A METHOD OF MULTIOBJECTIVE DECISION MAKING USING A VECTOR VALUE FUNCTION

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A decision situation with partial information on preferences by means of a vector value function is assumed. The concept of minimum value dispersion solution as a reference point joined with a pseudodistance function from such a point and a dispersion level ϵ, lead to the notion of $\epsilon-$dispersion set. The dispersion level represents the amount of "value" that the decision maker can be indifferent to, therefore he should choose his most preferred solution in this set. Convergence properties, as well as an interactive method based on the reduction of $\epsilon-$dispersion sets by means of parametric variation of ϵ, to aid decision making in discrete problems is considered. Detailed numerical examples are included.

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1. INTRODUCTION

In this paper we propose an interactive method to aid decision making in solving deterministic multiobjective problems. It tries to help a decision maker (DM) to come up with a decision in the value efficient set, combining optimization and satisficing points of view. The method we present, assumes a vector value function and makes use of an evaluation function obtained as the scalar product of the vector value function and a generic weights vector (scaling), which leads to assume an imprecise weighted additive evaluation function (see Debreu, 1960 or Krantz *et al.*, 1971). A minimum dispersion solution, which is the one with smaller value difference over an information set about component value weights (optimization), is obtained. Such solution, as a reference point to choose the DM, is theoretically provided. Next, a pseudodistance function from a minimum dispersion solution and a dispersion level, which is considered as the amount of value that the DM can be indifferent to, leads to the dispersion set (satisficing). The procedure uses a parametric variation of the dispersion level which produces an interactive reduction of the dispersion set, whose convergence is proved. Different parametric variations are possible and, in practice, the method stops when the dispersion set has been reduced enough for the DM to choose his most preferred solution.

Consider the problem of multiple objective (or vector) optimization: A set $X \subset \mathbb{R}^N$ of alternatives or decisions $\mathbf{x}$ called decision or attribute space and a set of objective functions $\mathbf{z} = (z_1, \ldots, z_n)$, which the DM wishes to maximize. In this way, a function $\mathbf{z}$ defined on Z which take values on the objective or solution space $\mathbb{R}^n$ is defined, with

$$\mathbf{z} = \mathbf{z}(\mathbf{x}) \in Z \quad \text{and} \quad Z = \mathbf{z}(X) \subseteq \mathbb{R}^n$$

where Z is the feasible region in the solution space.

Three different situations depending on the available information on preferences for the DM are considered.

1. The only one information in Z is "more is better" for each objective $z_i (i = 1, \ldots, n)$, which leads to the *null information problem* and we formulate it as

$$\max_{\mathbf{x} \in X} \mathbf{z}(\mathbf{x})$$

In this problem, which has been widely studied, arises in a natural way the concept of efficient (Pareto optimal or nondominated) point. Preference optimal points must be efficients and the set of efficient points, defined as

$$E(X, \mathbf{z}) = \{\mathbf{x} \in \mathbf{X} : \nexists \ \mathbf{x}' \in X \text{ with } \mathbf{z}(\mathbf{x}') \geq \mathbf{z}(\mathbf{x})\}$$

where $[\mathbf{z}(\mathbf{x}') \geq \mathbf{z}(\mathbf{x}) \Leftrightarrow z_i(\mathbf{x}') \geq z_i(\mathbf{x}) \forall i$ and $\mathbf{z}(\mathbf{x}') \neq \mathbf{z}(\mathbf{x})]$, will include the most preferred solution. There are several methods available to generate, under some conditions, all the elements of $E(X, \mathbf{z})$(see among others the books of Goicoechea *et al.*,1982; Chankong *et al.*, 1983; Steuer, 1986).

2. There is complete information on preferences in the sense that we may assess a value function (Debreu, 1954; Fishburn, 1964; Keeney *et al.*, 1993) denoted by v, which is a real-valued function defined on Z and represents the DM preferences by means of a strict weak order $\succ$ (asymmetric and negatively transitive), such that

$$\mathbf{z} \succ \mathbf{z}' \Leftrightarrow v(\mathbf{z}) \geq v(\mathbf{z}')$$

We call it *complete information problem* and we formulate it as

$$\max_{\mathbf{z} \in Z} v(\mathbf{z})$$

which it is a classical optimization problem and the optimal solution will be a point of the set

$$\mathrm{Opt}(Z, v) = \{\mathbf{z}' \in Z : v(\mathbf{z}') = \max v(\mathbf{z}), \mathbf{z} \in \mathbf{Z}\}$$

3. The DM provides information not enough to assess a value function, but to assess a vector value function $\mathbf{v} : Z \to \mathbb{R}^p$, which represents (Roberts, 1979; Rietveld, 1980; Ríos-Insua, 1980) a strict partial order $\succ$ (irreflexive and transitive) intersection of p strict weak orders, where

$$\mathbf{z} \succ \mathbf{z}' \Leftrightarrow \mathbf{v}(\mathbf{z}) \geq \mathbf{v}(\mathbf{z}')$$

We call this situation, which may be considered intermediate between 1 and 2, *partial information problem*. Note that, if there were no more information on preferences in the feasible region $\mathbf{v}(Z) \subseteq \mathbb{R}^p$, we would be in case 1. On other hand, if all the components v_i in $\mathbf{v}$ were equals, would be in case 2.

