Publ. Mat. **54** (2010), 317–339

NICE ELONGATIONS OF PRIMARY ABELIAN GROUPS

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Abstract

Suppose N is a nice subgroup of the primary abelian group G and $A = G/N$. The paper discusses various contexts in which G satisfying some property implies that A also satisfies the property, or visa versa, especially when N is countable. For example, if n is a positive integer, G has length not exceeding ω_1 and N is countable, then G is n-summable iff A is n-summable. When A is separable and N is countable, we discuss the condition that any such G decomposes into the direct sum of a countable and a separable group, and we show that it is undecidable in ZFC whether this condition implies that A must be a direct sum of cyclics. We also relate these considerations to the study of nice bases for primary abelian groups.

0. Introduction and Terminology

By the term "group" we will mean an abelian p-group, where p is a prime fixed for the duration. Our group theoretic terminology and notation will generally follow [**11**]. We will say a group G is Σ-*cyclic* if it is isomorphic to a direct sum of cyclic groups. We will also utilize the language of *valuated groups* and *valuated vector spaces* (see [**22**] and [**12**], respectively); for example, if Y is a valuated group, the letter v will be reserved for its valuation, and for any ordinal α, by $Y(\alpha)$ we will mean the subgroup $\{y \in Y : v(y) \geq \alpha\}$. We will implicitly assume that all valuated vector spaces are over $\mathbb{Z}_p$.

If $0 \to X \to G \to A \to 0$ is a short exact sequence, we will routinely identify X with an actual subgroup of G and A with the quotient group G/X; we will say that G is an *elongation* of A by X. We then say that G is a *nice-elongation* of A if X is a nice subgroup (i.e., for every $y \in G$, the coset $y + X$ has an element of maximum height). Further,

2000 *Mathematics Subject Classification.* 20K10.
Key words. Nice subgroups, elongations, abelian groups, ω_1-homomorphisms.

if X is countable, then we will say G is an $\aleph_0$-*elongation* of A by X. We can clearly combine these and speak of *nice*-$\aleph_0$-*elongations*.

The main purpose of this paper is to investigate how certain properties of groups are preserved under nice-$\aleph_0$-elongations, though we will occasionally prove results that extend beyond the countable case. For example, if $\lambda \leq \omega_1$ is an ordinal, then a group G is a C_λ-*group* if for every $\alpha < \lambda$, if H is a p^α-high subgroup of G (i.e., H is maximal with respect to the property that $H \cap p^\alpha G = \{0\}$), then H is a dsc-group (i.e., a *direct sum of countable groups*). We verify that if G is a nice-$\aleph_0$-elongation of A, then G is a C_λ-group iff A is a C_λ-group (Corollary 2.4).

In another direction, a valuated group Y will be said to be *Honda* if it is the ascending union of subgroups X_m whose value spectra $\{v(x) : x \in X_m\}$ are finite (i.e., they are *value-finite*). Note that if Y is actually countable, then it is clearly Honda, since it is the ascending union of finite subgroups.

If G is a group with a subgroup Y, then restricting the height function on G to Y gives a valuation on Y. A group G is *summable* if $G[p]$ is isometric to a free valuated vector space. More generally, following [**8**], if n is a positive integer, a reduced group G is n-*summable* if $G[p^n]$ is isometric to a valuated direct sum of countable valuated groups, and G is n-*Honda* if $G[p^n]$ is Honda as a valuated group. In [**8**] it was shown that if G is n-summable, then (a) it is summable, so that $p^{\omega_1}G = \{0\}$ (see, for example, [**11**, Theorem 84.3]); (b) the valuated group $G[p^n]$ is determined up to isometry by its Ulm function; (c) G is n-Honda iff it is n-summable and has countable length (i.e., $p^\alpha G = \{0\}$ for some countable α). In fact, these results were proven in the category of *valuated* p^n-*socles* (which extends the idea of a valuated vector space). Note that (c) generalizes the classical criterion (due to Honda) for the summability of a group of countable length (see, for example, [**11**, Theorem 84.1]).

