm{e}+1)\mu(J_r)$$
$$\le 2^2 \lambda_r^{-(n+1)} \mu(K_{r+3}) + 2^2 2^n \lambda_{r+1}^{-n} \le 2^{n+2} \lambda_r^{-(n+1)} (1 + \mu(K_{r+3})). \blacksquare$$

Proposition 1 a) *For every $m \in \mathbb{N}$, there is $C_m > 0$ such that*

$$|D^\alpha G_r(u+iv, f) - D^\alpha G_r(u, f)| \le C_m 2^{-m|\alpha|} \|f\|_m$$

for every $f \in \mathrm{BC}^\infty(\Omega)$, $u+iv \in D_\Omega$, $r \in \{1, \ldots, m\}$ and $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| \le m$.
b) *For every $f \in \mathrm{BC}^\infty(\Omega)$, m, $r \in \mathbb{N}$, $u+iv \in D_\Omega$ and $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| \le m$, one has*

$$|D^\alpha G_{m+r}(u+iv, f) - D^\alpha G_{m+r}(u, f)| \le 2^{-(m+r)} 2^{-m|\alpha|} \|f\|_m .$$

PROOF. a) Indeed for $r = 1$, the Lemma 1 leads immediately to

$$|\mathrm{D}^\alpha G_1(u+iv, f) - \mathrm{D}^\alpha G_1(u, f)|$$
$$\le \pi^{-n/2} \lambda_1^n \cdot d_{1,m}^2 2^{-m|\alpha|} \|f\|_m \cdot (1+\mathrm{e})\mu(K_4)$$

and, for $r \in \{2, \ldots, m\}$, it leads to

$$|\mathrm{D}^\alpha G_r(u+iv, f) - \mathrm{D}^\alpha G_r(u, f)|$$
$$\le \lambda_r^{-1}(1 + \mu(K_{r+3})) \cdot \pi^{-n/2} p_m^2 2^{n+2} \cdot 2^{-m|\alpha|} \|f\|_m .$$

b) Again the Lemma 1 leads to

$$|\mathrm{D}^\alpha G_{m+r}(u+iv, f) - \mathrm{D}^\alpha G_{m+r}(u, f)|$$
$$\le \lambda_{m+r}^{-1} \pi^{-n/2} 2^{n+2} (1 + \mu(K_{m+r+3})) p_{m+r}^2 \cdot 2^{-m|\alpha|} \|f\|_m$$

hence the conclusion by use of the requirement (3) of the definition of the numbers λ_r. $\blacksquare$

4. Main result

Let us now set $G(u+iv, f) = \sum_{r=0}^\infty G_r(u+iv, f)$ for every $f \in \mathrm{BC}^\infty(\Omega)$ and $u+iv \in \Omega^*$. By ([2], Proposition 3.8), we know that $G(\cdot, f)$ is a holomorphic function on Ω^* hence on D_Ω.

In fact a lot more can be said: everything is now in order to obtain the key result about the space $\mathrm{BC}^\infty(\Omega)$ in view of the extension theorems.

Theorem 2 *There is a continuous linear map T_Ω from $\mathrm{BC}^\infty(\Omega)$ into $\mathcal{H}_\infty(D_\Omega)$ such that for every $f \in \mathrm{BC}^\infty(\Omega)$, $\varepsilon > 0$ and $s \in \mathbb{N}$, there is a compact subset K of Ω such that*

$$|D^\alpha (T_\Omega f)(u+iv) - D^\alpha f(u)| \le \varepsilon$$

for every $u+iv \in D_\Omega$ and $\alpha \in \mathbb{N}_0^n$ verifying $u \in \Omega \setminus K$ and $|\alpha| \le s$.

PROOF. In fact we have just to set $(T_\Omega f)(u+iv) = G(u+iv, f)$ for every $f \in \mathrm{BC}^\infty(\Omega)$ and $u+iv \in D_\Omega$.

