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# PIVOT VARIABLES, TESTS AND STRONG CONSISTENCY

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*Let the random vector $\mathcal{Z}$ have a distribution depending on parameters $\gamma$. A function $\mathcal{U}$ of both $\mathcal{Z}$ and $\gamma$ is a pivot variable if its distribution does not depend on $\gamma$. Assuming that $\alpha_n$ is an estimate of $\alpha$ whose asymptotic distribution has an associated pivot variable and that $\gamma_n$ is a consistent estimate of $\gamma$, it is shown how to derive tests for hypotheses on $\alpha$. Conditions are obtained for these tests to be strongly consistent and to enjoy duality. The case in which $\alpha_n$ is asymptotically normal is considered and so the Wald and Rao score tests are shown to be strongly consistent.*

**Key words:** Pivot variable, strong consistency, duality, Wald and Rao score tests.

## 1. INTRODUCTION

Strongly consistent tests were first considered by TIAGO DE OLIVEIRA (1980). In this paper we are going to present a technique to obtain such tests.

Given a random vector $\mathcal{Z}^k$ with $k$ components and distribution $F(z^k\,\gamma^t)$, we put $\mathcal{Z}^k \sim F(z^k\,\gamma^t)$. If $\mathcal{U} = g(\mathcal{Z}^k, \gamma^k)$ has a distribution that does not depend on $\gamma^t$ it will be a *pivot variable*. Since the distribution of $|\mathcal{U}|$ will also not depend

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on $\gamma^t$ we take pivot variables to be non negative. $\mathcal{U}$ will be a *continuous pivot variable* if $g$ is a continuous function. Assuming $\mathcal{Z}_n^k = h(n)(\alpha_n^k - \alpha^k)$, with $h(n) \rightarrow \infty$, has limit distribution $F(z^k\,\gamma^t)$ and $\tilde{\gamma}_n^t$ to be consistent, we will show how to use $\mathcal{U}_n = g(\mathcal{Z}_n^k, \gamma_n^t)$ to test

$$H_0: \alpha^k = \alpha_0^k$$

establishing conditions for strong consistency. The case in which the distribution of $\mathcal{Z}_n^k$ is asymptotically normal is considered and the Wald and Rao score tests are shown to be strongly consistent.

## 2. LIMIT DISTRIBUTIONS

Let us establish

**Proposition 1**

If $\mathcal{V}_n^r$ has limit distribution $\overline{F}(v^r)$, $\mathcal{V}^r \sim \overline{F}(v^r)$, and $\ell(\mathcal{V}^r)$ is continuous, the limit distribution of $\ell(V_n^r)$ is the distribution of $\ell(V^r)$.

*Proof:*

A direct application of the HELLY-BRAY lemma shows that the characteristic function of $\ell(\vec{\mathcal{V}}_n^r)$ converges to the characteristic function of $\ell(\vec{\mathcal{V}}^r)$, thus the thesis follows.

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Constant vectors may be considered as degenerate random vectors that are independent from any other random vector. Let $\mathcal{P}(x^t)$ be the degenerate $t$ dimension distribution with all the probability concentrated in the origin, then $a^t \sim \mathcal{P}(x^t - a^t)$, and, if $\mathcal{V}^r \sim \overline{F}(v^r)$, the joint distribution of the pair $(\mathcal{V}^r, a^t)$ will be $\overline{F}(v^r) \cdot \mathcal{P}(x^t - a^t)$. If $\mathcal{V}_n^r$ has limit distribution $F$ and $\mathcal{X}_n^t \xrightarrow{p} a^t$, the limit distribution of the pair $(\mathcal{V}_n^r, \mathcal{X}_n^t)$ will be $\overline{F}(v^r) \cdot \mathcal{P}(x^t - a^t)$. We now prove

**Proposition 2**

The limit distribution of $\mathcal{U}_n$ will be $\overset{\circ}{F}(u)$, the distribution of $\mathcal{U}$.

*Proof:*

Since $\mathcal{Z}_n^k$ has limit distribution $F(z^k|\gamma^t)$ and $\tilde{\gamma}_n^t \xrightarrow{p} \gamma^t$ the limit distribution of the pair $(\mathcal{Z}_n^k, \tilde{\gamma}_n^t)$ will be $F(z^k|\gamma^t) \cdot \mathcal{P}(x^t - \gamma^t)$. Thus, according to *proposition 1*, the limit distribution of $\mathcal{U}_n$ will be the distribution of $\mathcal{U} = g(\mathcal{Z}^k, \gamma^t)$ with $\mathcal{Z}^k \sim F(z^k|\gamma^t)$. Since $\mathcal{U} \sim \overset{\circ}{F}(u)$ the thesis is established.