If G is a nice-elongation of A by N, we show that if N is countable and G is n-summable, then A is also n-summable (Theorem 1.1); and conversely, if N is Honda (as a valuated group using the height valuation from G), $p^{\omega_1}G = \{0\}$ and A is n-summable, then G is n-summable (Theorem 1.2). It follows that if G is a nice-$\aleph_0$-elongation of A and $p^{\omega_1}G = \{0\}$, then G is n-summable iff A is n-summable (Corollary 1.3).

If A is a separable group and G is an elongation of A by X, then X will always be nice in G, so that an elongation of A is always a nice-elongation. Next, if $X = p^\omega G$ (so that $A \cong G/p^\omega G$ is separable), we will say that G is an ω-*elongation* of A (this terminology agrees with [**20**]), so that any ω-elongation is also a nice-elongation; and if $X = p^\omega G$ is also countable, we will say G is an ω-$\aleph_0$-*elongation* of A.

The group G will be said to be *countable plus separable* (or cps for short), if it is isomorphic to $C \oplus S$, where C is countable and S is separable. Note that if $G = D \oplus R$, where D is divisible and R is reduced, then G is cps iff D is countable and R is cps; we may occasionally, therefore, assume that some cps group is actually reduced. We will say that the separable group A has the *cps-elongation property* if whenever the group G is a (nice-)$\aleph_0$-elongation of A, then G is cps; similarly we will say A has the *ω-cps-elongation property* if whenever the group G is an ω-$\aleph_0$-elongation of A, then G is cps. It is straightforward to show that any Σ-cyclic group has the cps-elongation property and that any group with the cps-elongation property has the ω-cps-elongations property (Proposition 3.1); the natural question is whether a group with one of these latter two properties must actually be Σ-cyclic. We show that these questions are independent of the standard set-theoretic axioms (ZFC) by showing that they are consequences of the axiom of constructibility ($V = L$), but that counter-examples can be constructed using Martin's Axiom and the denial of the Continuum Hypothesis (MA + $\neg$CH, Theorem 3.4).

We then apply these results to question of when a group G has a *nice basis*; that is, an ascending sequence of nice subgroups $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$ whose union is all of G such that each X_m is Σ-cyclic (see, for example, [**2**]). In particular, we note that MA + $\neg$CH implies that there is separable group A which is not Σ-cyclic with the property that whenever G is an ω-$\aleph_0$-elongation of A by X, then G has a nice basis (Example 4.7).

Referring to [**6**], a group homomorphism $\varphi\colon G \to A$ is said to be *ω_1-bijective* if both the kernel and the co-kernel of φ are countable. Thus, if there exists an ω_1-bijective homomorphism $\varphi\colon G \to A$, then $G/\ker\varphi \cong \varphi(G)$ and G is an $\aleph_0$-elongation of $\varphi(G)$ by $\ker\varphi$. In particular, if $\ker\varphi$ is a countable nice subgroup of G and φ is surjective, then G is a nice-$\aleph_0$-elongation of A; so there is a natural connection between these two concepts. We also establish some results pertaining to ω_1-bijective homomorphisms of special classes of abelian groups (e.g., Proposition 3.6 and Theorem 5.6).

1. *n*-Summable Groups

A subgroup X of a valuated group Y is *nice* if every coset $y + X$ has an element of maximal value; such a y is called *proper with respect to X*; and in this case, letting $v(y + X) = v(y)$ makes Y/X into a valuated group.

Theorem 1.1. *Suppose n is a positive integer and G is a nice-$\aleph_0$-elongation of A by N. If G is n-summable, then A is n-summable.*

Proof: Fix a valuated decomposition $G[p^n] = \oplus_{i\in I} C_i$, where each C_i is a countable valuated group. We claim there is a countable subset $J \subseteq I$ such that:

(a) $N[p^n] \subseteq \oplus_{i\in J} C_i$;

(b) $(N + \oplus_{i\in J} C_i) \| G[p^n]$, that is, for every $x \in N + \oplus_{i\in J} C_i$ and $y \in G[p^n]$ there is a $z \in (N + \oplus_{i\in J} C_i) \cap G[p^n] = \oplus_{i\in J} C_i$ such that $ht_G(x + z) \geq ht_G(x + y)$.

