RAC$\mathbb{SAM}$
Rev. R. Acad. Cien. Serie A. Mat.
VOL. **95** (1), 2001, pp. 13–17
Análisis Matemático / Mathematical Analysis
Comunicación Preliminar / Preliminary Communication

Complemented copies of c_0 in $C_0(\Omega)$

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Abstract. In this note we consider a class of locally compact Hausdorff topological spaces Ω with the property that the Banach space $C_0(\Omega)$ of all scalarly valued continuous functions defined on Ω vanishing at infinity, equipped with the supremum-norm, contains a norm-one complemented copy of c_0 whereas $C(\beta\Omega)$ contains a linearly isometric copy of ℓ_∞.

Copias complementadas de c_0 en $C_0(\Omega)$

Resumen. En esta nota consideramos una clase de espacios topológicos de Hausdorff localmente compactos Ω con la propiedad de que el espacio de Banach $C_0(\Omega)$ de todas las funciones continuas con valores escalares definidas en Ω que se anulan en el infinito, equipado con la norma supremo, contiene una copia de c_0 norma-uno complementada, mientras que $C(\beta\Omega)$ contiene una copia de ℓ_∞ linealmente isométrica.

1. Preliminaries

Some known results are given in this preliminaires in order to make the reading easier and also to present some new viewpoints. All topological spaces considered in this paper will be Hausdorff. In what follows K will stand for a compact topological space, Ω for a locally compact topological space and X for a Banach space. The Banach space over the field $\mathbb{K}$ of the real or complex numbers of all X-valued continuous functions defined on K equipped with the supremum norm $\|f\|_\infty = \sup\{\|f(\omega)\| : \omega \in K\}$ will be represented by $C(K, X)$ [by $C(K)$ if $X = \mathbb{K}$]. The Banach space over $\mathbb{K}$ of all continuous functions $f : \Omega \to \mathbb{K}$ vanishing at infinity (that is, for each $\epsilon > 0$ there is a compact set $K_{f,\epsilon} \subseteq \Omega$ such that $|f(\omega)| < \epsilon$ for each $\omega \in \Omega - K_{f,\epsilon}$) will be denoted by $C_0(\Omega)$. The linear subspace of $C_0(\Omega)$ consisting of all those functions f of compact support, supp f, will be represented by $C_c(\Omega)$. A compact space K is called Eberlein compact if K is homeomorphic to a weakly compact set of a Banach space. If I is an index set and $\sum(I)$ denotes the subset of $[0,1]^I$ of all those functions of countable support, a compact space K is called Valdivia compact [3, 7] if there exists a set I such that K is homeomorphic to a subset K_0 of $[0,1]^I$ with the property that $K_0 \cap \sum(I)$ is dense in K_0. Every compact metric space is Eberlein compact, and every Eberlein compact is Valdivia compact [3, Theorem 7.2]. A Banach space X is said to be a Grothendieck space if each bounded linear operator $T : X \to c_0$ is weakly compact or, equivalently [4, Chapter VII. Exercise 4], if each weak* null sequence in X^* is weakly null. Concerning complemented copies of c_0 in $C(K, X)$, we have the following [1, 6].

Presentado por Manuel Valdivia
Recibido: 9 de Mayo de 2001. *Aceptado*: 6 de Junio de 2001.
Palabras clave / Keywords: Basic sequence, copy of c_0, copy of ℓ_∞, Valdivia compact set
Mathematics Subject Classifications: 46E15, 46B25

Theorem 1 (Cembranos-Freniche) *If K is infinite and X infinite-dimensional, then $C(K,X)$ contains a complemented subspace isomorphic to* c_0.

Relative to copies of ℓ_∞ one has Drewnowski's fundamental result [5].

Theorem 2 (Drewnowski) $C(K,X)$ *contains a copy of ℓ_∞ if and only if $C(K)$ contains a copy of ℓ_∞ or X contains a copy of ℓ_∞.*

When X is finite-dimensional the situation in $C(K,X)$ becomes more troublesome. For instance, if $\mathbb{N}$ denotes the set of the positive integers provided with the discrete topology and $\beta\mathbb{N}$ stands for the Stone-Čech compactification of $\mathbb{N}$, by the preceding theorems $C(\beta\mathbb{N}\times\beta\mathbb{N})\cong C(\beta\mathbb{N},\ell_\infty)$ contains a complemented copy of c_0 and a copy of ℓ_∞. There are no known both necessary and sufficient conditions for $C(K)$ to contain a complemented copy of c_0 or a copy of ℓ_∞. Nevertheless, the following results [2, 3.4] are well known.