We already know that, for every $f \in \mathrm{BC}^\infty(\Omega)$, $G(\cdot, f)$ is a holomorphic function on Ω^* hence on D_Ω by ([2], Proposition 3.8). Moreover it is clear that the construction of $G(\cdot, f)$ is linearly depending on f. As for every $f \in \mathrm{BC}^\infty(\Omega)$, $m \in \mathbb{N}$, $u + iv \in D_\Omega$ and $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| \leq m$, we successively have

$$\begin{aligned}
|\mathrm{D}^\alpha G(u+iv, f)| &\leq |\mathrm{D}^\alpha G(u, f)| + \sum_{r=1}^{m} |\mathrm{D}^\alpha G_r(u+iv, f) - \mathrm{D}^\alpha G_r(u, f)| \\
&\quad + \sum_{r=m+1}^{\infty} |\mathrm{D}^\alpha G_r(u+iv, f) - \mathrm{D}^\alpha G_r(u, f)| \\
&\leq c_m \|f\|_{m+1} + mC_m 2^{-m|\alpha|} \|f\|_m + 2^{-m} 2^{-m|\alpha|} \|f\|_m \\
&\leq (c_m + mC_m + 2^{-m}) \|f\|_{m+1}
\end{aligned}$$

by use of ([2], Proposition 3.7) and of the Proposition 1 to get the second inequality, it is already established that T_Ω is a continuous linear map from $\mathrm{BC}^\infty(\Omega)$ into $\mathcal{H}_\infty(D_\Omega)$.

Let us now prove the second part of the statement: let $f \in \mathrm{BC}^\infty(\Omega)$, $\varepsilon > 0$ and $s \in \mathbb{N}$ be fixed.

The part b) of the Proposition 1 leads to

$$|\mathrm{D}^\alpha G_r(u+iv, f) - \mathrm{D}^\alpha G_r(u, f)| \leq 2^{-r} 2^{-s|\alpha|} \|f\|_s$$

for every $u + iv \in D_\Omega$, $r \in \mathbb{N}$, $\alpha \in \mathbb{N}_0^n$ such that $r \geq s + 1$ and $|\alpha| \leq s$. So it is possible to fix a positive integer $m \geq s$ such that

$$\sum_{r=m+1}^{\infty} |\mathrm{D}^\alpha G_r(u+iv, f) - \mathrm{D}^\alpha G_r(u, f)| \leq \frac{\varepsilon}{3}$$

for every $u + iv \in D_\Omega$ and $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| \leq s$.

As the sequence $(\varepsilon_r)_{r\in\mathbb{N}}$ decreases to 0, the Lemma 3.6 of [2] then provides $d_0 \in \mathbb{N}$ such that

$$|\mathrm{D}^\alpha G(u, f) - \mathrm{D}^\alpha f(u)| \leq \frac{\varepsilon}{3}$$

for every $u \in \Omega \setminus K_{d_0}$ and $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| \leq s$.

Now we turn our attention to the evaluation of

$$|\mathrm{D}^\alpha G_r(u+iv, f) - \mathrm{D}^\alpha G_r(u, f)|$$

for every $r \in \{1, \ldots, m\}$, $u + iv \in D_\Omega$ and $\alpha \in \mathbb{N}_0^n$ such that $u \in \Omega \setminus K_d$ with $d \geq d_0$ and $|\alpha| \leq s$. We already know that it is

$$\leq \pi^{-n/2} \lambda_r^n \cdot d_{r,s}^2 2^{-s|\alpha|} \|f\|_s \cdot I_{r,u+iv} \leq \pi^{-n/2} \lambda_m^n \cdot p_m^2 2^{-s|\alpha|} \|f\|_s \cdot I_{r,u+iv}$$

with

$$\begin{aligned}
I_{r,u+iv} &:= \int_{K_{r+3}\setminus K_r} \mathrm{e}^{-\lambda_r^2 |u-y|^2} \left| \mathrm{e}^{\lambda_r^2 \sum_{j=1}^n (v_j^2 - 2iv_j(u_j - y_j))} - 1 \right| dy \\
&\leq \int_{K_{r+3}\setminus K_r} \left| \mathrm{e}^{\lambda_r^2 |v|^2} \mathrm{e}^{-2i\lambda_r^2 \sum_{j=1}^n v_j(u_j - y_j)} - 1 \right| dy.
\end{aligned}$$