To begin, let J_0 be a countable subset of I such that $N[p^n] \subseteq \oplus_{i\in J_0} C_i$. If we have constructed a countable subset J_k of I, we want to construct another countable subset J_{k+1} of I containing J_k such that:

(c) for every $x \in N + \oplus_{i\in J_k} C_i$ and $y \in G[p^n]$ there is a $z \in \oplus_{i\in J_{k+1}} C_i$ such that $ht_G(x + z) \geq ht_G(x + y)$.

For a second, fix x to be an element of the countable subgroup $N + \oplus_{i\in J_k} C_i$. We claim that there is countable sequence of elements $y_{x,\ell}$ of $G[p^n]$ such that

$$\sup\{ht_G(x + y) : y \in G[p^n]\} = \sup\{ht_G(x + y_{x,\ell}) : \ell < \omega\}.$$

This (essentially well-known) fact is a consequence of the assumption that $G[p^n]$ is a valuated direct sum of countable valuated groups and such an object will always be complete in the (induced) ω_1-topology: To see this, note that if it failed, we could, for all $\alpha < \omega_1$, choose $y_\alpha \in G[p^n]$ such that $ht(x + y_\alpha) \geq \alpha$. It follows that if $\alpha < \alpha' < \omega_1$, then $ht(y_\alpha - y_{\alpha'}) \geq \alpha$. If for each $\alpha < \omega_1$, $y_\alpha = (z_{\alpha,i})_{i\in I} \in \oplus_{i\in I} C_i$, then $z_{\alpha,i}$ must eventually be constant for each $i \in I$; say it takes on the value w_i. We next verify that all but finitely many of the w_i must be 0: If this failed, and $\omega_1 > \alpha > ht(w_i)$ for an infinite number of indices i, it would easily follow that $z_{\alpha,i} \neq 0$ for this same infinite set of indices i, which cannot be. It follows that $w = (w_i)_{i\in I}$ is in $\oplus_{i\in J} C_i$, and that for all $\alpha < \omega_1$, $ht(x + w) = ht(x + y_\alpha + w - y_\alpha) \geq \alpha$, so that $x + w = 0$; which means that we could let $y_{x,\ell} = w$ for all $\ell < \omega$.

We then let J_{k+1} be a countable subset of I containing J_k such that $y_{x,\ell} \in \oplus_{i\in J_{k+1}} C_i$ for all $x \in N + \oplus_{i\in J_k} C_i$ and $\ell < \omega$.

If we then let $J = \cup_{k<\omega} J_k$, then (a) is immediate and (b) follows from (c).

Let $L = I - J$. If x is a non-zero element of $\oplus_{i\in L} C_i$, we claim that $ht_G(x) = ht_{G/N}(x + N)$: Note that if this failed, then since N is nice in G, there would be an element $y \in N$ such that $ht_G(x + y) > ht_G(x)$. Now, using (b), we could then find a $z \in \oplus_{i\in J} C_i$ such that

$ht_G(x+z) \geq ht_G(x+y) > ht_G(x)$, but this contradicts the observation that $G[p^n] = (\oplus_{i\in J} C_i) \oplus (\oplus_{i\in L} C_i)$ is a valuated direct sum.

It follows that $\oplus_{i\in L} C_i = V$ embeds isometrically in $A[p^n]$. By Theorem 2.6 of [**8**], we will be done if we can show that $A[p^n]/V$ is countable. To that end, suppose P is a countable pure subgroup of G containing N. Note that if $z + N \in A[p^n]$, then $p^n z \in N \subseteq P$, so $p^n z = p^n w$ for some $w \in P$. Therefore, $z - w \in G[p^n] = (\oplus_{i\in J} C_i) \oplus V$, so $z - w = t + u$ where $t \in \oplus_{i\in J} C_i$ and $u \in V$. It follows that $z = (w+t)+u \in (P + \oplus_{i\in J} C_i)+V$, so that $A[p^n] \subseteq (P+\oplus_{i\in J} C_i)/N + V$, and since $(P + \oplus_{i\in J} C_i)/N$ is clearly countable, $A[p^n]/V$ must be countable, as well. □

Recall that a countable valuated group is always Honda, so the following is essentially a converse of the above.

