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A NEW GEOMETRIC APPROACH TO BIMATRIX GAMES[1]

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In this paper we study some properties concerning the equilibrium points of a bimatrix game and describe a geometric method to obtain all the equilibria of a bimatrix game when one of the players has at most three pure strategies.

Keywords: Bimatrix Games, Nash Equilibria, Geometric Methods.

1. INTRODUCTION

In Borm *et al.* (1988), a geometric method for finding all equilibria in a $2 \times n$ (or $m \times 2$) bimatrix game is introduced. It is inspired in the double description method for $2 \times n$ matrix games described in Motzkin *et al.* (1953). As they argued, geometric methods are useful because they permit to visualize the situation that they solve, so one can have new views of the problem in question. Moreover, $2 \times n$ bimatrix games are quite important, at least from a theoretical point of view, because they provide many examples and counter examples of several facts. In this paper we describe a new version of the method, now based on a geometric interpretation of von Neumann's minimax theorem suggested in Luce and Raiffa (1957). It somewhat gives a dual version of the algorithm, which could inspire new ideas through the different visualization of the problem. Besides, our version can be also used to find the equilibria of $3 \times n$ bimatrix games.

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Notation

For every $n > 0$ we write Δ_n for the set

$$\left\{(x_1,\ldots,x_n) \in \mathbb{R}^n \Big/ x_i \geq 0 \quad \forall i \in \{1,\ldots,n\}, \sum_{i=1}^{n} x_i = 1\right\}$$

and $\{e_i / i \in \{1,\ldots,n\}\}$ for the canonical basis of $\mathbb{R}^n$. Given a subset A of $\mathbb{R}^n$, we denote by $\mathrm{conv}(A)$ its convex hull, by $\mathrm{Cl}(A)$ its closure and by $\mathrm{WPF}(A)$ its weak Pareto frontier, i.e.

$$\mathrm{WPF}(A) = \{(x_1,\ldots,x_n) \in A \,/\, \nexists\; (\overline{x}_1,\ldots,\overline{x}_n) \in A \text{ s.t. } \overline{x}_i > x_i \;\forall i \in \{1,\ldots,n\}\}\,.$$

Throughout this paper we identify every $x \in \mathbb{R}^n$ with an $n \times 1$ matrix. For any matrix A, we denote its transposed by A^t.

2. SOME RESULTS CONCERNING BIMATRIX GAMES

We devote the first part of this section to establish the notation and derive some useful results concerning bimatrix games. Then we describe an algorithm to find the set of equilibria of some special bimatrix games.

An $m \times n$ bimatrix game (A, B), A and B being $m \times n$ real matrices, is a two-person normal form game $(\Delta_m, \Delta_n, H_1, H_2)$, where Δ_m, Δ_n are the strategy spaces of player one and two respectively and H_1, H_2 are their payoff functions defined by

$$H_1(p,q) = p^t A q,\; H_2(p,q) = p^t B q.$$

For every $q \in \Delta_n$ we call $B_1(q)$ to the set of best replies of player one to q i.e.

$$B_1(q) = \left\{\hat{p} \in \Delta_m \Big/ \hat{p}^t A q = \max_p p^t A q\right\}.$$

Analogously we can define $B_2(p)$ for every $p \in \Delta_m$. A Nash equilibrium is a $(p,q) \in \Delta_m \times \Delta_n$ such that $p \in B_1(q)$ and $q \in B_2(p)$. We denote by $E(A,B)$ the set of Nash equilibria of (A,B).

Now consider the map g defined from Δ_n onto $S := \mathrm{conv}\{Be_1,\ldots,Be_n\} \subset \mathbb{R}^m$ given by $g(q) = Bq$. Analogously we can define f from Δ_m onto

$R := \text{conv}\{e_1^t A, \ldots, e_m^t A\} \subset \mathbb{R}^n$ by $f(p) = p^t A$. Next, we prove the following theorem.

Theorem 2.1

Let (A, B) be an $m \times n$ bimatrix game. Then, given $p \in \Delta_m$ and $q \in \Delta_n$,

a) $q \in \bigcup_{p \in \Delta_m} B_2(p) \Leftrightarrow g(q) \in \text{WPF}(S)$

b) $p \in \bigcup_{q \in \Delta_n} B_1(q) \Leftrightarrow f(p) \in \text{WPF}(R)$

Proof

We only demonstrate a). b) could be done analogously.

It is clear that, if $q \in B_2(p)$ for some $p \in \Delta_m$, then it is not strictly dominated and hence $g(q) \in \text{WPF}(S)$.