From a general point of view, the terminology, partial or incomplete information, is also used in the case under uncertainty for utility functions, probability distributions and the evaluation function. Several methodologies have been proposed to deal with this problem on partial information (Chankong *et al.*, 1983; Ríos *et al.*, 1989), but in in our context, we shall refer to the certainty case with a vector value function, that may be seen as a way for lack of precision of the true DM's (scalar) value function and is suitable for hierarchical structures which

often exhibits the multiple objective decision making problems. The problem formulation is formally analogous to 1, but in the solution space. We have

$$\max_{\mathbf{z} \in Z} \mathbf{v}(\mathbf{z})$$

and leads us to the value efficient set

$$E(Z, \mathbf{v}) = \{\mathbf{z} \in \mathbf{Z} : \not\exists \ \mathbf{z}' \in Z \text{ with } \mathbf{v}(\mathbf{z}') \geq \mathbf{v}(\mathbf{z})\}$$

where

$$E(X, \mathbf{z}) \supseteq \mathbf{z}^{-1}(E(Z, \mathbf{v})) = \{\mathbf{x} \in \mathbf{X} : \mathbf{z}(\mathbf{x}) \in E(Z, \mathbf{v})\}$$

assuming that $\mathbf{z}$ is an increasing function, that is,

$$\mathbf{z}(\mathbf{x} + \Delta\mathbf{x}) > \mathbf{z}(\mathbf{x}) \text{ for } \mathbf{x} \in \mathbf{X}, \Delta\mathbf{x} \geq 0 \text{ and } \mathbf{x} + \Delta\mathbf{x} \in \mathbf{X}$$

where $[\mathbf{z}(\mathbf{x} + \Delta\mathbf{x}) > \mathbf{z}(\mathbf{x}) \Leftrightarrow z_i(\mathbf{x} + \Delta\mathbf{x}) > z_i(\mathbf{x}) \forall i]$

Several important useful concepts have been developed by researchers for the case of partial information on value (or utility) functions as Fishburn (1964,1965), Sarin (1977), Hannan (1981), Kirkwood *et al.* (1985), Korhonen *et al.* (1984), White *et al.* (1984), Weber (1985), Malakooti (1989), Ríos-Insua (1990) and Kirkwood (1992) among others, and an interesting overview within a general framework is found in Weber (1987). Therefore, we shall not consider it here. On other hand, there are also a number of helpful computer programs which aids to assess value and utility functions (i.e., Keeney *et al*, 1976; Kirkwood *et al.*, 1986,1987; Logical Decision, 1992; Decision Pad, 1993).

The paper consist of five sections. In the first section the problem is formulated with a brief overview of the context. In the second and third sections, solution concepts, as well as, monotonicity and convergence properties are provided. The fourth section synthesize the method into an algorithm for solving discrete problems. Some numerical examples are given in the last section.

2. SOLUTION CONCEPTS

Given the partial information problem, it leads us to determine the value efficient set $E(Z, \mathbf{v})$ and so $\mathbf{z}^{-1}(E(Z, \mathbf{v}))$, and considering that such set, reduced from $E(Z, \mathbf{z})$, could be still too extensive for the DM to choose an alternative, we propose a method to aid him.

Let us consider a vector value function $\mathbf{v} : Z \longrightarrow \mathbb{R}^p$ and the set $K^\circ \equiv \mathbb{R}^p_+$, which is a convex cone.

Definition 1

Let be $K \supseteq K^0$ a constant, convex, closed and acute cone which we call *information cone*, and K^p its positive polar. The set

$$K_* = K^p \cap S_p$$

is called *information set* associated to K, where S_p is the unit sphere on R^p.

Let $E(Z, K)$ be the efficient set with respect to $K(\mathbf{z} \in E(Z, K)$ if there is no $\mathbf{z}' \in Z$ such that $\mathbf{z}' \in \mathbf{z} + K)$. It may be seen that, if K^p is a polyhedral cone it will be possible to determine its set of generators $\{\mathbf{k}^1, \ldots, \mathbf{k}^r\}$(Tamura. 1976). which normalized on S_p are denoted $\{\mathbf{k}_*^1, \ldots, \mathbf{k}_*^r\}$.

Definition 2

For each $\mathbf{z} \in \mathbf{Z}$, the numbers

$$v^*(\mathbf{z}) = \max_{\mathbf{k} \in K_*} (\mathbf{k} \cdot \mathbf{v})(\mathbf{z}) \qquad v_*(\mathbf{z}) = \min_{\mathbf{k} \in K_*} (\mathbf{k} \cdot \mathbf{v})(\mathbf{z})$$

are called *upper and lower indexes*, respectively.

Let be $E_K \equiv \{\mathbf{z} \in E(Z, \mathbf{v}): \mathbf{v}(\mathbf{z}) \in E(\mathbf{v}(Z), K)\}$ and $E = E_{K^0}$ in what follows. Assuming that $v^*(\mathbf{z})$ y $v_*(\mathbf{z})$ represents for each $\mathbf{z}$, the best and the worst values under the information cone K, a criterion to determine a solution in E_K consists of considering those solutions whereby the difference between both indexes will be as small as possible. This setting is based on the idea that carries to choose those points of the efficient set in which the information cone approximates the information given by a value function (scalar) in which both indexes obviously are equal. We introduce a function which measures such difference on E_K.

