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A NOTE ON THE APPLICATION OF INTEGRALS INVOLVING CYCLIC PRODUCTS OF KERNELS

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In statistics of stochastic processes and random fields, a moment function or a cumulant of an estimate of either the correlation function or the spectral function can often contain an integral involving a cyclic product of kernels. We define and study this class of integrals and prove a Young-Hölder inequality. This inequality further enables us to study asymptotics of the above mentioned integrals in the situation where the kernels depend on a parameter. An application to the problem of estimation of the response function in a Volterra system is given.

Keywords: Integral involving a cyclic product of kernels, cumulant, Young-Hölder inequality, cross-correlogram, asymptotic normality

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1. INTRODUCTION

The problem of identification in stochastic linear systems has been a matter of active research for the last four decades. One of the simplest models considers a «black box» which enjoys some input and gives a certain output. The input may be single or multiple, and one can choose between the assumption that it is perfect or noisy. The same applies to the output irrespective of the input. This variety of possibilities generates a great amount of models to be considered, and one still has a choice between a parametric or nonparametric framework. The scope of applications of these models is fairly extensive, ranging from signal processing and automatic control to econometrics (errors-in-variables models). For more details, see Schetzen (1980), Hannan and Deistler (1988), Söderström and Stoica (1989). Each of these books may be a good starting point for further bibliographic references. When only second-order statistics are involved, the question arises about the uniqueness of the solution of a parametric identification problem, see Green and Anderson (1986), Deistler and Anderson (1989). The use of higher order cumulant statistics leads to consistent estimates of the parameters, see Tugnait (1992), and also proves to be useful in nonparametric settings, see Akaike (1966).

Our interest in the topic was stimulated by the problem of estimation of the so-called impulse response function (also called the transfer function in the time domain) of an SISO (single-input single-output) system. One of the usual tools used for this estimation is the discrete-time cross-correlogram between the input and the output, and one of the basic methods applied to prove statistical properties of the estimate is Brillinger's method of cumulants, see Brillinger (1981). Our impression is that the application of this method would always lead us to a certain class of integrals. These integrals also appear in other nonparametric statistical problems, and the peculiarity of this kind of integrals is that they involve cyclic products of kernels, meaning that the internal structure of integrals is always the same. The ability to handle these integrals in general can give a clue to obtaining good upper bounds. If one switches to a spatial setting, the Rosenblatt approximation by quadratic forms is often applied, see Rosenblatt (1985). It is interesting that integrals involving cyclic products of kernels also appear in the cumulants of bilinear forms of Gaussian random vectors, challenging us even more.

Since the weak convergence of estimators is often proved using high-order cumulants (see, e. g., Zhang and Shaman (1991), Grimmett (1992), Brillinger (1996), Haberzettl (1997)), making a closer look at integrals involving cyclic products of kernels worthwhile. Earlier studies of integral representations of cumulants of second-order statistics of stationary stochastic processes were carried out by Lithuanian statisticians R. Bentkus (1972, 1976) and Statulevičius (1977). A later paper by R. Bentkus (1977) deals with cumulants of polynomial statistics. Other integral representations of cumulants of the periodogram of a homogeneous random field were considered by Guyon (1995), Benn and Kulperger (1998), Rosenblatt (2000). Interesting classes of multiple stochastic integrals were introduced by Surgailis (1981), Engel (1982), Fox and Taqqu (1987).