For $u + iv \in D_\Omega$ verifying $u \in K_{d+1} \setminus K_d$ with $d \geq \sup\{m + 2, d_0\}$, we have

$$\begin{aligned}
&\left| \mathrm{e}^{\lambda_r^2 |v|^2} \mathrm{e}^{-2i\lambda_r^2 \sum_{j=1}^n v_j(u_j - y_j)} - 1 \right| \\
&\qquad \leq \mathrm{e}^{\lambda_r^2 \lambda_{d+2}^{-2}} \left| \mathrm{e}^{-2i\lambda_r^2 \sum_{j=1}^n v_j(u_j - y_j)} - 1 \right| + (\mathrm{e}^{\lambda_r^2 \lambda_{d+2}^{-2}} - 1)
\end{aligned}$$

with $\exp(\lambda_r^2\lambda_{d+2}^{-2}) \leq \mathrm{e}$ and $\exp(\lambda_r^2\lambda_{d+2}^{-2}) - 1 \to 0$ if $d \to \infty$. Moreover we have

$$\left|2\lambda_r^2 \sum_{j=1}^{n} v_j(u_j - y_j)\right| \leq 2\lambda_r^2\lambda_{d+2}^{-1}(|u| + |y|)$$
$$\leq 2\lambda_r^2\lambda_{d+2}^{-1}(R_{d+1} + R_{r+3}) \leq 4\lambda_m^2\lambda_{d+2}^{-1}R_{d+1}.$$

So we can choose $d_1 \geq \sup\{m+2, d_0\}$ such that

$$|\mathrm{D}^\alpha G_r(u+iv, f) - \mathrm{D}^\alpha G_r(u, f)| \leq \frac{\varepsilon}{3m}$$

for every $r \in \{1, \ldots, m\}$, $\alpha \in \mathbb{N}_0^n$ and $u + iv \in D_\Omega$ such that $|\alpha| \leq s$ and $u \in \Omega \setminus K_{d_1}$.

Taking all these informations together leads then to the conclusion with $K = K_{d_1}$. ■

5. Existence of holomorphic extension maps

Case 1: F is compact or $\mathbb{R}^n \setminus F$ is bounded.

Definition 2 *Given a proper open subset Ω of $\mathbb{R}^n$, $\mathcal{H}_\infty \mathrm{C}^\infty(\Omega)$ designates the following Fréchet space. Its elements are the functions f defined on $\mathbb{R}^n \cup D_\Omega$ such that*
(1) $f|_{\mathbb{R}^n} \in \mathrm{C}^\infty(\mathbb{R}^n)$;
(2) $f|_{D_\Omega} \in \mathcal{H}_\infty(D_\Omega)$;
(3) $\lim_{z\to x} D^\alpha(f|_{D_\Omega}) = D^\alpha(f|_{\mathbb{R}^n})(x)$ for every $\alpha \in \mathbb{N}_0^n$ and $x \in \partial_{\mathbb{R}^n}\Omega$.
It is endowed with the countable system of semi-norms $\{|||\cdot|||_m : m \in \mathbb{N}\}$ defined by

$$|||f|||_m = \sup_{|\alpha|\leq m} \|D^\alpha(f|_{\mathbb{R}^n})\|_{b_m\setminus\Omega} + \sup_{|\alpha|\leq m} \|D^\alpha(f|_{D_\Omega})\|_{D_\Omega}$$

where $b_m := \{x \in \mathbb{R}^n : |x| \leq m\}$.

Theorem 3 *Let F be a proper closed subset of $\mathbb{R}^n$ and set $\Omega := \mathbb{R}^n \setminus F$.*

If F is compact or if Ω is bounded, then the existence of a continuous linear extension map E from $\mathcal{E}(F)$ into $\mathrm{C}^\infty(\mathbb{R}^n)$ implies the existence of a continuous linear extension map E_F from $\mathcal{E}(F)$ into $\mathcal{H}_\infty\mathrm{C}^\infty(\Omega)$.