Theorem 1.2. *Suppose n is a positive integer, G is a nice-elongation of A by N, $p^{\omega_1} G = \{0\}$ and N is Honda (as a valuated group using the height valuation from G). If A is n-summable, then G is n-summable.*

Proof: Since the value spectrum of N is countable, there is a countable ordinal λ such that $N(\lambda) = N \cap p^\lambda G = \{0\}$. Let G' be a $p^{\lambda+n}$-high subgroup of G containing N. Then $p^\lambda G'$ is p^n-bounded and there is a decomposition $p^\lambda G = p^\lambda G' \oplus X$ leading to an isometry $G[p^n] = G'[p^n] \oplus X[p^n]$, where the values in $X[p^n]$ are λ plus the heights of elements computed in X (cf. [**8**, Lemma 1.9]). Since $N(\lambda) = N \cap p^\lambda G = \{0\}$ and N is nice, the map $p^\lambda G \to p^\lambda A$ is an isomorphism, so that $p^\lambda G$, and hence X, is n-summable. We therefore only need to show that G' is n-summable. Towards this end, note that G' is isotype in G and N is nice in G'. It follows that $A' = G'/N$ embeds as an isotype subgroup of $A = G/N$. Since $p^{\lambda+n} A' = p^{\lambda+n}(G'/N) = [p^{\lambda+n} G' + N]/N = \{0\}$, it follows that $A'[p^n]$ is Honda. Without loss of generality, then, assume $G' = G$ has countable length $\mu = \lambda + n$ and $G/N = A$ is n-Honda.

Let M be the subgroup of G containing N satisfying $M/N = (G/N)[p^n] = A[p^n]$. The height function on G gives a valuation on M and M/N is the valuated direct sum $\oplus_{i\in I} V_i$, where V_i is countable. For each coset in M/N, choose an element which is proper with respect to N and let $M_p \subseteq M$ be the collection of all these proper elements.

Let O_j for $j < \omega$ be an ascending sequence of finite sets of ordinals with union μ.

For each $i \in I$, suppose $F_{i,j}$, for $j < \omega$ is an ascending chain of finite subgroups of V_i whose union is all of V_i such that the value spectrum of $F_{i,j}$ is contained in O_j. Let $F'_{i,j} = \langle x \in M_p : x + N \in F_{i,j} \rangle$, so that $F'_{i,j}$ is a finite group.

Suppose N is the ascending union of the subgroups X_j such that the value spectrum of X_j is contained in O_j. Define

$$Z_j = X_j + \langle F'_{i,k} : i \in I,\ k \leq j \text{ and } F'_{i,k} \cap N \subseteq X_j \rangle.$$

We claim that M is the ascending union of these Z_j, and that the value spectrum of Z_j is contained in O_j.

Claim 1. $Z_j \subseteq Z_{j+1}$.

First, note that $X_j \subseteq X_{j+1}$. Second, if $i \in I$, $k \leq j$, and $F'_{i,k} \cap N \subseteq X_j$, then $k \leq j+1$, and $F'_{i,k} \cap N \subseteq X_j \subseteq X_{j+1}$, so that the claim follows.

Claim 2. $M = \cup_{j<\omega} Z_j$.

Note $N = \cup_{j<\omega} X_j \subseteq \cup_{j<\omega} Z_j$. Further, if $i \in I$ and $k < \omega$, then since $F'_{i,k}$ is finite, we can find a $j \geq k$ such that $F'_{i,k} \cap N \subseteq X_j$, and it follows that $F'_{i,k} \subseteq Z_j$. It follows that $F_{i,k} \subseteq [\cup_{j<\omega} Z_j + N]/N = [\cup_{j<\omega} Z_j]/N$, and since this happens for all i and k, it follows that $M/N = [\cup_{j<\omega} Z_j]/N$, which implies the claim.

Claim 3. *The value spectrum of Z_j is contained in O_j.*

Suppose

$$z = x + g_{i_1} + \cdots + g_{i_m} \in Z_j$$

where $x \in X_j$, and for $1 \leq \ell \leq m$, each $i_\ell \in I$ and for some $k_\ell \leq j$, $g_{i_\ell} \in F'_{i_\ell,k_\ell}$, where $F'_{i_\ell,k_\ell} \cap N \subseteq X_j$. Note that each g_{i_ℓ} is congruent modulo $F'_{i_\ell,k_\ell} \cap N \subseteq X_j$ to some element of $F'_{i_\ell,k_\ell} \cap M_p$, and so we actually assume each g_{i_ℓ} is proper with respect to M.