Let us introduce the object we would like to focus on in what follows. Denote by $\mathbb{N}$ the set of positive integers, $\mathbb{Z}$ the set of all integers, $\mathbb{R}$ the set of real numbers, and let $\mathbb{N}_m := \{1,\ldots,m\}$ for $m \in \mathbb{N}$. Let $(\mathbb{T},\mu)$ be a measurable space endowed with a σ-finite measure μ. We define an integral involving a cyclic product of kernels as the following Lebesgue integral:

$$(1)\quad \overset{\frown}{I}_n\,(K_1,\ldots,K_n;\ \varphi_1,\ldots,\varphi_n) := \int\cdots\int_{\mathbb{T}^n}\left[\prod_{p=1}^{n} K_p(t_p,t_{p+1})\varphi_p(t_p)\right]\mu(dt_1)\ldots\mu(dt_n)$$

where $t_{n+1} = t_1$. The functions $K_p := (K_p(s,t),\ s,t \in \mathbb{T}),\ p \in \mathbb{N}_n$, are called kernels. In general, both the kernels $K_p,\ p \in \mathbb{N}_n$, and the functions $\varphi_p := (\varphi_p(t),\ t \in \mathbb{T})$ are complex-valued.

The remaining part of this paper is organized as follows: Section 2 gives an overview of the situations where integrals (1) appear. Section 3 states some new results on these integrals: a Young–Hölder inequality (Theorem 1) and the convergence to zero of an integral depending on a parameter (Theorem 2). These results are applied to prove asymptotic normality of a nonparametric estimate of the impulse response function in a Volterra system (Theorem 3).

2. SOME MOTIVATING EXAMPLES

Integral representation of a cumulant of a set of bilinear forms of Gaussian random vectors

Assume that $m \in \mathbb{N}$, $n_j \in \mathbb{N}$, $j \in \mathbb{N}_m$, and let $\vec{X}_{j,1} := (X_{j,1}(k),\ k \in \mathbb{N}_{n_j})$, $\vec{X}_{j,2} := (X_{j,2}(k),\ k \in \mathbb{N}_{n_j})$, $j \in \mathbb{N}_m$, be real-valued zero-mean jointly Gaussian random vectors. Consider the following set of bilinear forms: $S_j := \sum_{k=1}^{n_j} a_j(k)X_{j,1}(k)X_{j,2}(k)$, $j \in \mathbb{N}_m$, where $a_j(k) \in \mathbb{R}$ are nonrandom numbers for all $k \in \mathbb{N}_{n_j}$, $j \in \mathbb{N}_m$. Consider also the joint simple cumulant (see the definition, for example, in Brillinger (1981), Section 2.3), denoted by cum $(S_1,\ldots,S_m)$, of the random variables $S_1,\ldots,S_m$. By Theorem 2.3.2 in Brillinger (1981) and by some algebra, one can show that

$$(2)\qquad \operatorname{cum}(S_1,\ldots,S_m) = \sum_{(\vec{\jmath},\vec{\alpha})\in\{P,2\}_{m-1}}\ \sum_{k_1,\ldots,k_m=1}^{n_1,\ldots,n_m}\left[\prod_{p=1}^{m} a_{j_p}(k_{j_p})C_{j_p,j_{p+1}}^{\overline{\alpha}_p,\alpha_{p+1}}(k_{j_p},k_{j_{p+1}})\right]$$

where $\vec{\jmath} := (j_1, j_2,\ldots, j_m)$, $\vec{\alpha} := (\alpha_1,\alpha_2,\ldots,\alpha_m)$, with the convention that $j_1 = j_{m+1} = 1$, $\alpha_1 = \alpha_{m+1} = 2$, and where

$$C^{\overline{\alpha}_p,\alpha_{p+1}}_{j_p,j_{p+1}}(k_{j_p},k_{j_{p+1}}) := \mathsf{E}X_{j_p,\overline{\alpha}_p}(k_{j_p})X_{j_{p+1},\alpha_{p+1}}(k_{j_{p+1}}), \qquad \overline{\alpha}_k := \begin{cases} 2, & \text{if } \alpha_k = 1, \\ 1, & \text{if } \alpha_k = 2. \end{cases}$$

