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Specializations of Jordan superalgebras

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Abstract. We construct universal associative enveloping algebras for a large class of Jordan superalgebras.

Especializaciones de superálgebras de Jordan

Resumen. Construimos álgebras envolventes universales asociativas para varias superálgebras de Jordan.

1. Introduction

Let F be a ground field of characteristic $\neq 2$. A (linear) Jordan algebra is a vector space J with a binary bilinear operation $(x, y) \to xy$ satisfying the following identities:
(J1) $xy = yx$
(J2) $(x^2y)x = x^2(yx)$

For an element $x \in J$ let $R(x)$ denote the right multiplication $R(x) : a \to ax$ in J. If $x, y, z \in J$ then by $\{x, y, z\}$ we denote their Jordan triple product $\{x, y, z\} = (xy)z + x(yz) - y(xz)$.

A Jordan algebra J is called *special* if it is embeddable into an algebra of type $A^{(+)}$, where A is an associative algebra. The algebra $H_3(O)$ is exceptional. A homomorphism $J \to A^{(+)}$ is called a *specialization* of a Jordan algebra J. N. Jacobson [3] introduced the notion of a universal associative enveloping algebra $U = U(J)$ of a Jordan algebra J and showed that the category of specializations of J is equivalent to the category of homomorphisms of the associative algebra $U(J)$.

Let V be a Jordan bimodule over the algebra J (see [3]). We call V a one-sided bimodule if $\{J, V, J\} = (0)$. In this case, the mapping $a \to 2R_V(a) \in End_F V$ is a specialization. The category of one-sided bimodules over J is equivalent to the category of right (left) $U(J)$-modules.

N. Jacobson [3] found universal associative enveloping algebras for all simple finite dimensional Jordan algebras.

In this paper we study specializations and one-sided bimodules of Jordan superalgebras. Let us introduce the definitions.

By a superalgebra we mean a $Z/2Z$-graded algebra $A = A_{\bar{0}} + A_{\bar{1}}$. We define $|a| = 0$ if $a \in A_{\bar{0}}$ and $|a| = 1$ if $a \in A_{\bar{1}}$.

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For instance, if V is a vector space of countable dimension, and $G(V) = G(V)_{\bar{0}} + G(V)_{\bar{1}}$ is the Grassmann algebra over V, that is, the quotient of the tensor algebra over the ideal generated by the symmetric tensors, then $G(V)$ is a superalgebra. Its even part is the linear span of all products of even length and the odd part is the linear span of all products of odd length.

If A is a superalgebra, its *Grassmann enveloping algebra* is the subalgebra of $A \otimes G(V)$ given by $G(A) = A_{\bar{0}} \otimes G(V)_{\bar{0}} + A_{\bar{1}} \otimes G(V)_{\bar{1}}$.

Let $\mathcal{V}$ be a homogeneous variety of algebras, that is, a class of F-algebras satisfying a certain set of homogeneous identities and all their partial linearizations (see [20]).

Definition 1 *A superalgebra $A = A_{\bar{0}} + A_{\bar{1}}$ is called a $\mathcal{V}$ superalgebra if $G(A) \in \mathcal{V}$.*

C. T. C. Wall [19] showed that every simple finite-dimensional associative superalgebra over an algebraically closed field F is isomorphic to the superalgebra

$$M_{m,n}(F) = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, A \in M_m(F),\ D \in M_n(F) \right\} + \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, B \in M_{m\times n}(F), C \in M_{n\times m}(F) \right\}$$

or to the superalgebra

$$P(n) = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix},\ A \in M_n(F) \right\} + \left\{ \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix},\ B \in M_n(F) \right\}.$$

Jordan superalgebras were first studied by V. Kac [5] and I. Kaplansky [10,11]. In [5] V. Kac (see also I. L. Kantor [8,9]) classified simple finite dimensional Jordan superalgebras over an algebraically closed field of zero characteristic. In [16] this classification was extended to simple finite dimensional Jordan superalgebras, with semisimple even part, over characteristic $p > 2$; a few new exceptional superalgebras in characteristic 3 were added to the list. In [13] the remaining case of Jordan superalgebras with nonsemisimple even part was tackled.

Let's consider the examples that arise in these classifications.