Definition 3

The function $d_K : E_K \longrightarrow \mathbb{R}^+$, defined as

$$d_K(\mathbf{z}) = v^*(\mathbf{z}) - v_*(\mathbf{z})$$

is called *dispersion function* or *dispersion* associated to $\mathbf{z}$ under the information cone K.

Note that $v^*(\mathbf{z}) \geq v_*(\mathbf{z})$ for each $\mathbf{z} \in E_K$, therefore, it is always $d_K(\mathbf{z}) \geq 0$. On other hand. we can introduce the ordering $\geq_{(K)}$ on $E_K \times E_K$ defined by

$$\mathbf{z} \geq_{(K)} \mathbf{z}' \Leftrightarrow d_K(\mathbf{z}) \leq d_K(\mathbf{z}')$$

which is a strict linear order (asymmetric, transitive and complete) and leads to a total ordering on E_K. As the most preferred solution, we shall consider that with smaller dispersion.

Definition 4

A solution $\mathbf{z}' \in E_K$ such that

$$d_K(\mathbf{z}') = \min_{\mathbf{z} \in E_K} d_K(\mathbf{z})$$

is called a *minimum dispersion solution* under K.

The set

$$D(Z, K) = \{\mathbf{z}' \in E_K : d_K(\mathbf{z}') = \min d_K(\mathbf{z}), z \in E_K\}$$

is called *minimum dispersion set*.

Once $\mathbf{v}$ is assessed, if there is no more information on preferences on $\mathbf{v}(Z) \subseteq \mathbb{R}^p$, the information cone is $K^\circ \subseteq \mathbb{R}^p$ and the ordering will be $\geq_{(K^\circ)}$. However, the DM may provide more information on his preferences by means of a cone $K \supseteq K^\circ$, what leads to a smaller dispersion.

Proposition 1

Let be K' and K information cones such that $K' \supseteq K$ and $\mathbf{z} \in E_K$, then $d_{K'}(\mathbf{z}) \leq d_K(\mathbf{z})$ and

$$\min_{\mathbf{z} \in E_{K'}} d_{K'}(\mathbf{z}) \leq \min_{\mathbf{z} \in E_K} d_K(\mathbf{z})$$

Proof

We observe that if $K' \supseteq K$ it is $K'_* \subseteq K_*$, $E_{K'} \subseteq E_K$ and

$$\max_{\mathbf{k} \in K'_*} (\mathbf{k} \cdot \mathbf{v})(\mathbf{z}) \leq \max_{\mathbf{k} \in K_*} (\mathbf{k} \cdot \mathbf{v})(\mathbf{z}) \text{ and } \min_{\mathbf{k} \in K'_*} (\mathbf{k} \cdot \mathbf{v})(\mathbf{z}) \geq \min_{\mathbf{k} \in K_*} (\mathbf{k} \cdot \mathbf{v})(\mathbf{z})$$

hence $d_{K'}(\mathbf{z}) \leq d_K(\mathbf{z})$ for each $\mathbf{z} \in E_K$, and thus

$$\min_{\mathbf{z} \in E_K} (\max_{\mathbf{k} \in K'_*} (\mathbf{k} \cdot \mathbf{v})(\mathbf{z}) - \min_{\mathbf{k} \in K'_*} (\mathbf{k} \cdot \mathbf{v})(\mathbf{z})) \leq \min_{\mathbf{z} \in E_K} (\max_{\mathbf{k} \in K_*} (\mathbf{k} \cdot \mathbf{v})(\mathbf{z}) - \min_{\mathbf{k} \in K_*} (\mathbf{k} \cdot \mathbf{v})(\mathbf{z}))$$

which shows the second inequality.

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Let us now consider a result which states that if a solution $\mathbf{z}$ has minimum dispersion for an increasing sequence of information cones, whose information sets converge to a vector $\mathbf{k}^+$, such solution maximizes the value function $(\mathbf{k}^+ \cdot \mathbf{v})$.

Proposition 2

Let be $\mathbf{z} \in E$ and $\{K^n\}_{n=0}^{\infty}$ an increasing sequence of information cones such that $K_*^n \downarrow \mathbf{k}^+$. If $\mathbf{z} \in D(Z, K^n)$ for each $n = 0, 1, \ldots$, then $\mathbf{z} \in \mathrm{Opt}(Z, (\mathbf{k}^+ \cdot \mathbf{v}))$.

Proof

If $\mathbf{z} \in D(Z, K^n)$ for each n, then $\mathbf{v}(\mathbf{z}) \in E(\mathbf{v}(Z), K^n)$ for each n. Because of the nesting property of the efficient sets with respect to the sequence $\{K^n\}$, it will be $\mathbf{v}(\mathbf{z}) \in E(\mathbf{v}(Z), K^+)$, where K^+ has as polar positive the vector $\mathbf{k}^+$ and thus $\mathbf{z} \in \mathrm{Opt}(Z, (\mathbf{k}^+ \cdot \mathbf{v}))$.

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We note that $(\mathbf{k}^+ \cdot \mathbf{v})$ represents the value function when all the uncertainty on preferences has been removed and, consequently, we would have a complete information problem. Observe in this last case that the dispersion of all solutions would be zero, we have no objection to this, because we have a value function that provides a total ordering.