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## 3. TESTS AND CONDIFENCE REGIONS

Since $\mathcal{U}_n$ has limit distribution $\overset{\circ}{F}(u)$, if $u_p$ is the quantile for probability $p$ of $\overset{\circ}{F}(u)$, we will have $\mathrm{pr}(\mathcal{U}_n \leq u_{1-q}) \rightarrow 1 - q$, thus

$$H = \{\alpha^k | g(h(n))(\alpha_n^k - \alpha^k); \tilde{\gamma}_n) \leq u_{1-q}\} \tag{1}$$

may be considered as a $1 - q$ *limit level confidence region* for $\alpha^k$.

When $H_0$ holds, with $\mathcal{Z}_{n,0}^k = h(n)(\tilde{\alpha}_n^k - \alpha_0^k)$ and $\mathcal{U}_{n,0} = g(\mathcal{Z}_{n,0}^k, \tilde{\gamma}_n^t)$ we will have $\mathcal{Z}_{n,0}^k = \mathcal{Z}_n^k$, $\mathcal{U}_{n,0} = \mathcal{U}_n$ and $\mathrm{pr}(\mathcal{U}_{n,0} \leq u_{1-q}) \rightarrow 1 - q$. Thus with $\mathcal{U}_{n,0}$ as *test statistic* and $\mathrm{Rej} = ]u_{1-q}; +\infty[$ as *rejection region* we will have a *$q-$limit level test.*

This test is associated with the confidence region since $H_0$ is not rejected, by the $q-$limit level test, if and only if $\alpha_0^k$ belongs to the $1 - q$ limit level confidence region.

The existence of associated confidence regions points towards these tests having duality properties.

## 4. NORMS, SEMI-NORMS AND STRONG CONSISTENCY

Given $w^t, g(z^k, w^t)$ reduces to a function $\overline{g}(z^k|w^k)$ of $z^k$. With $E_n$ the event that occurs when

$$\ell(z^k|\tilde{\gamma}_n^t) = \sqrt{\overline{g}(z^k|\tilde{\gamma}_n^t)} \tag{2}$$

is a semi-norm, let us establish

**Lemma 1**

If $\ell(z^k|\gamma^t)$ is a norm and $\mathrm{pr}(E_n) \to 1$, whenever $\alpha^k \neq \alpha_0^k$ and $K_n h(n)^{-1} \to 0$, we have $\mathrm{pr}(\ell(\mathcal{Z}_{n,0}^k - \mathcal{Z}_n^k|\tilde{\gamma}_n^t) > K_n) \to 1$.

*Proof:*

When $E_n$ occurs,

$$\ell(\mathcal{Z}_{n,0}^k - \mathcal{Z}_n^k|\tilde{\gamma}_n^t) = \ell(h(n)(\alpha^k - \alpha_0^k|\tilde{\gamma}_n^t) = h(n)\ell(\alpha^k - \alpha_0^k|\tilde{\gamma}_n^t)$$

and, when $\alpha^p \neq \alpha_0^k$, $\ell(\alpha^k - \alpha_0^k|\tilde{\gamma}_n^t) \xrightarrow{p} \ell(\alpha^k - \alpha_0^k|\tilde{\gamma}^t) > 0$. Thus, to complete the proof, we have only to point out that $\mathrm{pr}(E_n) \to 1$ and $K_n h(n)^{-1} \to 0$.

This lemma enables us to prove

**Proposition 3**

Under the conditions of Lemma 1 we have $\mathrm{pr}(\mathcal{U}_{n,0} > K_n^2) \to 1$.

*Proof:*

Since $\ell(\mathcal{Z}_{n,0}^k|\tilde{\gamma}_n^t) = \sqrt{\mathcal{U}_{n,0}}$ we have only to show that $\mathrm{pr}(\ell(\mathcal{Z}_{n,0}^k|\tilde{\gamma}_n^t) > K_n) \to 1$. From propositions 1 and 2 we see that $\ell(\mathcal{Z}_n^k|\tilde{\gamma}_n^t)$ has as limit distribution the distribution of $\sqrt{\mathcal{U}}$ with $\mathcal{U} \sim \overset{o}{F}(u)$. When $E_n$ occurs,

$$\ell(\mathcal{Z}_{n,0}^k - \mathcal{Z}_n^k|\tilde{\gamma}_n^t) - \ell(\mathcal{Z}_n^t|\tilde{\gamma}_n^t) \leq \ell(\mathcal{Z}_n^k|\tilde{\gamma}_n^t)$$

It is now easy to use Lemma 1 to complete the proof.