If $A = A_{\bar{0}} + A_{\bar{1}}$ is an associative superalgebra then the superalgebra $A^{(+)}$, with the new product $a \cdot b = \frac{1}{2}(ab + (-1)^{|a||b|}ba)$ is Jordan. This leads to two superalgebras:

1) $M_{m,n}^{(+)}(F)$, $m \geq 1,\ n \geq 1$;

2) $P(n)^{(+)}$, $n \geq 2$;

If A is an associative superalgebra and $\star : A \to A$ is a superinvolution, that is, $(a^\star)^\star = a$, $(ab)^\star = (-1)^{|a||b|}b^\star a^\star$, then $H(A, \star) = H(A_{\bar{0}}, \star) + H(A_{\bar{1}}, \star)$ is a subsuperalgebra of $A^{(+)}$. The following two subalgebras of $M_{m,n}^{(+)}$ are of this type:

3) $Osp_{m,n}(F)$ if $n = 2k$ is even. The superalgebra consists of matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A^t = A \in M_m(F),\ C = J^{-1}B^t \in M_{n\times m}(F),\ D = J^{-1}D^tJ \in M_n(F)$, $J = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}$;

4) $Q(n) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix},\ D = A^t, B^t = B, C^t = -C \in M_n(F) \right\}$.

5) Let $V = V_{\bar{0}} + V_{\bar{1}}$ be a $Z/2Z$-graded vector space with a superform $<,>: V \times V \to F$ which is symmetric on $V_{\bar{0}}$, skewsymmetric in $V_{\bar{1}}$ and $< V_{\bar{0}}, V_{\bar{1}} >= (0) =< V_{\bar{1}}, V_{\bar{0}} >$.

The superalgebra $J = F1 + V = (F1 + V_{\bar{0}}) + V_{\bar{1}}$ is Jordan.

6) The 3-dimensional Kaplansky superalgebra, $K_3 = Fe + (Fx + Fy)$, with the multiplication $e^2 = e, \quad ex = \frac{1}{2}x, ey = \frac{1}{2}y, \quad [x, y] = e.$

7) The 1-parametric family of 4-dimensional superalgebras D_t is defined as $D_t = (Fe_1 + Fe_2) + (Fx + Fy)$ with the product: $e_i^2 = e_i, e_1e_2 = 0, e_ix = \frac{1}{2}x, e_iy = \frac{1}{2}y, xy = e_1 + te_2,\ i = 1, 2.$

The superalgebra D_t is simple if $t \neq 0$. In the case $t = -1$, the superalgebra D_{-1} is isomorphic to $M_{1,1}(F)$.

8) The 10-dimensional Kac superalgebra (see [5]) has been proved to be exceptional in [15]. In characteristic 3 this superalgebra is not simple, but it has a subalgebra of dimension 9 that is simple (the degenerated Kac superalgebra. There are two other examples of simple Jordan superalgebras in ch F = 3, both of them exceptional (see [16]).

9) We will consider now Jordan superalgebras defined by a bracket.

If $A = A_{\bar{0}} + A_{\bar{1}}$ is an associative commutative superalgebra wit a bracket on A, $\{,\} : A \times A \to A$, the Kantor Double of $(A, \{,\})$ is a the superalgebra $J = A + Ax$ with the $Z/2Z$ gradation $J_{\bar{0}} = A_{\bar{0}} + A_{\bar{1}}x$, $J_{\bar{1}} = A_{\bar{1}} + A_{\bar{0}}x$ and the multiplication in J given by: $a(bx) = (ab)x, \quad (bx)a = (-1)^{|a|}(ba)x$, $(ax)(bx) = (-1)^{|b|}\{a, b\}$, and the product (in J) of two elements of A is just the product of them in A.

A bracket on A is called a *Jordan bracket* if the Kantor Double $J(A, \{,\})$ is a Jordan superalgebra. Every Poisson bracket is a Jordan bracket.

10) Let Z be a unital associative commutative algebra with a derivation $d : Z \to Z$. Consider the superalgebra $CK(Z, d) = A + M$, where $A = J_{\bar{0}} = Z + \sum_{i=1}^{3} w_iZ$, $M = J_{\bar{1}} = xZ + \sum_{i=1}^{3} x_iZ$ are free Z-modules of rank 4. The multiplication on A is Z-linear and $w_iw_j = 0, i \neq j, w_1^2 = w_2^2 = 1, w_3^2 = -1.$

Denote $x_{i\times i} = 0, \quad x_{1\times 2} = -x_{2\times 1} = x_3 \qquad x_{1\times 3} = -x_{3\times 1} = x_2, \quad -x_{2\times 3} = x_{3\times 2} = x_1.$

The bimodule structure and the bracket on M are defined via the following tables:

	g	w_jg
xf	$x(fg)$	$x_j(fg^d)$
x_if	$x_i(fg)$	$x_{i\times j}(fg)$

	xg	x_jg
xf	$f^dg - fg^d$	$-w_j(fg)$
x_if	$w_i(fg)$	0

The superalgebra $CK(Z, d)$ is simple if and only if Z does not contain proper d-invariant ideals.