The notation $(\vec{\jmath},\vec{\alpha}) \in \{P,2\}_{m-1}$ applied to the vectors $\vec{\jmath} = (j_1, j_2, \ldots, j_m)$ and $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ is interpreted as follows:

$$j_1 = 1, \quad \alpha_1 = 2, \quad \text{and} \quad ((j_2,\ldots,j_m),(\alpha_2,\ldots,\alpha_m)) \in \mathrm{Perm}\{2,\ldots,m\} \times \{1,2\}^{m-1}$$

where $\mathrm{Perm}\{2,\ldots,m\}$ denotes the set of all permutations of $\{2,\ldots,m\}$. If $\mathrm{card}(A)$ is cardinality of the set A, then it is clear that $\mathrm{card}\left(\mathrm{Perm}\{2,\ldots,m\} \times \{1,2\}^{m-1}\right) = 2^{m-1} \cdot \cdot (m-1)!$

Let $j, j' \in \mathbb{N}_m$ and $\alpha, \alpha' \in \{1,2\}$. Assume further that there exists a Lebesgue integrable complex-valued function $G^{\alpha,\alpha'}_{j,j'}(u,v)$, $(u,v) \in \mathbb{R}^2$, such that

$$\mathsf{E}X_{j,\alpha}(k)X_{j',\alpha'}(k') = \int\int_{\mathbb{R}^2} \exp\{i(uk+u'k')\}G^{\alpha,\alpha'}_{j,j'}(u,u')du du'.$$

In this case (2) implies that

$$\text{(3)} \qquad \mathrm{cum}\,(S_1,\ldots,S_m) = \sum_{(\vec{\jmath},\vec{\alpha})\in\{P,2\}_{m-1}} \int\cdots\int_{\mathbb{R}^m} \left[\prod_{p=1}^{m} Q^{(\vec{\jmath},\vec{\alpha})}_p(v_p,v_{p+1})\right] dv_1 \ldots dv_m$$

where $v_{m+1} = v_1$ and

$$Q^{(\vec{\jmath},\vec{\alpha})}_p(s,u) := \int_{\mathbb{R}} G^{\overline{\alpha}_p,\alpha_{p+1}}_{j_p,j_{p+1}}(s,t)A_{j_{p+1}}(t,u)dt, \quad p \in \mathbb{N}_m;$$

$$A_j(t,u) := \sum_{k=1}^{n_j} a_j(k)\exp\{i(t+u)k\}, \quad j \in \mathbb{N}_m.$$

Each integral on the right-hand side of (3) involves a cyclic product of kernels and the cumulant $\mathrm{cum}\,(S_1,\ldots,S_m)$ is the sum of these integrals.

Let $j, j' \in \mathbb{N}_m$ and $\alpha, \alpha' \in \{1,2\}$. Assume that there exists a complex-valued measure $M^{\alpha,\alpha'}_{j,j'}$ on $([-\pi,\pi],\ \mathcal{B}([-\pi,\pi]))$ such that $\mathsf{E}X_{j,\alpha}(k)X_{j',\alpha'}(k') = \int_{-\pi}^{\pi} e^{i(k-k')\lambda}M^{\alpha,\alpha'}_{j,j'}(d\lambda)$. Here and in what follows, $\mathcal{B}(A)$ denotes the Borel σ-algebra on A. Then

$$\text{(4)} \qquad \mathrm{cum}\,(S_1,\ldots,S_m) = \sum_{(\vec{\jmath},\vec{\alpha})\in\{P,2\}_{m-1}} \int\cdots\int_{[-\pi,\pi]^m} \left[\prod_{p=1}^{m} A^{(\vec{\jmath},\vec{\alpha})}_p(\lambda_p,\lambda_{p+1})\right] \cdot$$

$$\cdot \mu^{(\vec{\jmath},\vec{\alpha})}_1(d\lambda_1)\ldots\mu^{(\vec{\jmath},\vec{\alpha})}_m(d\lambda_m)$$