Reciprocally, let us consider $q \in \Delta_n$ such that $g(q) \in \text{WPF}(S)$. Take $T_q = \{x \in \mathbb{R}^m \, / x_i > e_i^t g(q) \, \forall i \in \{1, \ldots, m\}\}$. As $g(q) \in \text{WPF}(S)$, it is clear that $T_q \cap S = \emptyset$. Moreover, as S and T_q are convex sets, we can apply a separation theorem and then consider and hyperplane (u, c) such that $u^t x \geq c \, \forall x \in \text{Cl}(T_q)$ and $u^t x \leq c \, \forall x \in S$. Note that $g(q) \in \text{Cl}(T_q) \cap S$ and hence $u^t g(q) = c$. Now taking $g(q)+e_i \in \text{Cl}(T_q)$ for every i, we know that $u^t(g(q)+e_i) \geq c = u^t g(q)$ and then $u_i \geq 0$. Consequently, as $u \neq 0$, $\sum_{i=1}^{n} u_i > 0$ and hence $p = \left(\sum_{i=1}^{n} u_i\right)^{-1} \cdot u \in \Delta_m$. Moreover, it is clear that $q \in B_2(p)$. This fact completes the proof.

■

Let us give now some more definitions. For every $q \in \Delta_n$ we call $PB_1(q)$ to the set of pure best replies of player 1 to q, i.e. $PB_1(q) = B_1(q) \cap \{e_1, \ldots, e_m\}$, and $C(q)$ to the carrier of q, i.e. $C(q) = \{j \in \{1, \ldots, n\} \, / q_j > 0\}$. Analogously we define $PB_2(p)$ and $C(p)$ for every $p \in \Delta_m$.

Next, consider the following two correspondences, both defined from $g^{-1}(\text{WPF}(S)) := \{q \in \Delta_n \, / g(q) \in \text{WPF}(S)\}$, given by

$$\begin{aligned} V(q) &= \{i \in \{1, \ldots, m\} \, / e_i \in PB_1(q)\} \\ v(q) &= \{p \in \Delta_m \, / q \in B_2(p)\} . \end{aligned}$$

Observe that, from theorem 2.1, $v(q) \neq \emptyset \, \forall q \in g^{-1}(\mathrm{WPF}(S))$ and then v is well defined. Analogously we can consider U and u defined from $f^{-1}(\mathrm{WPF}(R))$ and given by

$$\begin{aligned} U(p) &= \{j \in \{1, \ldots, n\} \, / e_j \in PB_2(p)\} \\ u(p) &= \{q \in \Delta_n \, / p \in B_1(q)\} \, . \end{aligned}$$

Then we can state the following result which is an immediate consequence of theorem 2.1 and the definition of Nash equilibrium.

Theorem 2.2

For every $(p, q) \in \Delta_m \times \Delta_n$, the next three statements are equivalent:

a) (p, q) is a Nash equilibrium.

b) $g(q) \in \mathrm{WPF}(S), p \in v(q)$ and $C(p) \subset V(q)$.

c) $f(p) \in \mathrm{WPF}(R), q \in u(p)$ and $C(q) \subset U(p)$.

Now, from theorem 2.2 above we can describe the following method to find all equilibria in an $m \times n$ bimatrix game:

Step 1. Obtain the set $Q := g^{-1}(\mathrm{WPF}(S))$.

Step 2. (For all $q \in Q$). Compute $V(q)$.

Step 3. (For all $q \in Q$). Compute $v(q)$ and check, for any $p \in v(q)$, if $C(p) \subset V(q)$. In such a case, (p, q) is a Nash equilibrium.

Remark 2.1

As Q is in general an infinite set, step 2 apparently has infinitely many stages. However, Q is a finite union of sides of a polytope whose vertices $x_1, \ldots, x_v$ correspond to pure strategies of player two. Moreover, as H is a continuous function, there is a finite number of regions $R_1, \ldots, R_r$ in Q (these regions being the intersection of Q with a finite number of hyperplanes and/or a finite number of open semi spaces), such that, for every $R_i, V(q) = V(q') \, \forall q, q' \in R_i$. Then, step 2 has actually a finite number of stages.

Remark 2.2

It is not easy to find $v(q)$ for a $q \in Q$. We can act as in the proof of theorem 1 finding separation hyperplanes between Q and T_q and then obtaining $v(q)$ from their normal vectors. But to do that, we need to visualize Q. Hence this method seems to be useful only for $2 \times n$ or $3 \times n$ bimatrix games. By other side, as Q is in general an infinite set, again step 3 apparently has infinitely many stages. However, as Q is a finite union of sides of a polytope, we only have to analyze separately the relative interior of every side, the relative interior of every intersection of two sides, and so on. Then we have to make only a finite number of investigations. Moreover, although $v(q)$ could be an infinite set for some q, it is a convex set with a finite number of extreme points and then to find all the $p \in v(q)$ such that $C(p) \subset V(q)$ is a finite process.