Theorem 3 (Grothendieck) *If K is extremally disconnected, then $C(K)$ contains no complemented copies of c_0.*

Theorem 4 (Rosenthal) *If K is extremally disconnected, then $C(K)$ contains a copy of ℓ_∞.*

According to Borsuk-Dugundji's extension theorem [11], which states that if M is a closed metrizable subspace of a compact space K there is a norm-one extension linear operator T from $C(M)$ into $C(K)$, then $C(K)$ contains a complemented copy of $C(M)$. In fact, the mapping $P: C(K)\to T(C(M))$ defined by $Ph=T(h|_M)$ is a norm-one bounded linear projection operator from $C(K)$ onto $T(C(M))$. In particular, if K contains a nontrivial convergent sequence $\{t_n\}$ and $t=\lim t_n$, setting $M=\{t,t_n:n\in\mathbb{N}\}$ then M is a closed metrizable subspace of K and hence $C(M)$ has a complemented copy in $C(K)$. Since $C(M)$ is isometric to c, the Banach space of all convergent sequences endowed with the supremum-norm, it follows that $C(K)$ contains a complemented copy of c_0. This can be obtained straightforwardly without appealing to the Borsuk-Dugundji theorem as follows.

Proposition 1 *Assume K is a compact topological space. If K contains a nontrivial convergent sequence, then $C(K)$ contains a complemented copy of c_0.*

PROOF. Suppose that $\{t_n\}$ is an injective convergent sequence in K, and let t be its limit. Choose a sequence $\{U_n\}$ of pairwise disjoint open subsets of $K-\{t\}$ such that $t_n\in U_n$ for each $n\in\mathbb{N}$ and $t_m\notin U_n$ if $m\neq n$, and use Urysohn's lemma to select $f_n\in C(K)$, with $0\le f_n\le 1$, such that $f_n(t_n)=1$ and $\operatorname{supp} f_n\subseteq U_n$. Then $\{f_n\}$ is a normalized basic sequence in $C(K)$ equivalent to the unit vector basis $\{e_n\}$ of c_0 with 1 as basis constant. If δ_ω denotes the Dirac measure at $\omega\in K$, since $t_n\to t$ in K then $\delta_{t_n}\to\delta_t$ weakly* in $C(K)$, so that $\{\delta_{t_n}-\delta_t\}$ is a weak* null sequence in $C(K)^*$ such that $\langle\delta_{t_i}-\delta_t,f_j\rangle=\delta_{ij}$ for each $i,j\in\mathbb{N}$. So $[f_n]$ is a complemented copy of c_0 as a consequence of [2, Proposition 1.1.2]. Explicitly, for each $n\in\mathbb{N}$ define the linear functional $u_n: C(K)\to\mathbb{K}$ by $u_n(f)=\langle\delta_{t_n}-\delta_t,f\rangle$ and, given $f\in C(K)$, note that $|u_n(f)|\le 2\|f\|_\infty$. Consequently, the linear operator $P: C(K)\to[f_n]$ defined by $Pf=\sum_{n=1}^\infty u_n(f)\,f_n$ satisfies that $\|Pf\|_\infty\le\sup_{n\in\mathbb{N}}\|\sum_{i=1}^n u_i(f)\,f_i\|=\sup_{i\in\mathbb{N}}|u_i(f)|\le 2\|f\|_\infty$. Since $\delta_t\in[f_n]^\perp$, we have $Pf_j=\sum_{i=1}^\infty u_i(f_j)\,f_i=\sum_{i=1}^\infty\langle\delta_{t_i}-\delta_t,f_j\rangle\,f_i=f_j$ for each $j\in\mathbb{N}$. Thus P is a bounded linear projection operator from $C(K)$ onto $[f_n]$. ■

Corollary 1 *If K is a compact space containing a nontrivial convergent sequence, then $C(K)$ is not a Grothendieck space.*

PROOF. This is an obvious consequence of [1, Corollary 2]. ■

Remark 1 If K is an infinite Valdivia compact set in $[0,1]^I$ and $\{x_n\}$ is an injective sequence in $K \cap \sum(I)$, since $J := \bigcup_{n=1}^{\infty} \text{supp } x_n$ is countable there is a subsequence $\{x_{n_i}\}$ which converges to some $x \in K \cap \sum(I)$. So each infinite Valdivia compact set contains a nontrivial convergent sequence. On the other hand, Helly's space H, i.e. the set of all nondecreasing functions of the product space $[0,1]^I$, with $I = [0,1]$, is an infinite compact, sequentially compact (hence H contains a nontrivial convergent sequence), separable and nonmetrizable space [8, Chapter 5, Problem M], which is not Valdivia compact. Indeed, the set $\sum([0,1]) \cap H$ contains only functions of the form

$$f_\lambda(t) = \begin{cases} 0 & \text{for } 0 \leq t < 1 \\ \lambda & \text{for } t = 1 \end{cases}$$

with $0 \leq \lambda \leq 1$, so $\sum([0,1]) \cap H$ is not dense in H.