We shall not consider here effective ways to asses information sets (cones) based on preferences, but some methods can be found in Malakooti (1989).

3. PSEUDODISTANCE ON DISPERSION

The above criterion, based on minimum dispersion, may be too strict in some cases, so we are going to introduce a less restrictive criterion in the idea of the satisficing approach, that combined with the former, leads to an aid decision making method.

In this satisficing criterion an amount $\epsilon > 0$ is considered, which means the maximum "amount of value" that the DM can ignore or be indifferent to, and the solution(s) to the problem will be those whose distance from a minimum dispersion solution will be smaller than ϵ.

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Definition 5

We call *pseudodistance on dispersion* between $\mathbf{z}, \mathbf{z}' \in E_K$, to the mapping $\rho_K : E_K \times E_K \to \mathbb{R}^+$ where

$$\rho_K(\mathbf{z}, \mathbf{z}') = |d_K(\mathbf{z}) - d_K(\mathbf{z}')| .$$

Proposition 3

The mapping ρ_K is a pseudodistance on E_K.

Proof

a) As a consequence of the definition is immediate that $\rho_K(\mathbf{z}, \mathbf{z}') \geq 0$.

b) For the triangular inequality, note that

$$\rho_K(\mathbf{z}, \mathbf{z}'') = |d_K(\mathbf{z}) - d_K(\mathbf{z}'')| =$$

adding and subtracting $d_K(\mathbf{z}')$

$$= |d_K(\mathbf{z}) - d_K(\mathbf{z}') + d_K(\mathbf{z}') - d_K(\mathbf{z}'')| \leq |d_K(\mathbf{z}) - d_K(\mathbf{z}')| + |d_K(\mathbf{z}') - d_K(\mathbf{z}'')|$$
$$= \rho_K(\mathbf{z}, \mathbf{z}') + \rho_K(\mathbf{z}', \mathbf{z}'')$$

with the last inequality, because of definition 5, proves the triangular inequality for all $\mathbf{z}, \mathbf{z}', \mathbf{z}'' \in E_K$.

c) It is trivial from definition 5 that $\rho_K(\mathbf{z}, \mathbf{z}') = \rho_K(\mathbf{z}', \mathbf{z})$.

d) We shall see with an example that can exist two solutions $\mathbf{z}, \mathbf{z}'$ with $\mathbf{z} \neq \mathbf{z}'$, but $\rho_K(\mathbf{z}, \mathbf{z}') = 0$ and so ρ_K will be a pseudodistance.

Let us consider a problem where $\mathbf{v}(\mathbf{z})$ is defined on $\mathbb{R}^2$ and a polyhedral information cone K whose information set K_*, is given by the generators $\{(1/\sqrt{5}, 2/\sqrt{5}), (2/\sqrt{5}, 1/\sqrt{5})\}$. Let be $\mathbf{z}, \mathbf{z}'$ with $\mathbf{z} \neq \mathbf{z}'$ and $\mathbf{v}$ an undefined vector value function such that

$$\mathbf{v}(\mathbf{z}) = (1, 0) \text{ and } \mathbf{v}(\mathbf{z}') = (0, 1)$$

From previous definitions we have

$$v^*(\mathbf{z}) = \max_{\mathbf{k} \in K_*} (k_1 \cdot 1 + k_2 \cdot 0) = \max_{\mathbf{k} \in K_*} k_1 = \frac{2}{\sqrt{5}}$$

and analogously

$$v_*(\mathbf{z}) = \min_{\mathbf{k} \in K_*} k_1 = \frac{1}{\sqrt{5}}$$

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Hence

$$d_K(\mathbf{z}) = \frac{2}{\sqrt{5}} - \frac{1}{\sqrt{5}} = \frac{1}{\sqrt{5}}$$

In a similar way, we obtain for $\mathbf{z}'$

$$v^*(\mathbf{z}') = \max_{\mathbf{k} \in K_*} k_2 = \frac{2}{\sqrt{5}} \quad \text{and} \quad v_*(\mathbf{z}') = \min_{\mathbf{k} \in K_*} k_2 = \frac{1}{\sqrt{5}}$$

and $d_K(\mathbf{z}') = 1/\sqrt{5}$. Then, $\rho_K(\mathbf{z}, \mathbf{z}') = 0$ but $\mathbf{z} \neq \mathbf{z}'$.

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Definition 6

Given $\mathbf{z}^+ \in D(Z, K)$ and a real number $\epsilon > 0$, a solution $\mathbf{z} \in E_K$ such that $\rho_K(\mathbf{z}, \mathbf{z}^+) \leq \epsilon$, is called $\epsilon-dispersion\ solution$.

The number ϵ is called *dispersion level* and the set

$$D_\epsilon(Z, K, \mathbf{z}^+) = \{\mathbf{z} \in E_K : \rho_K(\mathbf{z}, \mathbf{z}^+) \leq \epsilon\}$$

$\epsilon-dispersion\ set$ (with respect to $\mathbf{z}^+$).

