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Linearity in non-linear problems

Richard Aron, Domingo García and Manuel Maestre

Abstract. We study some situations where one encounters a problem which, at first glance, appears to have no solutions at all. But, actually, there is a large linear subspace of solutions of that problem.

Linealidad en problemas no lineales

Resumen. Estudiamos algunas situaciones donde encontramos un problema que, a primera vista, parece no tener solución. Pero, de hecho, existe un subespacio vectorial grande de soluciones del mismo.

The basic premise for this note is the following:

> *In many different settings one encounters a problem which, at first glance, appears to have no solution at all. And, in fact, it frequently happens that there is a* **large** *linear subspace of solutions to the problem.*

In the following paragraphs, we will give brief descriptions of a number of such situations, and we will also indicate a few related open problems. We should state that by **large** we will usually mean an infinite dimensional vector space. Depending on the setting of the problem, this vector space may in addition be dense, or else closed.

1. Zeros of polynomials

Although it may seem surprising that there is even a $2-$dimensional subspace on which a polynomial in many complex variables is constant, the fact is that given an arbitrary polynomial P of degree d on a complex Banach space E (of infinite or finite dimension) there is a "large" subspace of E, whose dimension depends only on d, on which P is constant:

Theorem 1 (A. Plichko & A. Zagorodnyuk [16]) *Given any infinite dimensional complex Banach space E and any polynomial $P : E \to \mathbb{C}$, there is an infinite dimensional subspace $F \subset E$ such that $P|_F \equiv P(0)$.*

In fact, M. P. Rueda and the first author have proved the following extension of this result (a version of which may be known to algebraic geometers).

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Corollary 1 [4] *There is a function $\Phi : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$, $\Phi(m, d) = n$, with the following property:*
For every complex polynomial $P : \mathbb{C}^n \to \mathbb{C}$ of degree d, there is a subspace $X \subset \mathbb{C}^n$ of dimension m such that $P|_X \equiv P(0)$.

The real analogue of this result is obviously false, as can be seen by considering $P(x) = \sum x_j^2$. Despite this, a number of positive results hold. For example, one can show:

Theorem 2 [3] *There is a function $\theta : \mathbb{N}_0 \to \mathbb{N}_0$, $\theta(m) = n$, with the following property:*
For every real polynomial $P : \mathbb{R}^n \to \mathbb{R}$ which is homogeneous of degree 3, there is a subspace $X \subset \mathbb{R}^n$ of dimension m such that $P|_X \equiv 0$.

Theorem 3 [2] *If a real infinite dimensional Banach space E does not admit a $2-$homogeneous positive definite polynomial, then every $2-$homogeneous polynomial $P : E \to \mathbb{R}$ is identically 0 on an infinite dimensional subspace of E.*

We should mention that much " finer" information remains to be uncovered concerning the size of these subspaces. For instance, it is known [4] that every complex polynomial of degree 2 in n complex variables is constant on a subspace of dimension $[\frac{n}{2}]$, and that this constant is best possible. However, similar results for polynomials of higher degree are unknown. In addition, if in the result of Plichko and Zagorodnyuk, the Banach space E is non-separable, can one choose F to be non-separable as well?

2. Hypercyclic operators

A continuous linear operator $S : E \to E$ is said to be *hypercyclic* if there is some vector $x_0 \in E$ such that the set $\{x_0, S(x_0), S^2(x_0), ...\}$ of iterates of x_0 is dense in E. The vector x_0 is called a hypercyclic vector associated to the hypercyclic operator S. One's intuition notwithstanding, it was observed by G. Birkhoff [7] over 70 years ago that the translation operator T on $H(\mathbb{C})$, given by $T(f)(z) = f(z+1) \quad (z \in \mathbb{C}, f \in H(\mathbb{C}))$, is hypercyclic. (Throughout, $H(\mathbb{C}^n)$ has the standard compact-open topology). Around 20 years later, G. R. MacLane [14] proved that the differentiation operator, $D : f \in H(\mathbb{C}) \rightsquigarrow D(f) = f' \in H(\mathbb{C})$, is also hypercyclic. G. Godefroy and J. Shapiro generalized this, with the following result involving what are called *convolution operators*:

Theorem 4 [11] *Let $S : H(\mathbb{C}^n) \to H(\mathbb{C}^n)$ be a continuous linear operator such that $S(f)(z+b) = S(\tau_b(f))(z)$ (where for $f \in H(\mathbb{C}^n)$ and $z \in \mathbb{C}^n$, $\tau_b(f)(z) :\equiv f(z+b)$). If S is not a multiple of the identity, then S is hypercyclic.*

In 1969, S. Rolewicz extended this notion to Hilbert space. In fact, it is now known that there are hypercyclic operators on every separable Banach space E [1, 6]. For example, the Rolewicz weighted backward shift operator $S : \ell_2 \to \ell_2$ given by $S(x_1, x_2, x_3, ...) = \lambda(x_2, x_3, ...)$ is hypercyclic whenever $\lambda \in \mathbb{C}, |\lambda| > 1$.