where $\lambda_{m+1} = \lambda_1$ and $A_p^{(\vec{\jmath},\vec{\alpha})}(u,v) := \sum_{k=1}^{n_{j_p}} a_{j_p}(k)\exp\{-ik(u-v)\},\ p \in \mathbb{N}_m;\ \mu_p^{(\vec{\jmath},\vec{\alpha})} := M_{j_p,j_{p+1}}^{\overline{\alpha}_p,\alpha_{p+1}},\ p \in \mathbb{N}_m$. Suppose now that for any $(\vec{\jmath},\vec{\alpha})$ and any p we have $\mu_p^{(\vec{\jmath},\vec{\alpha})}(B) = \int_B f_p^{(\vec{\jmath},\vec{\alpha})}(\lambda)d\lambda$, $B \in \mathcal{B}([-\pi,\pi])$. Then

$$(5)\quad \mathrm{cum}\,(S_1,\ldots,S_m) = \sum_{(\vec{\jmath},\vec{\alpha})\in\{P,2\}_{m-1}} \int\cdots\int_{[-\pi,\pi]^m} \left[\prod_{p=1}^m A_p^{(\vec{\jmath},\vec{\alpha})}(\lambda_p,\lambda_{p+1}) f_p^{(\vec{\jmath},\vec{\alpha})}(\lambda_p)\right]\cdot \cdot d\lambda_1\ldots d\lambda_m$$

where $\lambda_{m+1} = \lambda_1$. Formulas (4) and (5) give further representations of the cumulant $\mathrm{cum}\,(S_1,\ldots,S_m)$ as a finite sum of integrals involving cyclic products of kernels.

The book by Mathai and Provost (1992) is a good reference on quadratic forms, and the papers of Mathai (1992) and Holmquist (1996) focus on quadratic and bilinear forms in normal variables.

Inequalities for a cumulant of the cross-correlogram in a Gaussian time series

Let $Y(t), t \in \mathbb{Z}$, be a real-valued zero-mean stationary Gaussian process with covariance function $C(t) := \mathsf{E}Y(t)Y(0)$, $t \in \mathbb{Z}$, and spectral function $F(\lambda)$, $\lambda \in [-\pi,\pi]$. Consider the sample correlogram $\hat{C}(\tau;N) := \sum_{k=1}^N a(k;\ \tau,N)Y(\tau+k)Y(k)$, $\tau \in \mathbb{Z}$, $N \in \mathbb{N}$ where $a(k;\ \tau,N)$ are nonrandom real-valued weights for all $k \in \mathbb{N}_N$, $\tau \in \mathbb{Z}$, $N \in \mathbb{N}$. Formula (4) implies the following inequality:

$$(6)\ \left|\mathrm{cum}\,(\hat{C}(\tau_1;N),\ldots,\hat{C}(\tau_m;N))\right|$$

$$\leq 2^{m-1}(m-1)! \int\cdots\int_{[-\pi,\pi]^m} \left[\prod_{p=1}^m \left|A^{(N)}(\lambda_p-\lambda_{p+1};\ \tau_{p+1})\right|\right] dF(\lambda_1)\ldots dF(\lambda_m)$$

where $\lambda_{m+1} = \lambda_1$ and $A^{(N)}(u;\tau) := \sum_{k=1}^N a(k;\ \tau,N)\exp\{-iku\}$, $\tau \in \mathbb{Z}$, $u \in [-\pi,\pi]$. If, for example, $a(k;\ \tau,N) \equiv 1/N$, then (6) implies that

$$\left|\mathrm{cum}\,(\hat{C}(\tau_1;N),\ldots,\hat{C}(\tau_m;N))\right|$$

$$\leq N^{-m}2^{m-1}(m-1)! \int\cdots\int_{[-\pi,\pi]^m} \left[\prod_{p=1}^m \left|\frac{\sin\left(\frac{N(\lambda_p-\lambda_{p+1})}{2}\right)}{\sin\left(\frac{\lambda_p-\lambda_{p+1}}{2}\right)}\right|\right] dF(\lambda_1)\ldots dF(\lambda_m).$$

A representation carried out by R. Bentkus of a cumulant of a second-order spectral estimate in a stationary time series