Remark 2.3

From remarks 2.1 and 2.2, it is clear that the method we are presenting is useful to find all the equilibria of an $m \times n$ bimatrix game with $m \leq 3$. Obviously, we can reformulate the algorithm in such a way that it is useful for $m \times n$ bimatrix games with $n \leq 3$. It would be like this:

Step 1. Obtain the set $P := f^{-1}(\mathrm{WPF}(R))$.

Step 2. (For all $p \in P$). Compute $U(p)$.

Step 3. (For all $p \in P$). Compute $u(p)$ and check, for any $q \in u(p)$, if $C(q) \subset U(p)$. In such a case, (p, q) is a Nash equilibrium.

3. APPLICATION TO 2×N BIMATRIX GAMES

In this section we develop our method for the special case of $2 \times n$ bimatrix games. Analogously it could be developèd for $3 \times n, m \times 2$ and $m \times 3$ bimatrix games.

Consider a $2 \times n$ bimatrix game $\Gamma = (\Delta_2, \Delta_n, A, B)$. It is clear that $\mathrm{WPF}(S)$ is a finite union of sets of the form $\mathrm{conv}(\{Be_r, Be_g\}) \subset \mathbb{R}^2$. Suppose that all columns in B are different. Then Q is a finite union of convex hulls of pairs of pure strategies of player two. Now, for every $e_j \in Q, V(e_j) = \{1\}$ if $a_{1j} > a_{2j}, V(e_j) = \{2\}$ if $a_{1j} < a_{2j}$ and $V(e_j) = \{1, 2\}$ if $a_{1j} = a_{2j}$. For any other $q \in Q$, there exists an only pair e_r, e_g such that $q = \alpha e_r + (1 - \alpha)e_g$, being $\alpha \in [0, 1]$. In such a case, one and only one of the three following cases is verified:

1. $V(e_r) = V(e_g)$. Then $V(q) = V(e_r) = V(e_g)$.

2. $V(e_r) \neq V(e_g)$ and $V(e_r)$ or $V(e_g)$ are equal to $\{1,2\}$. Then $V(q)$ is equal to $V(e_g)$ or $V(e_r)$ respectively.

3. $V(e_r) \neq V(e_g)$ and both are different from $\{1,2\}$. Then $V(q) = V(e_r)$ if $\alpha > \beta := (a_{2g} - a_{1g})/(a_{1r} - a_{1g} - a_{2r} + a_{2g})$, $V(q) = V(e_g)$ if $\alpha < \beta$ and $V(q) = \{1,2\}$ if $\alpha = \beta$.

If B has some identical columns, then some $q \in Q$ can be a convex combination of more than two columns. Apart from this, the process of finding $V(q)$ is completely similar. Now, to compute $v(q)$ for any q, we must look for the separation hyperplane(s) between Q and T_q. Note that this is an easy (and finite) task. Observe that $v(q)$ will contain e_1 if and only if $e_1^t g(q) \geq e_1^t g(q') \forall q' \in \Delta_n$ and that $v(q)$ will contain e_2 if and only if $e_2^t g(q) \geq e_2^t g(q'), \forall q' \in \Delta_n$. Finally, according to theorem 2.2, to obtain all the Nash equilibria of the game, we have to find all the (p,q) such that $q \in Q, p \in v(q)$ and $C(p) \subset V(q)$. Then, in view of all we have seen in this section, we can state the following theorem.

Theorem 3.1

$(p,q) \in \Delta_2 \times \Delta_n$ is a Nash equilibrium of Γ if and only if at least one of the following statements is fulfilled:

- $q \in Q$, $e_1^t g(q) \geq e_1^t g(q') \forall q' \in \Delta_n$, $V(q) \supset \{1\}$ and $p = e_1$.
- $q \in Q$, $e_2^t g(q) \geq e_2^t g(q') \forall q' \in \Delta_n$, $V(q) \supset \{2\}$ and $p = e_2$.
- $q \in Q$, $V(q) = \{1,2\}$ and $p \in v(q)$.

To finalize, we make an example to illustrate the method.

Example 3.1

We are going to compute the set of Nash equilibria of the following 2×5 bimatrix game.

$$(A,B) = \begin{pmatrix} (3,6) & (0,6) & (5,4) & (2,3) & (5,5) \\ (0,0) & (2,1) & (5,5) & (3,2) & (0,4) \end{pmatrix}$$

In Figure 1 below, we have drawn S and have marked its weak Pareto frontier WPF(S) with a bold trace.