The $\epsilon-$dispersion concept is more general than the one of minimum dispersion

Proposition 4

Given $\mathbf{z}^+ \in D(Z, K)$ and a dispersion level ϵ, then

$$D(Z, K) \subseteq D_\epsilon(Z, K, \mathbf{z}^+)$$

Proof

If $\mathbf{z} \in D(Z, K)$ then

$$d_K(\mathbf{z}) = d_K(\mathbf{z}^+) = \min_{\mathbf{z} \in E_K} d_K(\mathbf{z})$$

hence $\rho_K(\mathbf{z}, \mathbf{z}^+) \leq \epsilon$ for all $\epsilon > 0$.

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Now, we consider approximation and convergence principles for the ϵ-dispersion sets.

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Theorem 1

Given dispersion levels ϵ_1, ϵ_2, where $\epsilon_1 < \epsilon_2$, then for all $\mathbf{z}^+ \in D(Z, K)$ is

$$D_{\epsilon_1}(Z, K, \mathbf{z}^+) \subseteq D_{\epsilon_2}(Z, K, \mathbf{z}^+).$$

Proof

It is immediate.

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Theorem 2

Let be $\mathbf{z}^+ \in D(Z, K)$ and $\{\epsilon_n\}_{n \in \mathbb{N}}$ a sequence of dispersion levels such that $\epsilon_n \downarrow 0$, then

$$D_{\epsilon_n}(Z, K, \mathbf{z}^+) \downarrow D(Z, K)$$

(in the sense that $\bigcap_{n \in \mathbb{N}} D_{\epsilon_n}(Z, K, \mathbf{z}^+) = D(Z, K)$).

Proof

Let us call $D_{\epsilon_n}(Z, K, \mathbf{z}^+) = D_n$ and $D(Z, K) = D$.

Because $D \subset D_n$ for all n, then $D \subset \bigcap_n D_n$.

To show the other content, note that if $\mathbf{z}' \in \bigcap_n D_n$ then

$$\rho_K(\mathbf{z}', \mathbf{z}^+) \leq \epsilon_n \text{ for all } n$$

so, taking limits and because ρ_K is non negative, we have

$$\rho_K(\mathbf{z}', \mathbf{z}^+) = 0$$

thus, we obtain that

$$d_K(\mathbf{z}') = d_K(\mathbf{z}^+) = \min_{\mathbf{z} \in E_K} d_K(\mathbf{z})$$

and $\mathbf{z}' \in D$.

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4. ALGORITHM FOR DISCRETE PROBLEMS

Although the above developments are theoretically valid for continuous problems, due to the difficulty to determine E_K, specially in the nonlinear case, a method for multicriteria evaluation for discrete problems with its corresponding algorithm is now considered.

Firstly, the algorithm computes the minimum dispersion set $D \equiv D(Z, K)$ as well as the associated dispersion value. Then, for a solution $\mathbf{z}^+ \in D$ and an initial dispersion level ϵ, the ϵ-dispersion set $D_\epsilon \equiv D_\epsilon(Z, K, \mathbf{z}^+)$ is computed and then presented to the DM to make a choice if possible or otherwise, take a smaller dispersion level and repeat the process.

Let $E_K = \{\mathbf{z}^1, \ldots, \mathbf{z}^q\}$, i the iteration index, M a large positive number, j the index for the dispersion level sequence $\epsilon_j (= \epsilon j)$, $q(j) = \mathrm{card}(D_{\epsilon j})$ and F a subset of E_K. We assume that the indexes of the solutions in $D_{\epsilon j}$ are appropriately renumbered from 1 to $q(j)$ in each iteration ($q(1) = q$). The algorithm is as follows

Step 1. Set $i = 0, d = M$ and $D = \emptyset$.

Step 2. Set $i = i + 1$.

Step 3. Choose $\mathbf{z}^i \in E_K$ and compute $d^i = d_K(\mathbf{z}^i)$. If $d^i > d$, go to step 4. If $d^i = d$, set $D = D \cup \{\mathbf{z}^i\}$ and go step 4. Otherwise, set $d = d^i$, $D = \{\mathbf{z}^i\}$ and go to step 4.

Step 4. If $i < q$, go to step 2. Otherwise, D and d are identified and let be $\mathbf{z}^+$ a solution in D.

Step 5. Set $j = 1$ and $F = \emptyset$.

Step 6. Set $i = 0, D_{\epsilon j} = E_K \backslash F, \epsilon_j = \epsilon/j$ and $q = q(j)$.

Step 7. Set $i = i + 1$.

Step 8. Choose $\mathbf{z}^i \in D_{\epsilon j}$ and compute $\rho_K(\mathbf{z}^i, \mathbf{z}^+)$. If $\rho_K(\mathbf{z}^i, \mathbf{z}^+) \leq \epsilon_j$ go to step 9. Otherwise set $D_{\epsilon j} = D_{\epsilon j} \backslash \{\mathbf{z}^i\}$ and go to step 9.

Step 9. If $i < q$, go to step 7. Otherwise $D_{\epsilon j}$ has been identified.

Step 10. If $D_{\epsilon j}$ is satisfactory for the DM to choose one solution, stop. Otherwise, set $F = D_{\epsilon j}^c (= E_K \backslash D_{\epsilon j})$.

Step 11. Set $j = j + 1$ and go to step 6.

Let us now consider some computational aspects of the method.

[1.] In the first part of the algorithm it is necessary to compute the indexes $v^*(\mathbf{z})$ and $v_*(\mathbf{z})$ to determine the solution's dispersion. In the case of polyhedral cones, what is usually considered, we shall propose a method based in the Lagrange multipliers, easy to implement.