Let $Y(t)$, $t \in \mathbb{Z}$, be a weak sense stationary zero-mean real-valued time series. Consider the second-order spectral estimate:

$$\hat{E}^{(N)}(g_j) = \int_{-\pi}^{\pi} g_j(x) I^{(N)}(x)dx, \quad j \in \mathbb{N}_m, \quad m \in \mathbb{N},$$

where $g_j \in L_1[-\pi,\pi]$, $j \in \mathbb{N}_m$, and where

$$I^{(N)}(x) := (2\pi N)^{-1} \left| \sum_{s=1}^{N} Y(s) e^{-isx} \right|^2, \quad x \in [-\pi,\pi], \quad N \in \mathbb{N},$$

is the so-called periodogram (Brillinger (1981), Section 5.2). Assume that the process $Y(\cdot)$ has n-th order spectral density $\varphi_n\left(\lambda_1,\ldots,\lambda_{n-1},-\sum_{j=1}^{n-1}\lambda_j\right)$, $(\lambda_1,\ldots,\lambda_{n-1}) \in [-\pi,\pi]^{n-1}$, for any $n = 2,\ldots,2m$. Then a result by R. Bentkus (1977) after some algebra implies that

$$\begin{aligned}
&\operatorname{cum}(\hat{E}^{(N)}(g_1),\ldots,\hat{E}^{(N)}(g_m)) \\
&\quad = \frac{2}{(2\pi)^m N} \int\cdots\int_{[-\pi,\pi]^{2m}} \prod_{p=1}^{2m} \left[\frac{\sin(N(v_p - v_{p+1})/2)}{\sin((v_p - v_{p+1})/2)}\right] G(v_1,\ldots,v_{2m}) dv_1 \ldots dv_{2m}
\end{aligned}$$

where $v_{2m+1} = v_{2m}$, and where the function $G(\cdot)$ is expressed in terms of $g_1(\cdot),\ldots,$ $g_m(\cdot)$ and $\varphi_1(\cdot),\ldots,\varphi_{2m}(\cdot)$. This formula shows that $\operatorname{cum}(\hat{E}^{(N)}(g_1),\ldots,\hat{E}^{(N)}(g_m))$ is reduced to an integral involving a cyclic product of kernels.

Long-range dependence. The Rosenblatt distribution

Let $Y(t)$, $t \in \mathbb{Z}$, be a stationary Gaussian real-valued zero-mean stochastic process with covariance function $C(t), t \in \mathbb{Z}$. Assume that $C(0) = 1$ and $\lim_{|t|\to\infty} \alpha^{-1}|t|^{-\beta}C(t) = 1$, where $\alpha > 0$ and $\beta > 0$. If $\beta \in (0,1/2)$, then $Y(\cdot)$ is called a process with long-range dependence. Consider the sample correlogram $\hat{C}(\tau;N) = N^{-1}\sum_{k=1}^{N} Y(\tau+k)Y(k)$, $\tau \in \mathbb{Z}$, $N \in \mathbb{N}$. For a process with index $\alpha = 1$, the limit distribution as $N \to \infty$ of $N^{\beta}(\hat{C}(0;N) - 1)$ is non-normal, see Rosenblatt (1985), p. 72. This distribution is often called the Rosenblatt distribution. The characteristic function $\varphi(u), u \in \mathbb{R}$, of the Rosenblatt distribution has the following form: $\varphi(u) = \exp\left\{(1/2)\sum_{k=2}^{N}(2iu)^k \mathbf{I}_\beta(k)/k\right\}$, $u \in \mathbb{R}$, where

$$\mathbf{I}_\beta(k) = \int\cdots\int_{[0,1]^k} \left[\prod_{j=1}^{k} |x_j - x_{j+1}|^{-\beta}\right] dx_1 \ldots dx_n$$

for any $k \geq 2$ and where $x_{k+1} = x_1$. It is clear that each of the integrals $\mathbf{I}_\beta(k)$, $k \geq 2$, involves a cyclic product of kernels.