Once again, it seems counter-intuitive that there are entire subspaces of hypercyclic operators:

Theorem 5 [11] *Let $S : H(\mathbb{C}^n) \to H(\mathbb{C}^n)$ be a convolution operator which is not a multiple of the identity. Then there is an infinite dimensional subspace $X \subset H(\mathbb{C}^n)$ such that for every $f \in X, f \neq 0$, f is hypercyclic for S.*

Theorem 6 [15] *There is an infinite dimensional subspace $X \subset \ell_2$, such that every non-zero vector in X is hypercyclic for the Rolewicz operator.*

The authors are grateful to José M. Ansemil for bringing the following result of C. Blair and L. Rubel (and, therefore, the question which follows) to their attention:

Theorem 7 [9] *There is an entire function $f \in H(\mathbb{C})$ which is hypercyclic with respect to both differentiation D and translation T.*

Problem 1 *Is there an infinite dimensional subspace of such functions f? That is, is there some infinite dimensional subspace $X \subset H(\mathbb{C})$ such that if $f \in X, f \neq 0$, then f is hypercyclic with respect to both D and T?*

Another basic question in this area is to characterize when there exist *closed* subspaces consisting of hypercyclic vectors and when there exist *dense* subspaces of such vectors. For example, B. Beauzamy has constructed an operator on Hilbert space which admits a dense subspace of (non-zero) hypercyclic vectors [5]. Also, in [15], A. Montes-Rodríguez has proved that under certain conditions, there is an infinite dimensional subspace of hypercyclic vectors, and that for the Rolewicz operator, there is no closed, infinite dimensional subspace consisting of such vectors.

3. Spaces of non-extendible holomorphic functions

We will discuss two settings involving non-extendibility of holomorphic functions. First, let $U \subset \mathbb{C}^n$ be a domain of holomorphy. By the Cartan-Thullen theorem (see, e.g, [13, 2.5.5]) it follows that there exists an analytic function $f \in H(U)$ which cannot be holomorphically continued beyond any point of the boundary of U. The construction of such a function f is somewhat involved, and it is certainly false that the sum of two such functions yields another analytic function whose domain of existence is U. Despite this, one can show the following:

Theorem 8 *There are a dense subspace $X \subset H(U)$ and an infinite dimensional closed subspace $Y \subset H(U)$ such that if f is in either X or Y, $f \neq 0$, then f cannot be holomorphically continued beyond any point of the boundary of U.*

To show the existence of a dense subspace X, take an increasing sequence $(K_j) \subset U$ of holomorphically convex compact sets which exhausts U, and let $(z_j) \subset U$ be a sequence without limit point in U such that every point of the boundary of U is an accumulation point of some subsequence. For fixed j_0 and for any $j \geq j_0$, choose $f_j \in H(U)$ such that $f_j(z_j) = 1$, and $\sup_{z \in K_j} |f_j(z)| < 2^{-j}$. For sufficiently large ℓ, the product

$$h_\ell(z) :\equiv \Pi_{j=\ell}^{\infty}(1 - f_j)$$

is analytic on U, uniformly close to 1 on K_{j_0}, and vanishes on all $z_j,\ j \geq \ell$. Now, if $g \in H(U)$ is arbitrary, then $h_\ell g$ is very close to g on K_{j_0} and, moreover, $h_\ell g$ vanishes on all but a finite number of points z_j. Thus, if we define $X \subset H(U)$ to be the set of analytic functions which vanish on all but a finite number of points z_j, we see that X satisfies our requirements. Note in particular that the result applies to any domain in $\mathbb{C}$.

To show the existence of a closed subspace Y is somewhat easier, since in this case all we need do is to take $Y = \{f \in H(U) \ : \ f \text{ vanishes on all } z_j\}$.

A similar type of result holds in the context of the classical Banach algebra $H^\infty(D)$, where D is the complex unit disc. Indeed, let us suppose that $(z_n) \subset D$ is a sequence with the following properties:

(i) For every θ, there is a subsequence of (z_{2n}) converging to $e^{i\theta}$.

(ii) (z_n) is an interpolating sequence for $H^\infty(D)$.