In an infinite locally compact topological space Ω there are always two disjoint nonempty open sets such that one of them is infinite. Indeed, choose a nondense open subset A of Ω. If $\Omega - \overline{A}$ is infinite, take A and $\Omega - \overline{A}$, otherwise take $\Omega - \overline{A}$ and $\overline{A}$. Consequently, in each infinite compact topological space K there exists a sequence $\{U_n\}$ of nonempty pairwise disjoint open sets. So, selecting $t_n \in U_n$ and $f_n \in C(K)$ for each $n \in \mathbb{N}$ as in the proof of Proposition 1, then $\{f_n\}$ is a basic sequence in $C(K)$ equivalent to the unit vector basis of c_0. If K is a Valdivia compact set and Z is a separable subspace of $C(K)$, it follows easily from [9, Lemma] that there exists a norm-one linear projection operator P on $C(K)$ such that $P(C(K))$ is separable and $Z \subseteq P(C(K))$. Hence, according to Sobczyk's theorem, this implies that each copy of c_0 in $C(K)$ is complemented. This fact prevents $C(K)$ to contain a copy of ℓ_∞ whenever K is Valdivia compact. However, a compact topological space K may contain nontrivial convergent sequences and still $C(K)$ to contain a copy of ℓ_∞. For instance, if Q denotes the weak* dual ball of ℓ_∞, then ℓ_∞ is embedded in $C(Q)$ and Q contains nontrivial weak* null sequences.

2. Complemented copies of c_0 in $C_0(\Omega)$

Let us consider a wide class of locally compact topological spaces Ω such that $C_0(\Omega)$ contains a complemented copy of c_0.

Theorem 5 (Main result) *Let Ω be a locally compact topological space. If $\Omega = \bigcup_{n=1}^{\infty} U_n$, where $\{U_n : n \in \mathbb{N}\}$ is a sequence of nonempty pairwise disjoint open sets, then $C_0(\Omega)$ contains a norm-one complemented copy of c_0.*

PROOF. For each $i \in \mathbb{N}$ choose $\omega_i \in U_i$, use local compactness to select a compact neighborhood V_i of ω_i in Ω such that $\omega_i \in V_i \subseteq U_i$ and pick a regular Borel probability measure μ in Ω such that $\mu(V_i) > 0$. Take for instance $\mu = \sum_{i=1}^{\infty} \frac{1}{2^i} \delta_{\omega_i}$ (pointwise convergence). According to Uryshon's lemma, for each $i \in \mathbb{N}$ there is $f_i \in C_c(\Omega)$ such that $0 \leq f_i \leq 1$, $f_i(\omega) = 1$ for every $\omega \in V_i$ and supp $f_i \subseteq U_i$. Then $\{f_n\}$ is a normalized basic sequence in $C_0(\Omega)$ equivalent to the unit vector basis $\{e_n\}$ of c_0, with 1 as basis constant. Actually, note that if $f = \sum_{i=1}^{\infty} a_i f_i \in [f_n]$ with $a_i \to 0$ and $\epsilon > 0$, choosing $k \in \mathbb{N}$ such that $|a_i| < \epsilon$ for each $i > k$ then $Q = \bigcup_{i=1}^{k} \text{supp } f_i$ is compact and $|f(\omega)| < \epsilon$ for each $\omega \in \Omega - Q$, hence $f \in C_0(\Omega)$. Since each $f \in C_0(\Omega)$ is μ-integrable, we may define the linear functionals $v_i : C_0(\Omega) \to \mathbb{K}$ in terms of the conditional expectation value of f relative to the event V_i by

$$v_i(f) = \mathbf{E}_{V_i}(f) = \frac{1}{\mu(V_i)} \int_{V_i} f \, d\mu$$

for each $i \in \mathbb{N}$. Given $f \in C_0(\Omega)$, for each $\epsilon > 0$ there exists a compact set $K_{f,\epsilon} \subseteq \Omega$ such that $|f(\omega)| < \epsilon$ for all $\omega \in \Omega - K_{f,\epsilon}$. Since $\{K_{f,\epsilon} \cap U_i : i \in \mathbb{N}\}$ is a covering of the compact topological subspace $K_{f,\epsilon}$ by open sets (in the relative topology of $K_{f,\epsilon}$) there must be a $j \in \mathbb{N}$ such that $K_{f,\epsilon} \cap U_i = \emptyset$ for each $i \geq j$. Therefore, since $V_i \subseteq \text{supp } f_i \subseteq U_i$, one has