Let K be a cone and K_* its positive polar (on the unit sphere) with generators $G = \{\mathbf{k}^1, \ldots, \mathbf{k}^r\}$ and $G_* = \{\mathbf{k}^1_*, \ldots, \mathbf{k}^r_*\}$, respectively. Let be $E_K = \{\mathbf{z}^1, \ldots \mathbf{z}^q\}$ as before. The scheme is as follows: Let be

$$E^1_K = \{\mathbf{z}^i \in E_K : \mathbf{H}\cdot(\mathbf{v}(\mathbf{z}^i))^T \geq \mathbf{0}^T\}$$

where $\mathbf{H}$ is a $r \times p$ matrix with $\mathbf{k}^j$ being the jth row of $\mathbf{H}$, and

$$E^2_K = E_K \backslash E^1_K$$

In E^1_K and E^2_K we shall compute the lower index

$$v_*(\mathbf{z}^i) = \min_{\mathbf{k} \in G_*} (\mathbf{k} \cdot \mathbf{v})(\mathbf{z}^i)$$

and the upper index as

$$v^*(\mathbf{z}^i) = (\mathbf{k}^u \cdot \mathbf{v})(\mathbf{z}^i)$$

with $\mathbf{k}^u = \mathbf{v}(\mathbf{z}^i)$ in E^1_K, and in E^2_K as

$$v^*(\mathbf{z}^i) = \max_{\mathbf{k} \in G_*} (\mathbf{k} \cdot \mathbf{v})(\mathbf{z}^i)$$

To explain why the maximum is reached on such a point, and the minimum on G_*, we use the Lagrange method. Let us consider the problem

$$\max(\mathbf{k} \cdot \mathbf{v})(\mathbf{z}) = k_1 v_1(\mathbf{z}) + \cdots + k_p v_p(\mathbf{z})$$

subject to

$$k_1^2 + \cdots + k_p^2 = 1$$

(that is, $(k_1, \ldots, k_p) \in S_p$) and consider the Lagrange function

$$L(k_1, \ldots, k_p) = k_1 v_1(\mathbf{z}) + \cdots + k_p v_p(\mathbf{z}) + \lambda(k_1^2 + \cdots + k_p^2 - 1)$$

Given the partial derivatives of L, we obtain the system

(1) $$\begin{cases} v_i(\mathbf{z}) + 2\lambda k_i = 0, \quad i = 1, \ldots, p \\ k_1^2 + \cdots + k_p^2 = 1 \end{cases}$$

From any of the above equations where $v_i(\mathbf{z}) \neq 0$, we obtain

$$\lambda = -\frac{v_i(\mathbf{z})}{2k_i} \quad \text{and} \quad k_j = \frac{k_i v_j(\mathbf{z})}{v_i(\mathbf{z})}$$

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for any $j \neq i$. If we substitute each k_j in (1), we have

$$k_i^2 \left(\frac{v_1^2(\mathbf{z})}{v_i^2(\mathbf{z})} + \cdots + \frac{v_i^2(\mathbf{z})}{v_i^2(\mathbf{z})} + \cdots + \frac{v_p^2(\mathbf{z})}{v_i^2(\mathbf{z})} \right) = 1$$

so

$$k_i = \frac{\pm v_i(\mathbf{z})}{\sqrt{v_i^2(\mathbf{z}) + \cdots + v_p^2(\mathbf{z})}}$$

and the two critical points will be

$$\left(\frac{v_1(\mathbf{z})}{\sqrt{v_1^2(\mathbf{z}) + \cdots + v_p^2(\mathbf{z})}}, \ldots, \frac{v_p(\mathbf{z})}{\sqrt{v_1^2(\mathbf{z}) + \cdots + v_p^2(\mathbf{z})}} \right)$$

and the opposite.

2. To start the method it is necessary an initial value ϵ to iterate. A way to have an idea about this value may be to put

$$\epsilon = \max_{\mathbf{z}} \rho_K(\mathbf{z}, \mathbf{z}^+)$$

Note that with this initial value, the first D_ϵ set would be E_K.

3. In the algorithm, the considered parameterization is $\epsilon_j = \epsilon/j$, however, others parameterizations could be proposed depending on the characteristics of the problem under consideration (i.e., number of alternatives, speed convergence).

5. EXAMPLES

A) Let the structure of the DM's vector value function be linear

$$\mathbf{v}(\mathbf{z}) = (v_1(\mathbf{z}), v_2(\mathbf{z})) = (4z_1 + z_2, z_1 + 6z_2)$$

that must be maximized and let be $Z \subseteq \Xi^2$ the set of solutions whose efficient set $E(Z, K^\circ)$ for the Pareto order is shown in table 1.

Table 1

	$\mathbf{z}^1$	$\mathbf{z}^2$	$\mathbf{z}^3$	$\mathbf{z}^4$	$\mathbf{z}^5$	$\mathbf{z}^6$	$\mathbf{z}^7$	$\mathbf{z}^8$	$\mathbf{z}^9$
z_1^i	3	4	8	8.2	9	11	12	15	15.2
z_2^i	12	8	7	6.1	6	5	3	2	1

Table 2

	z^1	z^2	z^3	z^4	z^5	z^6	z^7	z^8	z^9
$v_1(z^i)$	24	24	39	38.9	42	49	51	62	61.8
$v_2(z^i)$	75	52	50	44.8	45	41	30	27	21.2

We determine for each point z^i their values v_1 and v_2 as shown in table 2 and figure 1a), and thus $E = \{z^1, z^3, z^5, z^6, z^7, z^8\}$, because z^1 dominates z^2, z^3 dominates z^4 and z^8 dominates z^9.