One example of such a sequence can be found by taking $(z_j) \subset D$ such that the set of accumulation points of $\{z_{2j}\}$ is the unit circle, and which has the further property that $\sup_n \frac{1-|z_n|}{1-|z_{n-1}|} < 1$ (see, e.g., [12, p. 203]).

Theorem 9 *Under the above hypotheses, the space $Y \equiv \{f \in H^\infty(D) \ : \ f(z_{2n}) = 0 \text{ for every } n\}$ is a non-separable closed subspace of $H^\infty(D)$, every non-zero element of which is non-extendible.*

It is clear that Y is a closed subspace, so we only need to show that Y is non-separable. Indeed, let J be an arbitrary set of odd positive integers. By the Carleson-Newman theorem, there is an element $f_J \in Y$ such that $f_J(z_n) = 1$ for $n \in J$ and $f_J(z_n) = 0$ for all other n, and moreover such that $\|f_J\| \leq M$ for some absolute constant M. It follows that there is an uncountable bounded set of functions $\{f_J\}$ in $H^\infty(D)$ such that if $J \neq J'$, $\|f_J - f_{J'}\| \geq 1$. The non-separability of Y is an immediate consequence of this.

4. Spaces of nowhere differentiable functions

A result given in an introductory class in Calculus is the construction of a continuous, nowhere differentiable, function on $[0, 1]$. Whether it is given by means of the Weierstrass construction, or by use of the Baire category theorem, or by probabilistic means, the existence of such a function is always somewhat surprising and a little involved. Therefore, the following result may be quite startling.

Theorem 10 [17] *Any separable Banach space is isomorphic to a subspace of $\mathcal{C}[0, 1]$ consisting of functions which, with the sole exception of the 0 function, are nowhere differentiable.*

Fonf, Gurariy, and Kadets have made a number of refinements to this result (see, e.g., [10]). What may be of additional interest is the fact that there are no closed, infinite dimensional subspaces of $C[0, 1]$ consisting solely of differentiable functions!

5. Spaces of nowhere quasi-analytic functions

In this section, we mention a result in the same spirit as the rest of this note, by J. Schmets and M. Valdivia.

We first recall some definitions. Let (M_n) be a sequence of positive numbers, and let $U \subset \mathbb{R}$. We denote by $\mathcal{C}^{(M_n)}(U)$ the space of all $\mathcal{C}^\infty$ functions $f : U \to \mathbb{K}$ such that for some positive constants A and h,

$$\|f^{(n)}\| \leq Ah^n M_n$$

for all $n \in \mathbb{N}$. We say that $\mathcal{C}^{(M_n)}(U)$ is a *quasi-analytic class* if given $f \in \mathcal{C}^{(M_n)}(U)$ such that for some $x \in U$, $f^{(n)}(x) = 0$ for every n, then necessarily $f \equiv 0$. A function f is said to be *quasi-analytic at x_0* if there is a neighborhood U of x_0 such that $f|_U$ is in some quasi-analytic class $\mathcal{C}^{(M_n)}(U)$. Once again, functions which are *not* quasi-analytic are not so easy to come by. Despite this, J. Schmets and M. Valdivia have proved the following.

Theorem 11 [19] *Every space $\mathcal{C}^{(M_n)}(U)$ which is not quasi-analytic contains an infinite dimensional subspace Y, with the following property: If $f \in Y$, $f \not\equiv 0$, then f is nowhere quasi-analytic.*

6. The Bishop-Phelps Theorem

In this final section, we state what seems to be a very interesting open problem related to the Bishop-Phelps theorem:

Theorem 12 [8] *Let E be a Banach space. The set E'_{na}, of norm-attaining continuous linear functionals on E, is norm dense in E'.*

Problem 2 *Is there an infinite dimensional subspace $X \subset E'_{na}$? Is there such a subspace which is also dense?*

In every known situation, the answer is yes. Note that the corresponding question with *closed* instead of *dense,* has a negative answer. Indeed, it is easy to see that ${c_0}'_{na} = \cup_{n=1}^{\infty} \ell_1(n)$, where $\ell_1(n) = \{x = (x_j) \in \ell_1 \;:\; x_j = 0$ for all $j > n\}$, and this set contains no infinite dimensional closed subspace.

In conclusion, we would like to invite the reader to participate in this article, by advising the authors of other situations in which the "phenomenon" discussed here occurs, as well in which it fails to occur.

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R. Aron
Department of Mathematical Sciences
Kent State University
OH 44242 - USA.
aron@mcs.kent.edu

D. García and M. Maestre
Departamento de Análisis Matemático
Universidad de Valencia
46100 Burjasot (Valencia) Spain
domingo.garcia@uv.es
manuel.maestre@uv.es