3. SOME RESULTS ON INTEGRALS INVOLVING CYCLIC PRODUCTS OF KERNELS

The Young–Hölder inequality for an integral involving a cyclic product of kernels

For $K = (K(s,t),\ s,t \in \mathbb{T})$ we put $\|K\|_{\infty,p} := \mu$–ess $\sup_{s\in\mathbb{T}} \|K(s,\cdot)\|_p$, where $\|\cdot\|_p$ is the standard norm in $L_p(\mathbb{T}),\ p \in [1,\infty]$. We also introduce another norm: $|||K|||_p := \max\{\|K\|_{\infty,p}, \|K'\|_{\infty,p}\}$ where K' is the dual function of K, that is $K'(s,t) := K(t,s)$ for any $(s,t) \in \mathbb{T}^2$. Put $L_p^\infty(\mathbb{T}^2) := \{K :\ |||K|||_p < \infty\}$. An inequality having the form

$$|\ \overset{\frown}{I}\ (K_1,\dots,K_n;\ \varphi_1,\dots,\varphi_n)| \le \prod_{j=1}^{n}(|||K_j|||_{p_j} \cdot \|\varphi_j\|_{q_j}) \tag{7}$$

with $1 \le p_j, q_j \le \infty$, $j \in \mathbb{N}_n$, will be called a Young inequality for integral (1). If each q_j is the real conjugate exponent of p_j, $j \in \mathbb{N}_n$, that is $(1/p_j) + (1/q_j) = 1$ for all $j \in \mathbb{N}_n$, then (7) is called a Young–Hölder inequality for integral (1).

Theorem 1. *Let $n \in \mathbb{N}$, $n \ge 2$. Assume that $1 \le q_1,\dots,q_n \le \infty$ and*

$$\max_{j_1\ne j_2}((1/q_{j_1}) + (1/q_{j_2})) \ge 1. \tag{8}$$

If $K_j \in L_{p_j}^\infty(\mathbb{T}^2)$, $\varphi_j \in L_{q_j}(\mathbb{T})$ for each $j \in \mathbb{N}_n$, where $p_j^{-1} + q_j^{-1} = 1$, $j \in \mathbb{N}_n$, then

$$\left|\overset{\frown}{I}_n\ (K_1,\dots,K_n;\ \varphi_1,\dots,\varphi_n)\right| \le \prod_{j=1}^{n} |||K_j|||_{p_j} \|\varphi_j\|_{q_j}. \tag{9}$$

Sketch of the proof

Let

$$j_1 \in \operatorname{Arg\,max}\left(\frac{1}{q_j},\ j \in \mathbb{N}_n\right), \qquad j_2 \in \operatorname{Arg\,max}\left(\frac{1}{q_j},\ j \ne j_1\right). \tag{10}$$

Then we obtain by (8)

$$\frac{1}{q_{j_1}} + \frac{1}{q_{j_2}} \ge 1. \tag{11}$$

Put

$$p_j = q_j', \quad j \in \mathbb{N}_n, \tag{12}$$

and consider the following collections:

$$(\vec{p}_{j_1,j_2-1},\vec{q}_{j_1+1,j_2-1}) = (q'_{j_1}q_{j_1+1}q'_{j_1+1}\dots q_{j_2-1}q'_{j_2-1}),$$
$$(\vec{p}_{j_2,j_1-1},\vec{q}_{j_2+1,j_1-1}) = (q'_{j_2}q_{j_2+1}q'_{j_2+1}\dots q_{j_2-1}q'_{j_1-1}).$$

By (10) and (12), we have

$$\frac{1}{p_{j_1}} = \frac{1}{q'_{j_1}} = 1 - \frac{1}{q_{j_1}} = \min_{j\in\mathbb{N}_n}\frac{1}{p_j}$$

and

$$\frac{1}{p_{j_2}} = \frac{1}{q'_{j_2}} = 1 - \frac{1}{q_{j_2}} = \min_{j\in[j_2,j_1-1]}\frac{1}{p_j}\,,$$

that is

$$p_{j_1} = \max_{j\in[j_1,j_2-1]} p_j, \qquad p_{j_2} = \max_{j\in[j_2,j_1-1]} p_j. \tag{13}$$