In [5], [8] it was shown that simple finite dimensional Jordan superalgebras over an algebraically closed field F of zero characteristic are those of examples 1) - 8) and the Kantor Double (example 9) of the Grassmann algebra with the bracket $\{f, g\} = \sum(-1)^{|f|}\dfrac{\partial f}{\partial \xi_i}\dfrac{\partial g}{\partial \xi_i}$.

The examples 9), 10) are related to infinite dimensional *superconformal* Lie superalgebras (see [6], [7]). In particular, the superalgebras $CK(Z, D)$ correspond to an important superconformal algebra discovered in [1] and [2].

In [13] it was shown that the only simple finite dimensional Jordan superalgebras over an algebraically closed field of characteristic $p > 2$ with nonsemisimple even part are superalgebras 9),10) built on truncated polynomials.

2. Universal enveloping algebras

In what follows the ground field F is assumed to be algebraically closed.

1. Let U be a universal associative enveloping algebra of a special Jordan superalgebra J, $u : J \to U$ a universal specialization. The algebra U is equipped with a natural superinvolution $\star$ leaving all elements from $u(J)$ fixed. Then $u(J) \subseteq H(U, \star)$. We call a superalgebra J *reflexive* if $u(J) = H(U, \star)$.

Theorem 1 *All superalgebras of examples 1) - 4) are reflexive except the following ones:* $M_{1,1}^{(+)}(F)$, $Osp(1,2) \simeq D(-2)$, $Q(2)$. *Hence,*

$$\begin{cases} U(M_{m,n}^{(+)}(F)) \simeq M_{m,n}(F) \oplus M_{m,n}(F) & \textit{for } (m,n) \neq (1,1); \\ U(P^{(+)}(n)) = P(n) \oplus P(n), \quad n \geq 2; \\ U(osp(m,n)) \simeq M_{m,n}(F), \quad (m,n) \neq (1,2); \\ U(Q(n)) \simeq M_{n,n}(F), \quad n \geq 3. \end{cases}$$

2. Let Z be an associative commutative algebra with a derivation $D : Z \to Z$. Let $W =< Z, D >$ and let $u : CK(Z,D) \to \quad M_{2,2}(W)$ be the embedding found in [12]

The embedding u extends the embedding of Kantor doubles of brackets of vector type found in [14]

Theorem 2 $U(CK(Z,D)) = M_{2,2}(W)$, *the embedding* u *is universal.*

3. The superalgebra of $CK(Z,D)$ spanned over F by the elements $1, w_1, w_2, w_3, x, x_1, x_2, x_3$ is isomorphic to $Q(2)$.

Theorem 3 *The restriction of the embedding* u *(see above) to* $Q(2)$ *is a universal specialization;*

$$U(Q(2)) \simeq M_{2,2}(F[t]),$$

where $F[t]$ *is a polynomial algebra in one variable.*

4. Let us describe the universal associative enveloping superalgebra of $M_{1,1}(F)$. Consider the ring of polynomials and the field of rational functions in two variables, $F[z_1, z_2] \subseteq F(z_1, z_2)$. Let K be the quadratic extension of $F(z_1, z_2)$ generated by a root of the equation $a^2 + a - z_1 z_2 = 0$. Consider the subring $A = F[z_1, z_2] + F[z_1, z_2]a$ and the subspaces $M_{12} = F[z_1, z_2] + F[z_1, z_2]a^{-1}z_2$, $M_{21} = F[z_1, z_2]z_1 + F[z_1, z_2]a$ of K. Then $U = \begin{pmatrix} A & M_{12} \\ M_{21} & A \end{pmatrix}$ is a subring of $M_2(K)$.

Theorem 4 $U(M_{1,1}(F)) \simeq \begin{pmatrix} A & M_{12} \\ M_{21} & A \end{pmatrix}$. *The mapping*

$$u : \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \to \begin{pmatrix} \alpha_{11} & \alpha_{12} + \alpha_{21} a^{-1} z_2 \\ \alpha_{12} z_1 + \alpha_{21} a & \alpha_{22} \end{pmatrix}$$

is a universal specialization.