We consider three cases corresponding each one to a different information set (figures 1b), c) and d)).
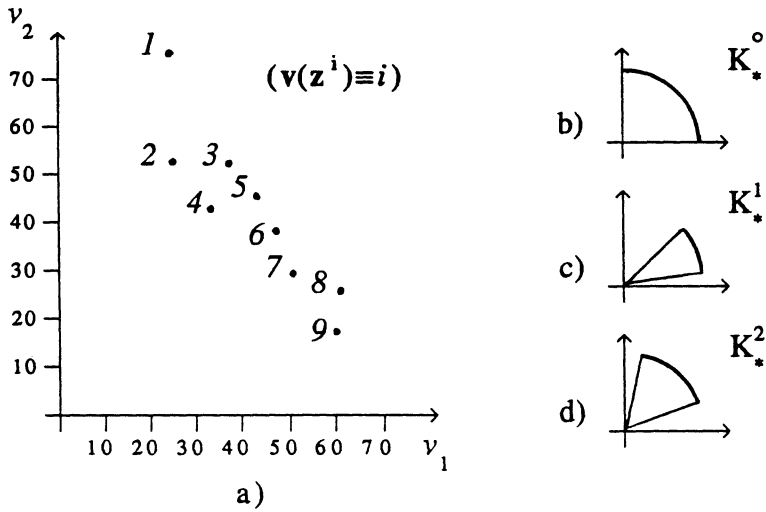
Figure 1.

a) Graphical illustration of the solutions on the $v_1 v_2$ space. b), c) and d), three possible information sets.

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A1. Assume that K_*^0 is the (null) information set (figure 1*b*), with generators $\{(1,0),(0,1)\}$ and let us compute D and d. Initially, let $M = 10^8$. Choosing $\mathbf{z}^1 \in E$, its dispersion value is

$$d^1 = d_{K0}(\mathbf{z}^1) = v^*(\mathbf{z}^1) - v_*(\mathbf{z}^1) = 78.7 - 24 = 54.7$$

Because $d^1 = 54.7 < 10^8 = M$, we set $d = 54.7$ and continue to proceed finally in iteration six of the first part of the algorithm (steps 1 to 4), to obtain $\mathbf{z}^+ = \mathbf{z}^5$ and hence

$$D = \{\mathbf{z}^5\} \text{ and } d = d^5 = 19.6$$

The value indexes, as well as the, dispersion values of all solutions in E are shown in table 3, from which we see that the preference ordering will be

$$\mathbf{z}^5 \succ \mathbf{z}^6 \succ \mathbf{z}^3 \succ \mathbf{z}^7 \succ \mathbf{z}^8 \succ \mathbf{z}^1$$

Table 3

	$\mathbf{z}^1$	$\mathbf{z}^3$	$\mathbf{z}^5$	$\mathbf{z}^6$	$\mathbf{z}^7$	$\mathbf{z}^8$
$v^*(\mathbf{z}^i)$	78.7	63.4	61.6	63.9	59.2	67.6
$v_*(\mathbf{z}^i)$	24	39	42	41	30	27
d_{K0}	54.7	24.4	19.6	22.9	29.2	40.6
$\rho_K(\mathbf{z}^i,\mathbf{z}^+)$	35.1	4.8	0	3.3	9.6	21

In the second part of the algorithm (steps 5 to 11), because $\max \rho_K(\mathbf{z}^i,\mathbf{z}^+) = 35.1$, we initially take $\epsilon = 36$ and choose the parametric variation $\epsilon_j = \epsilon/2^{j-1}$. In the first iteration $D_{36} = E$ and, in the fifth one, the algorithm stops because we obtain the unitary set $D_{2.25} = \{\mathbf{z}^5\}$. The iterations, with their respective ϵ-dispersion sets, are shown in table 4, where we see the monotonicity property.

Table 4

Iteration	ϵ	D_ϵ
1	36	E
2	18	$\mathbf{z}^3, \mathbf{z}^5, \mathbf{z}^6, \mathbf{z}^7$
3	9	$\mathbf{z}^3, \mathbf{z}^5, \mathbf{z}^6$
4	4.5	$\mathbf{z}^5, \mathbf{z}^6$
5	2.25	$\mathbf{z}^5$

If the DM states that 4.5 is a satisficing dispersion level he must choose the most preferred solution in $D_{4.5} = \{\mathbf{z}^5, \mathbf{z}^6\}$.