PROOF. If F is compact, we choose a function $\psi \in \mathrm{C}^\infty(\mathbb{R}^n)$ identically 1 on a neighbourhood of F with compact support and check that the map

$$E_1 : \mathcal{E}(F) \to \quad \varphi \mapsto \psi.E\varphi$$

also is a continuous linear extension map. So up to substituting E_1 to E itself, we may very well suppose that E is a continuous linear extension map from $\mathcal{E}(F)$ into $\mathrm{C}^\infty(\mathbb{R}^n)$ such that $(E\ \cdot)|_\Omega$ is a continuous linear map from $\mathcal{E}(F)$ into $\mathrm{BC}^\infty(\Omega)$.

Now to every jet $\varphi \in \mathcal{E}(F)$, we associate the function $E_F\varphi$ defined on $\mathbb{R}^n \cup D_\Omega$ as follows

$$\begin{cases} (E_F\varphi)(x) &= (E\varphi)(x), & \forall x \in F, \\ (E_F\varphi)(z) &= T_\Omega((E\varphi)|_\Omega)(z), & \forall z \in D_\Omega. \end{cases}$$

By use of the key Theorem 2, it is a direct matter to check that E_F so defined is a linear extension map from $\mathcal{E}(F)$ into $\mathcal{H}_\infty\mathrm{C}^\infty(\Omega)$. Let us prove that it also is continuous. For this purpose, we just need to prove that for every continuous semi-norm $|||\cdot|||_m$ on $\mathcal{H}_\infty\mathrm{C}^\infty(\Omega)$, there is a continuous semi-norm p on $\mathcal{E}(F)$ such that

$$|||E_F\varphi|||_m \leq p(\varphi), \quad \forall \varphi \in \mathcal{E}(F).$$

This is easy: we just have to note that we have

$$|||\varphi|||_m = \sup_{|\alpha|\leq m} \|\mathrm{D}^\alpha(E\varphi)|_{\mathbb{R}^n}\|_{b_m\setminus D_\Omega} + \sup_{|\alpha|\leq m} \|\mathrm{D}^\alpha(T_\Omega(E\varphi)|_\Omega)\|_{D_\Omega}$$

with

$$\sup_{|\alpha|\leq m} \|\mathrm{D}^\alpha(E\varphi)|_{\mathbb{R}^n}\|_{b_m\setminus D_\Omega} = \sup_{|\alpha|\leq m} \|\varphi_\alpha\|_{b_m\cap F}$$

and

$$\sup_{|\alpha|\leq m} \|\mathrm{D}^\alpha(T_\Omega(E\varphi)|_\Omega)\|_{D_\Omega} \leq q((E\varphi)|_\Omega) \leq p(\varphi)$$

for some continuous semi-norms q on $\mathrm{BC}^\infty(\Omega)$ since T_Ω is a continuous linear map from $\mathrm{BC}^\infty(\Omega)$ into $\mathcal{H}_\infty(D_\Omega)$ and for some continuous semi-norm p on $\mathcal{E}(F)$ since $(E\cdot)|_\Omega$ is a continuous linear map from $\mathcal{E}(F)$ into $\mathrm{BC}^\infty(\Omega)$. ■

Case 2: F is a closed subset of $\mathbb{R}^n$.

Definition 3 *Let us abbreviate "connected component" by "c.c.". Given a proper open subset Ω of $\mathbb{R}^n$, let us set*

$$\Omega_1 := \cup\{\omega\colon \omega = c.c.\ of\ \Omega, \omega \cap B_1 \neq \emptyset\},$$

introduce by recursion the sets

$$\Omega_j := \cup\{\omega\colon \omega = c.c.\ of\ \Omega, \omega \cap B_j \neq \emptyset, \omega \cap (\cap_{k=1}^{j-1}\Omega_k) = \emptyset\}$$

for $j = 2, 3, \ldots$ and write $J := \{j \in \mathbb{N}\colon \Omega_j \neq \emptyset\}$. For every $j \in J$, the construction of Paragraph 2. applied to Ω_j provides an open subset D_{Ω_j} of $\mathbb{C}^n$ such that $\mathbb{R}^n \cap D_{\Omega_j} = \Omega_j$ and $(u+iv \in D_{\Omega_j} \Rightarrow u \in \Omega_j)$.