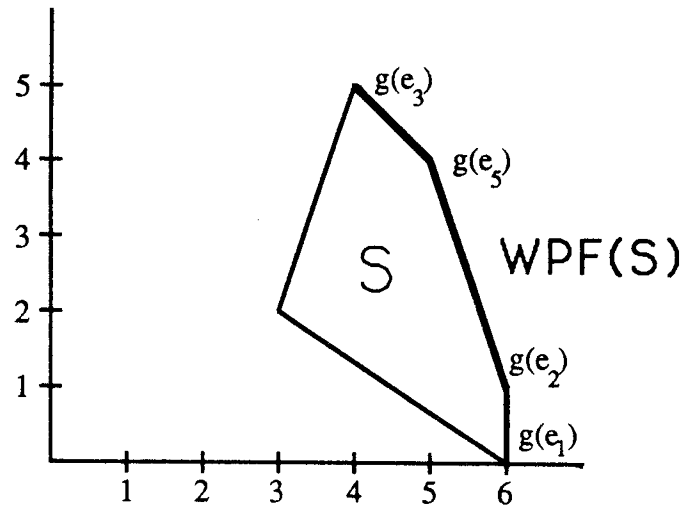

Figure 1.

It is clear from the picture that

$$Q = \text{conv}(\{e_3, e_5\}) \cup \text{conv}(\{e_5, e_2\}) \cup \text{conv}(\{e_2, e_1\}).$$

Now, we can determine $V(q)$ for all $q \in Q$.

$$\begin{aligned}
&V(e_3) = \{1, 2\} \\
&V(\lambda e_3 + (1-\lambda)e_5) = \{1\} \quad (\lambda \in [0,1)) \\
&V(\lambda e_5 + (1-\lambda)e_2) = \{1\} \quad (\lambda \in (2/7, 1]) \\
&V(2/7 \cdot e_5 + 5/7 \cdot e_2) = \{1, 2\} \\
&V(\lambda e_5 + (1-\lambda)e_2) = \{2\} \quad (\lambda \in [0, 2/7)) \\
&V(\lambda e_2 + (1-\lambda)e_1) = \{2\} \quad (\lambda \in (3/5, 1]) \\
&V(3/5 \cdot e_2 + 2/5 \cdot e_1) = \{1, 2\} \\
&V(\lambda e_2 + (1-\lambda)e_1) = \{1\} \quad (\lambda \in [0, 3/5))
\end{aligned}$$

Then, in view of theorem 3.1, the set of equilibria of this game is the union of the next three sets:

$$\{(p, e_3)/p \in v(e_3)\}$$

$$\{(p, 2/7 \cdot e_5 + 5/7 \cdot e_2)/p \in v(2/7 \cdot e_5 + 5/7 \cdot e_2)\}$$

$$\{(e_1, q)/q \in \text{conv}(\{e_1, 3/5 \cdot e_2 + 2/5 \cdot e_1\})\}.$$

The straight line joining $g(e_3)$ and $g(e_5)$ is $x + y = 9$ and then $v(e_3) = \text{conv}(\{e_2, 1/2 \cdot e_1 + 1/2 \cdot e_2\})$. Besides, the straight line joining $g(e_5)$ and $g(e_2)$ is $3x + y = 19$ and then $v(2/7 \cdot e_5 + 5/7 \cdot e_2) = \{3/4 \cdot e_1 + 1/4 \cdot e_2\}$. Hence, $E(A, B)$ is the union of the next three sets:

$$\{(p, e_3)/p \in \text{conv}(\{e_2, 1/2 \cdot e_1 + 1/2 \cdot e_2\})\}$$

$$\{(3/4 \cdot e_1 + 1/4 \cdot e_2, 2/7 \cdot e_5 + 5/7 \cdot e_2)\}$$

$$\{(e_1, q)/q \in \text{conv}(\{e_1, 3/5 \cdot e_2 + 2/5 \cdot e_1\})\}.$$

4. REFERENCES

[1] **P. Borm, A. Gijsberts** and **S.H. Tijs** (1988). "A Geometric Combinatorial Approach to Bimatrix Games". *Methods of Operations Research,* **59**.

[2] **R.D. Luce** and **H. Raiffa** (1957). *Games and Decisions.* Wiley. New York.

[3] **T. Motzkin, H. Raiffa, G. Thompson** and **R. Thrall** (1953). "The Double Description Method". In: H.W. Kuhn and A.W. Tucker, eds., Contributions to the Theory of Games II, *Annals of Mathematics Studies,* **28**, 51–73.