A2. Now, if the DM states that the information set is given by the vectors $\mathbf{k}^1 = (0.8, 0.2)$ and $\mathbf{k}^2 = (0.5, 0.5)$, which normalized on S_2 gives us the extreme vectors for K_*^1(figure 1c)

$$\mathbf{k}_*^1 = (0.97, 0.24) \text{ and } \mathbf{k}_*^2 = (0.71, 0.71)$$

we shall have $E_{K^1} = \{\mathbf{z}^1, \mathbf{z}^6, \mathbf{z}^8\}$ because under the corresponding information, $\mathbf{z}^6$ dominates $\mathbf{z}^3, \mathbf{z}^5, \mathbf{z}^7$. The results are shown in table 5 in which we see that

$$D = \{\mathbf{z}^8\} \text{ and } d = 4.4$$

and the preference ordering will be

$$\mathbf{z}^8 \succ \mathbf{z}^6 \succ \mathbf{z}^1$$

Table 5

	$\mathbf{z}^1$	$\mathbf{z}^6$	$\mathbf{z}^8$
$v^*(\mathbf{z}^i)$	70.3	63.9	67.6
$v_*(\mathbf{z}^i)$	41.3	57.4	63.2
d_K	29.0	6.5	4.4
$\rho_K(\mathbf{z}^i, \mathbf{z}^+)$	24.6	2.1	0

Again, we can continue the process, starting with a dispersion level $\epsilon = 25(\max \rho_K (\mathbf{z}^i, \mathbf{z}^+) = 24.6)$ and for the parametric variation $\epsilon_j = \epsilon/5^{j-1}$, the ϵ-dispersion sets are shown in table 6.

Table 6

Iteration	ϵ	D_ϵ
1	25	E_{K^1}
2	5	$\mathbf{z}^6, \mathbf{z}^8$
3	1	$\mathbf{z}^8$

A3. Finally, if the DM states that the information set is given by the vectors $\mathbf{k}^1 = (0.7, 0.3)$ and $\mathbf{k}^2 = (0.1, 0.9)$ which normalized in S_2 give the extreme vectors for K_*^2(figure 1c)

$$\mathbf{k}_*^1 = (0.92, 0.39) \text{ and } \mathbf{k}_*^2 = (0.11, 0.99)$$

lead to $E = \{\mathbf{z}^1, \mathbf{z}^3, \mathbf{z}^5, \mathbf{z}^6, \mathbf{z}^8\}$, because $\mathbf{z}^6$ dominates $\mathbf{z}^7$. In this case, we obtain $D = \{\mathbf{z}^3\}$ and $d = 9.6$ and the preference ordering will be

$$\mathbf{z}^3 \succ \mathbf{z}^5 \succ \mathbf{z}^6 \succ \mathbf{z}^1 \succ \mathbf{z}^8$$

If $\epsilon = 7$, it will be easy to show that $D_7 = \{\mathbf{z}^3, \mathbf{z}^5\}$.

B) In the case of a nonlinear vector value function, as it will be the case of quadratic components. let us see that it is possible to apply in the same way the method. Let us consider quadratic components for $\mathbf{v}$ such that

$$v_1(\mathbf{z}) = -(z_1^2 + z_2^2 - 72z_1 - 20z_2) \text{ and } v_2(\mathbf{z}) = -(z_1^2 + z_2^2 - 16z_1 - 60z_2)$$

and let the information set be defined by the vectors $\mathbf{k}^1 = (0.8, 0.2)$ and $\mathbf{k}^2 = (0.4, 0.6)$. which normalized in S_2 give us the extreme vectors

$$\mathbf{k}_*^1 = (0.97, 0.24) \text{ and } \mathbf{k}_*^2 = (0.55, 0.83)$$

In this case we obtain $E = \{\mathbf{z}^3, \mathbf{z}^6, \mathbf{z}^8\}$ under the corresponding information cone, because $\mathbf{z}^3$ dominates $\mathbf{z}^1, \mathbf{z}^2, \mathbf{z}^4$ and $\mathbf{z}^6$ dominates $\mathbf{z}^5, \mathbf{z}^7$. Also, note that $\mathbf{z}^9$ is not a value efficient solution. The results are shown in table 7. from which we obtain the preference ordering

$$\mathbf{z}^3 \succ \mathbf{z}^6 \succ \mathbf{z}^8$$

with $D = \{\mathbf{z}^3\}$ and $d = 54.2$.

Table 7

	$\mathbf{z}^3$	$\mathbf{z}^6$	$\mathbf{z}^8$
$v^*(\mathbf{z}^i)$	743.5	815.7	895.7
$v_*(\mathbf{z}^i)$	689.3	684.2	598.8
d_K	54.2	131.5	296.9
$\rho_K(\mathbf{z}^i, \mathbf{z}^+)$	0	77.3	242.7

The last row of table 7 is the pseudodistance on dispersion and we can see that for a dispersion level $\epsilon = 80$. it is $D_{80} = \{\mathbf{z}^3, \mathbf{z}^6\}$.

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6. CONCLUSIONS

In this paper we have considered multiobjective decision making under partial information on preferences what may leads to determine the DM's vector value function instead of a scalar one. From this vector function, the value efficient set is obtained and considering such set too extensive to choice a solution, a method based on one hand on a minimum dispersion solution concept over a preference information set and on the other, on a dispersion function with respect to a minimum dispersion solution, which it is a pseudodistance, are considered. The idea of dispersion level leads to its associated set which fulfils monotonicity and convergence properties. This set will contain all the indifferent "in value" solutions for a given dispersion level and de DM will choose his solution in this set or will ask for a smaller one. An algorithm for discrete problems with different computational issues supports the method which is proved with some numerical illustrations.

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