Then we set $D := \cup_{j\in J} D_{\Omega_j}$ and introduce the following Fréchet space $\mathcal{H}\mathrm{C}^\infty(\Omega)$. Its elements are the functions f defined on $\mathbb{R}^n \cup D$ such that
(1) $f|_{\mathbb{R}^n} \in \mathcal{E}(\mathbb{R}^n)$;
(2) $f|_D \in \mathcal{H}(D)$;
(3) f and its derivatives are bounded on each one of the sets D_{Ω_j};
(4) $\lim_{z\to x} D^\alpha(f|_D)(z) = D^\alpha(f|_{\mathbb{R}^n})(x)$ for every $\alpha \in \mathbb{N}_0^n$ and $x \in \partial_{\mathbb{R}^n}\Omega$.
It is endowed with the countable system of semi-norms $\{|||\cdot|||_m : m \in \mathbb{N}\}$ defined by

$$|||f|||_m = \sup_{|\alpha|\leq m} \|D^\alpha(f|_{\mathbb{R}^n})\|_{b_m\setminus\Omega} + \sup_{j\leq m} \sup_{|\alpha|\leq m} \|D^\alpha f\|_{D_{\Omega_j}}.$$

Theorem 4 *Let F be a proper closed subset of $\mathbb{R}^n$ and set $\Omega = \mathbb{R}^n \setminus F$.*

If there is a continuous linear extension map $E\colon \mathcal{E}(F) \to \mathcal{E}(\mathbb{R}^n)$, the following assertions are equivalent:
(1) *there is a continuous linear extension map E_1 from $\mathcal{E}(F)$ into $\mathcal{H}\mathrm{C}^\infty(\Omega)$;*
(2) *for every bounded subset B of $\mathbb{R}^n$, the boundary of the union of the connected components of Ω having non empty intersection with B is compact.*

PROOF. $(1) \Rightarrow (2)$. One has just to follow the argument of Frerick and Vogt in [1]. For the sake of completeness, we repeat it. If it is not the case, there is $r > 0$ such that the boundary of

$$\omega_r = \cup\{\omega\colon \omega = \text{ c.c. of } \Omega, \omega \cap b_r \neq \emptyset\}$$

is unbounded. As $\|\cdot\|_{b_r}$ is a continuous semi-norm on $\mathcal{H}\mathrm{C}^\infty(\Omega)$, the continuity of the map E_1 provides the existence of $m \in \mathbb{N}$ such that $m > r$ and

$$\|E_1\varphi\|_{b_r} \leq m\,|\varphi|_m\,, \quad \forall \varphi \in \mathcal{E}(F),$$

where $|\cdot|_m$ denotes the m-th continuous semi-norm on $\mathcal{E}(F)$ (i.e. corresponding to the compact set $F \cap b_m$). Now we choose $x_0 \in (\partial_{\mathbb{R}^n}\omega_r) \setminus b_{m+1}$ and $\psi_0 \in \mathrm{C}^\infty(\mathbb{R}^n)$ such that $\psi_0(x_0) = 1$, and finally consider the jet $\varphi_0 = ((\mathrm{D}^\alpha\psi_0)|_F)_{\alpha \in \mathbb{N}_0^n} \in \mathcal{E}(F)$. On one hand, as $|\varphi_0|_m = 0$, $E_1\varphi_0$ is identically 0 on b_r. On the other hand, $E_1\varphi_0$ is not identically 0 on any neighbourhood of x_0. Hence a contradiction.

$(2) \Rightarrow (1)$. If F is compact or if Ω is bounded, the condition (2) is automatically satisfied and the previous theorem provides a better result.

If F is not compact and if Ω is not bounded, we proceed as follows.

If $n \geq 2$, as F is not compact, the condition (2) implies that all the connected components of Ω are bounded and, as Ω is not bounded, J is infinite. If $n = 1$, as F is not compact, the condition (2) implies that one and only one connected component ω of Ω may be unbounded: it is of the type $]-\infty, a[$ or $]b, +\infty[$. We then choose a function $\psi \in \mathrm{C}^\infty(\mathbb{R})$ identically 1 on a neighbourhood of $[a, +\infty[$ or $]-\infty, b]$ and 0 on $]-\infty, a-1]$ or $[b+1, +\infty[$ respectively and check that

$$E_2 \colon \mathcal{E}(F) \to \mathrm{C}^\infty(\mathbb{R}) \quad \varphi \mapsto \psi.E\varphi$$

is a continuous linear extension map such that $(E_2\cdot)|_\omega$ is a continuous linear map from $\mathcal{E}(F)$ into $\mathrm{BC}^\infty(\omega)$.