$$|v_i(f)| = |\mathbf{E}_{V_i}(f)| \leq \epsilon$$

whenever $i \geq j$. This establishes that $v_i(f) \to 0$. Consequently the linear operator $P : C_0(\Omega) \to [f_i]$ given by $Pf = \sum_{i=1}^{\infty} v_i(f)\ f_i$ is well-defined. Furthermore, since each v_i is a bounded linear functional on $C_0(\Omega)$, with $|v_i(f)| \leq \|f\|_\infty$ for each $f \in C_0(\Omega)$, it follows immediately that P is a norm-one linear operator. Finally, given that

$$Pf_j = \sum_{i=1}^{\infty} v_i(f_j)\ f_i = \sum_{i=1}^{\infty} \mathbf{E}_{V_i}(f_j)\ f_i = f_j$$

due to the fact that the f_i are disjointly supported and $f_j(\omega) = 1$ for each $\omega \in V_j$, we conclude that P is a norm-one linear projection operator from $C_0(\Omega)$ onto $[f_i]$. ■

Example 1 Consider the set $\mathbb{N}$ of positive integers equipped with its discrete topology. If Σ denotes the σ-algebra of all subsets of $\mathbb{N}$, the Stone space S_Σ of Σ, that is, the collection of all nontrivial $\{0,1\}$-valued additive measures defined on Σ provided with the relative topology of the product space $\{0,1\}^\Sigma$, coincides with the Stone-Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$. If $\{A_n : n \in \mathbb{N}\}$ is a partition of $\mathbb{N}$ by nonempty sets, then $\Omega = \bigcup_{n=1}^{\infty} \widehat{A}_n$, where $\widehat{A}$ represents the clopen set $\{\mu \in S_\Sigma : \mu(A) = 1\}$, is a dense locally compact subspace of $\beta\mathbb{N}$ formed by a disjoint union of open sets. So, although $C(\beta\mathbb{N})$ contains no complemented copy of c_0, according to the previous theorem $C_0(\Omega)$ contains a complemented copy of c_0.

Corollary 2 *If Ω is a locally compact topological space such that $\Omega = \bigcup_{n=1}^{\infty} U_n$, where $\{U_n : n \in \mathbb{N}\}$ is a sequence of nonempty pairwise disjoint open sets, then $C_0(\Omega)$ is not a Grothendieck space.*

PROOF. Since, as a consequence of Theorem 5, c_0 is a quotient of $C_0(\Omega)$ via some linear bounded operator T, then T^* embeds ℓ_1 isomorphically into $C_0(\Omega)^*$ and $\{T^* e_n^*\}$, where $\{e_n^*\}$ stands for the unit vector basis of ℓ_1, is a weakly* null sequence in $C_0(\Omega)^*$ which is not weakly null. ■

3. Copies of ℓ_∞ in $C(\beta\Omega)$

Theorem 6 *Let Ω be a locally compact topological space. If $\Omega = \bigcup_{n=1}^{\infty} U_n$, where $\{U_n : n \in \mathbb{N}\}$ is a sequence of nonempty pairwise disjoint open sets, then $C(\beta\Omega)$ contains a subspace linearly isometric to ℓ_∞.*

PROOF. It suffices to prove the theorem by assuming that $C(\beta\Omega)$ is a real Banach space. So, if $\{\xi_n\}$ is a real bounded sequence, the function $f_\xi : \Omega \to \mathbb{R}$ defined by $f_\xi(\omega) = \xi_n$ whenever $\omega \in U_n$ is continuous on Ω. Since Ω is C^*-embedded in $\beta\Omega$, there is a (unique) continuous extension f_ξ^β of f_ξ on $\beta\Omega$ such that $\left\|f_\xi^\beta\right\|_\infty = \|f_\xi\|_\infty = \|\xi\|_\infty$. Consequently, the mapping $\varphi : \ell_\infty \to C(\beta\Omega)$ defined by $\varphi\xi = f_\xi^\beta$ is a linear isometry from ℓ_∞ into $C(\beta\Omega)$. ■

Acknowledgement. This paper has been partially supported by DGICYT grant PB97-0342

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