5. Let $V = V_{\bar{0}} + V_{\bar{1}}$ be a $Z/2Z$-graded vector space, dim $V_{\bar{0}} = m$, $dim V_{\bar{1}} = 2m$; let $<,>: V \times V \to F$ be a supersymmetric bilinear form on V. The universal associative enveloping algebra of the Jordan algebra $F1 + V_{\bar{0}}$ is the Clifford algebra $Cl(m) = \langle 1, e_1, \ldots, e_m | e_i e_j + e_j e_i = 0, i \neq j, e_i^2 = 1 \rangle$ (see [3]). Assuming the generators $e_1, \ldots, e_m$ to be odd, we get a $Z/2Z$-gradation on $CL(m)$.

In $V_{\bar{1}}$ we can find a basis $v_1, w_1, \ldots, v_n, w_n$ such that $< v_i, w_j >= \delta_{ij}$, $< v_i, v_j >=< w_i, w_j >= 0$. Consider the Weyl algebra $W_n =< 1, x_i, y_i, 1 \leq i \leq n | [x_i, y_j] = \delta_{ij}, [x_i, x_j] = [y_i, y_j] = 0 >$. Assuming

$x_i, y_i, 1 \leq i \leq n$ to be odd, we make W_n a superalgebra. The universal associative enveloping algebra of $F1 + V$ is isomorphic to the (super)tensor product $Cl(m) \otimes_F W_n$.

6. Let $osp(1,2)$ denote the Lie subsuperalgebra of $M_{1,2}(F)$ which consists of skewsymmetric elements with respect to the orthosympletic superinvolution. Let x, y be the standard basis of the odd part of $osp(1,2)$.

Theorem 5 (I. Shestakov) *The universal enveloping algebra of K_3 is isomorphic to $U(osp(1,2)/id([x,y]^2 - [x,y]))$, where $U(osp(1,2))$ is the universal associative enveloping algebra of $osp(1,2)$ and $id([x,y]^2 - [x,y]))$ is the ideal of $U(osp(1,2))$ generated by $[x,y]^2 - [x,y]$.*

Clearly, if $\mathrm{ch}F = 0$ then K_3 does not have nonzero specializations that are finite dimensional algebras. If $\mathrm{ch}F = p > 0$ then K_3 has such specializations.

7. Let us consider the superalgebras $D(t)$. We will assume that $t \neq -1, 0, 1$, because $D(-1) \simeq M_{1,1}(F)$; $D(0) \simeq K_3 + F$; $D(1)$ is a Jordan superalgebra of a superform.

Theorem 6 (I. Shestakov) *The universal enveloping algebra of $D(t)$ is isomorphic to*

$$U(osp(1,2)/id([x,y]^2 - (1+t)[x,y] + t).$$

Corollary 1 *If $chF = 0$ then all finite dimensional one-sided bimodules over $D(t)$ are completely reducible.*

Indeed, it is known (see [4]) that finite dimensional representations of the Lie superalgebra $osp(1,2)$ are completely reducible.

Now we will assume thar $\mathrm{ch}F = 0$ and will classify irreducible finite dimensional one-sided bimodules over $D(t)$. Let us first consider four infinite dimensional Verma type right modules over $U(D(t)$. Each of these bimodules is generated by an even highest weight element v.

$V_1(t) = vU(d(t))$. Defining relations: $v(xy + yx) = (2t+1)v$, $vy^2 = 0$, $ve_1 = v$, $ve_2 = 0$. Basis: $v,\ vy,\ vx^i,\ i \geq 1$.

$V_2(t) = vU(d(t))$. Defining relations: $v(xy + yx) = (2t+1)v$, $vy = 0$, $ve_1 = v$, $ve_2 = 0$. Basis: $v,\ vx^i,\ i \geq 1$.

Changing parity we get two new bimodules $V_1(t)^{op}$ and $V_2(t)^{op}$.

Each of these bimodules has the unique irreducible homomorphism image $W_1(t)$ or $W_2(t)$ or $W_1(t)^{op}$ or $W_2(t)^{op}$ respectively.

Theorem 7 *If $t = \frac{-(m+1)}{m}$, $m \geq 1$, then $D(t)$ has two irreducible finite dimensional one sided bimodules $W_1(t)$ and $W_1(t)^{op}$.*

If $t = \frac{-m}{m+1}$, $m \geq 1$, then $D(t)$ has two irreducible finite dimensional one sided bimodules $W_2(t)$ and $W_2(t)^{op}$.

If t can not be represented as $-\frac{m+1}{m}$ or $\frac{-m}{m+1}$, where m is a positive integer, then $D(t)$ does not have nonzero finite dimensional specializations.

If $\mathrm{ch}F = p > 2$ then for an arbitrary t the superalgebra $D(t)$ can be embedded into a finite dimensional associative superalgebra. It suffices to notice that $D(t) \subseteq CK(F[t|t^p = 0], d/dt)$.

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