So up to a substitution, we may very well suppose that, for every $j \in J$, $(E\cdot)|_{\Omega_j}$ is a continuous linear map from $\mathcal{E}(F)$ into $\mathrm{BC}^\infty(\Omega_j)$.

Now we apply the Theorem 1 for every $j \in J$ and get continuous linear extension maps T_{Ω_j} from $\mathrm{BC}^\infty(\Omega_j)$ into $\mathcal{H}_\infty(D_j)$ such that for every $f \in \mathrm{BC}^\infty(\Omega_j)$, $\varepsilon > 0$ and $s \in \mathbb{N}$, there is a compact subset K_j of Ω_j such that $\left|\mathrm{D}^\alpha(T_{\Omega_j}f)(u+iv) - \mathrm{D}^\alpha f(u)\right| \leq \varepsilon$ for every $u + iv \in D_j$ and $\alpha \in \mathbb{N}_0^n$ such that $u \in \Omega_j \setminus K_j$ and $|\alpha| \leq s$.

To every jet $\varphi \in \mathcal{E}(F)$, we then associate the function $E_1\varphi$ defined on $\mathbb{R}^n \cup D$ by

$$\begin{cases} (E_1\varphi)(x) &= (E\varphi)(x), & \forall x \in F, \\ (E_1\varphi)(z) &= T_{\Omega_j}((E\varphi)|_{\Omega_j})(z), & \forall z \in D_j, j \in J. \end{cases}$$

It is a rather classic matter to check that E_1 so defined is a linear extension map from $\mathcal{E}(F)$ into $\mathcal{H}C^\infty(\Omega)$.

To conclude, we still have to establish its continuity. As it is a linear map, we just need to prove that for every $m \in \mathbb{N}$, there is a continuous semi-norm p on $\mathcal{E}(F)$ such that $|||E_1\varphi|||_m \leq p(\varphi)$ for every $\varphi \in \mathcal{E}(F)$. This is a direct matter since we have

$$|||E_1\varphi|||_m = \sup_{|\alpha| \leq m} \|\mathrm{D}^\alpha((E_1\varphi)|_{\mathbb{R}^n})\|_{b_m \setminus \Omega} + \sup_{j \leq m} \sup_{|\alpha| \leq m} \|\mathrm{D}^\alpha(E_1\varphi)\|_{D_{\Omega_j}}$$

with

$$\sup_{|\alpha| \leq m} \|\mathrm{D}^\alpha((E_1\varphi)|_{\mathbb{R}^n})\|_{b_m \setminus \Omega} \leq \sup_{|\alpha| \leq m} \|\varphi_\alpha\|_{b_m \cap F}$$

and

$$\sup_{j \leq m} \sup_{|\alpha| \leq m} \|\mathrm{D}^\alpha(E_1\varphi)\|_{D_{\Omega_j}} \leq \sup_{j \leq m} q_j((E\varphi)|_{\Omega_j}) \leq \sup_{j \leq m} p_j(\varphi)$$

for some continuous semi-nors q_j on $\mathrm{BC}^\infty(\Omega_j)$ since T_{Ω_j} is a continuous linear map from $\mathrm{BC}^\infty(\Omega_j)$ into $\mathcal{H}_\infty\mathrm{C}^\infty(D_{\Omega_j})$ and some continuous semi-norms p_j on $\mathcal{E}(F)$ since $(E\cdot)|_{\Omega_j}$ is a continuous linear map from $\mathcal{E}(F)$ into $\mathrm{BC}^\infty(\Omega_j)$. ■

Remark 1 *The same method applies to the case of the ultradifferentiable Whitney jets of the Beurling or Roumieu type (cf. [3]).*

Acknowledgement. Jean Schmets is partially supported by the Belgian C.G.R.I. Manuel Valdivia is partially supported by DGES pr. 96–0758 and the Convenio Hispano-Belga.

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