Furthermore

$$\frac{1}{q_j} + \frac{1}{p_j} = \frac{1}{q_j} + \frac{1}{q'_j} = 1, \qquad j\in\mathbb{N}_n\,, \tag{14}$$

and moreover

$$\left(\textstyle\sum_{j\in[j_1,j_2-1]}\dfrac{1}{p_j}\right) + \left(\textstyle\sum_{j\in[j_1+1,j_2-1]}\dfrac{1}{q_j}\right) \;=\; \frac{1}{q'_{j_1}} + \textstyle\sum_{j\in[j_1+1,j_2-1]}\left(\dfrac{1}{q_j}+\dfrac{1}{q'_j}\right) =$$
$$= \frac{1}{q'_{j_1}} + (|j_2-j_1|-1).$$

Therefore

$$\frac{1}{a_{j_1,j_2-1}} = \frac{1}{q'_{j_1}} \in [0,1]. \tag{15}$$

In a similar manner, we can prove that

$$\frac{1}{a_{j_2,j_1-1}} = \frac{1}{q'_{j_2}} \in [0,1]. \tag{16}$$

We also have

$$\left(\sum_{j=1}^{n}\frac{1}{p_j}\right) + \left(\sum_{j=1}^{n}\frac{1}{q_j}\right) = \sum_{j=1}^{n}\left(\frac{1}{q'_j}+\frac{1}{q_j}\right) = n. \tag{17}$$

Now Theorem 1 can be obtained from Theorem 9.5.1 in Edwards (1965), and formulas (11), (13)–(17) using the techniques employed in Section 5 of Buldygin *et al*. (2000). □

Corollary 1. *Let* $n \geq 2$. *Assume that* $1 \leq q_1, \ldots, q_n \leq \infty$ *and* (8) *holds. Then there exist numbers* $p_j \in (1, \infty]$, $j \in \mathbb{N}_n$, *satisfying* $\sum_{j=1}^{n}(p_j^{-1} + q_j^{-1}) = n$ *and such that* (9) *holds whenever* $K_j \in L_{p_j}(\mathbb{T}^2)$ *for each* $j \in \mathbb{N}_n$.

Convergence to zero of an integral involving a cyclic product of kernels, with the kernels depending on a parameter

Assume that the kernels $K_1, \ldots, K_n$ appearing in the integrals $\overset{\frown}{I}_n$ depend on a parameter σ, that is $K_j = K_j^{(\sigma)}$ for all $j \in \mathbb{N}_n$, $\sigma \in \mathfrak{S}$. We then write $\overset{\frown}{I}{}_n^{(\sigma)} := \overset{\frown}{I}\,(K_1^{(\sigma)}, \ldots, K_n^{(\sigma)};\, \varphi_1, \ldots, \varphi_n)$. Consider the following majorant condition: *for any* $p \in (1, \infty]$, *there exists a constant* $C_p \geq 0$ *not depending on* $\sigma \in \mathfrak{S}$ *such that* $\max_j |||K_j^{(\sigma)}|||_p \leq C_p[\rho(\sigma)]^{1/2-1/p}$ *where* $\rho(\sigma) > 0$, $\sigma \in \mathfrak{S}$.

Theorem 2. *Let* $n \in \mathbb{N}$, $n \geq 3$. *Assume that:* (i) *the kernels* $K_j^{(\sigma)}$, $j \in \mathbb{N}_n$, *satisfy the majorant condition*; (ii) *among the functions* $\varphi_1, \ldots, \varphi_n$, *there exist* $n_1 \geq 0$ *functions belonging to the space* $L_1(\mathbb{R}^d)$, $n_\infty = n_1$ *functions belonging to the space* $L_\infty(\mathbb{R}^d)$, *and* $n_2 = n - 2n_1 \geq 0$ *functions belonging to the space* $L_2(\mathbb{R}^d)$. *Then* $\lim_{\rho(\sigma)\to\infty} I_n^{(\sigma)} = 0$.