When $K_n \to \infty, q_n = \overset{o}{F}(u)(K_n^2) \to 1$, then, if the conditions in Lemma 1 hold, we get, for any alternative, rejection with limit probability 1 while the first type error tends to 0. Following TIAGO DE OLIVEIRA (1980) and (1982), we say that, then, the tests will be *strongly consistent.*

Let us now consider the confidence regions associated with these tests. According to propositions 1 and 2 the limit distribution of $\mathcal{V}_n = \ell(\mathcal{Z}_n^k|\tilde{\gamma}_n^t)$ will be the distribution of $\sqrt{\mathcal{U}}$ with $\mathcal{U} \sim \overset{o}{F}(u)$. Then, with $K_n h(n)^{-1} \to 0$ and $K_n \to \infty$ we will have $\mathrm{pr}(\mathcal{V}_n \leq K_n) \to 1$. Now, when $E_n$ occurs, $\mathcal{V}_n = h(n)\ell(\tilde{\alpha}_n^k - \alpha^k|\tilde{\gamma}_n^t)$ and, since $\mathrm{pr}(E_n) \to 1$.

$$\text{(3)} \qquad \text{pr}(\ell(\alpha_n^k - \alpha^k|\tilde{\gamma}_n^t) \leq \frac{K_n}{h(n)} \xrightarrow[n\to\infty]{} 1$$

thus we will have a limit level 1 confidence region for $\alpha^k$ whose size "shrinks to zero".

## 5. THE NORMAL CASE

If $\mathcal{Z}^k$ has the normal distribution with mean vector $\eta^k$ and variance-covariance matrix $\mathcal{M}$ with characteristic $k'$ and MOORE-PENROSE inverse $\mathcal{M}^g$,

$$\text{(4)} \qquad \mathcal{U} = (\mathcal{Z}^k - \eta^k)^T \mathcal{M}^g (\mathcal{Z}^k - \eta^k)$$

will be a continuous pivot variable since, see MEXIA (1990), it is a central chi-square with $k'$ degrees of freedom. If $\eta^k$ is null and $\mathcal{M}$ is regular, we will have $\mathcal{U} = \mathcal{Z}^T \mathcal{M}^{-1} \mathcal{Z}$ and, with $\mathcal{M}^{-1} = [m^{i,j}]$, we can take as components of $\gamma^t$ the $m^{i,j}$ with $i \leq j$. Let $\mathcal{W}(v^t)$ be the matrix thus reconstructed from the components of $v^t$.

Then, with $g(z^k, v^t) = z^T \mathcal{W}(v^t) z$, $\ell(z^k|v^t) = \sqrt{g(z^k|v^t)}$ will be a semi-norm [norm] whenever $\mathcal{W}(v^t)$ is positive semi-definite [definite]. We point out that, see WILKS (1961, pp. 80 to 82), regular variance-covariance matrices are, as well as their inverses, positive definite. Let us establish

### Proposition 4

If $\mathcal{W}(\gamma^t)$ is positive definite, $\text{pr}(E_n) \to 1$.

*Proof:*

Given $\mathcal{A} = [a_{ij}]$ a $k \times k$ matrix and $\Delta(\mathcal{A}) = \sum_{i=1}^{k}\sum_{j=1}^{k} |a_{i,j}|$ it is easy to see that $|z^T \mathcal{A} z| \leq \Delta(\mathcal{A}) \left\|z^k\right\|^2$, $\left\|z^k\right\|$ being the euclidean norm of $z^k$. The smallest eigenvalues of $\mathcal{W}(\tilde{\gamma}_n^t)$ and $\mathcal{W}(\gamma^t)$ will be $c_n = \min\left\{z^T \mathcal{W}(\tilde{\gamma}_n^t) z \middle| \left\|z^k\right\| = 1\right\}$ and $c = \min\left\{z^T \mathcal{W}(\gamma^t) z \middle| \left\|z^k\right\| = 1\right\}$. Thus, with $d_n = \Delta(\mathcal{W}(\tilde{\gamma}_n^t) - \mathcal{W}(\gamma^t))$, we will have $|c_n - c| \leq \max\left\{|z^T \mathcal{W}(\tilde{\gamma}_n^t) z - z^T \mathcal{W}(\gamma^t) z| \middle| \left\|z^k\right\| = 1\right\} \leq d_n$. Now the elements of $\mathcal{W}(\tilde{\gamma}_n^t)$ and $\mathcal{W}(\gamma^t)$ are components of $\tilde{\gamma}_n^t$ and $\gamma^t$ and, since $\tilde{\gamma}_n^t \xrightarrow{p} \gamma^t$, $d_n \xrightarrow{p} 0$, thus $c_n \xrightarrow{p} c$, with $c > 0$, since $\mathcal{W}(\gamma^t)$ is positive definite. When $c_n > 0, \mathcal{W}(\tilde{\gamma}_n^t)$ is positive definite and $E_n$ occurs so that we will have $\text{pr}(E_n) \to 1$.

In many instances the conditions here required are met, see AMEMYA (1985, pp. 105 to 121), by *extremum estimators.* These estimators are obtained maximizing or minimizing a function defined over the parameter space. Maximum likelihood and least square estimators belong to this class. We also point out that the tests that are obtained, in this way, for the normal case include the Wald and Rao score tests, see AMEMYA (1985, p. 142). Thus these tests will be strongly consistent.

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