The proof of Theorem 2 follows the lines of that of Part B) of Theorem 5.3 in Buldygin *et al*. (2000) with slight modifications related to the form of the majorant condition above.

Asymptotic normality of a cross-correlogram estimate of the response function in an SISO system

Take a time-invariant continuous SISO (single-input single-output) linear system with a real-valued impulse response function $H := (H(\tau), \tau \in \mathbb{R})$, also called the transfer function in the time domain. This means that the system is an input-output type «black box», and the response of the system that enjoys an input $x(t)$, $t \in \mathbb{R}$, has the following form: $y(t) = \int_{-\infty}^{\infty} H(t-s)x(s)ds$, where the function $H(\cdot)$ is unknown. The problem is to estimate H from observations after the input and the output interpreted as stochastic processes X and Y, respectively. Suppose that a family of continuous spectral densities f_Δ, $\Delta > 0$, satisfies conditions (1a)–(1g) in Buldygin *et al*. (1998). Let $X_\Delta := (X_\Delta(t),\, t \in \mathbb{R})$, $\Delta > 0$, be a family of separable stationary zero-mean Gaussian processes with spectral densities $f_{X_\Delta} = f_\Delta$, $\Delta > 0$. Consider a stochastic process Y_Δ representing the response of the system to the input X_Δ. Define the sample cross-correlogram:

$$\hat{H}_{\Delta,N}(\tau) := \frac{1}{N} \sum_{n=1}^{N} Y_\Delta(nh(\Delta)) X_\Delta(nh(\Delta) - \tau)$$

where $h(\Delta) := \pi/(2\lambda_0(\Delta))$, $\lambda_0(\Delta) := \sup\{\lambda > 0 : \ f_\Delta(\lambda) > 0\} < \infty$, and $\lambda_0(\Delta) \to \infty$ as $\Delta \to \infty$, see condition (1b) in Buldygin *et al*. (1998). By (6) and Theorem 2, the following statement can be proved.

Theorem 3. *If $H \in L_2(\mathbb{R})$, then for any $n \geq 1$ and all $\tau_1, \ldots, \tau_n \in \mathbb{R}$ one has*

$$\mathsf{E}\left[\prod_{j=1}^{n} \hat{Z}_{\Delta,N}(\tau_j)\right] \to \mathsf{E}\left[\prod_{j=1}^{n} Z(\tau_j)\right] \quad \text{as } (\Delta, Nh(\Delta)) \to \infty.$$

In particular, all finite-dimensional distributions of the process $\hat{Z}_{\Delta,N} = (\hat{Z}_{\Delta,N}(\tau),\ \tau \in \mathbb{R})$ converge weakly to the corresponding finite-dimensional distributions of the Gaussian process Z. Here, $\hat{Z}_{\Delta,N}(\tau) := \sqrt{Nh(\Delta)}\ \left[\hat{H}_{\Delta,N}(\tau) - \mathsf{E}\hat{H}_{\Delta,N}(\tau)\right]$, $Z = (Z(\tau),\ \tau \in \mathbb{R})$ is a zero-mean Gaussian process whose correlation function is

$$\mathsf{E}Z(\tau_1)Z(\tau_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[e^{i(t_1-t_2)\lambda} |H^*(\lambda)|^2 + e^{i(t_1+t_2)\lambda} (H^*(\lambda))^2\right] d\lambda, \quad \tau_1, \tau_2 \in \mathbb{R},$$

and H^* is the L_2 Fourier transform of H.

The proof presents no technical difficulties and can be obtained similarly to the proof of Theorem 4.1 in Buldygin *et al*. (2000) with the reference to Part B) of Theorem 5.3 replaced by that to the above stated Theorem 2.

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