Note

$$\begin{aligned} v(z) &\leq v(z+N) \\ &= v(g_{i_1} + \cdots + g_{i_m} + N) \\ &= \min\{v(g_{i_1}+N), \ldots, v(g_{i_m}+N)\} \\ &= \min\{v(g_{i_1}), \ldots, v(g_{i_m})\} \end{aligned}$$

which implies that

$$v(x) = v(z - (g_{i_1} + \cdots + g_{i_m})) \geq v(z),$$

which further implies that

$$v(z) = \min\{v(x), v(g_{i_1}), \ldots, v(g_{i_m})\} \in O_j.$$

It follows from Claims 2 and 3 that M is Honda. We now observe that $G[p^n]$ is contained in M, and since M is Honda, so is $G[p^n]$, so that G must be n-summable, as required. □

Since a countable valuated group is always Honda, the following is an immediate consequence of Theorems 1.1 and 1.2:

Corollary 1.3. *Suppose n is a positive integer and G is a nice-$\aleph_0$-elongation of A with $p^{\omega_1}G = \{0\}$. Then G is n-summable iff A is n-summable.*

Example 1.4. The hypothesis of niceness is necessary in Theorem 1.1.

As in Example 6.8 of [**6**], let G_0 be a group satisfying the following:

(a) G_0 is summable, B is a countable unbounded Σ-cyclic group and there is an embedding of the torsion completion $\overline{B}$ in G_0 such that $p^\omega G_0 = \overline{B}[p]$ and $G_0/\overline{B}$ is Σ-cyclic. [To construct such a G_0, let H be a dsc-group of length $\omega + 1$ such that there is a group isomorphism $\phi\colon p^\omega H \to \overline{B}[p]$, and let $G_0 = [H \oplus \overline{B}]/\{(x, \phi(x)) : x \in p^\omega H\}$, so G_0 is the sum of H and $\overline{B}$ along ϕ.]

Now, suppose M is a countable group such that there is an isomorphism $f\colon p^\omega M \to B$. Let $G = M \oplus G_0$, $X = \{(x, f(x)) : x \in p^\omega M\} \subseteq G$, and $A = G/X$. It follows that

(b) G is summable (since both M and G_0 are summable);

(c) X is countable (since M is countable);

(d) $p^\omega A \cong \overline{B}$, so that A is not summable.

To verify (d), note that clearly $B \cong B' = [(\{0\} \oplus B) + X]/X \subseteq p^\omega A$, and since $A/B' \cong (\overline{B}/B) \oplus (G_0/\overline{B}) \oplus (M/p^\omega M)$, where the first term of this decomposition is divisible and the last two terms are Σ-cyclic and hence separable, (d) then follows. □

Example 1.5. Theorem 1.1 does not hold if the nice subgroup N is only assumed to be Honda, as opposed to countable.

Suppose A is any separable group which is not Σ-cyclic and $N \to G$ is a pure-projective resolution of A (i.e., G is Σ-cyclic and N is pure in G), then N is clearly Honda (since it is Σ-cyclic and the height valuation on G and N agree), G is summable, but A is not summable. □

Example 1.6. The hypothesis $p^{\omega_1}G = \{0\}$ is necessary in Theorem 1.2.

Suppose G is a totally projective group with $N = p^{\omega_1}G$ countably infinite and $A = G/N$. Then A is a dsc-group, and hence A is summable. Since $p^{\omega_1}G \neq \{0\}$, however, G will not be summable. □

Example 1.7. The hypothesis of niceness is necessary in Theorem 1.2.

In Example 2.3 of [**6**], a separable group G was constructed which was not Σ-cyclic but had a (pure) countable subgroup X such that $A = G/X$ was a dsc-group of length $\omega + 1$. It follows that this A is summable, but G is not; in fact G is $p^{\omega+1}$-projective. □

2. Totally projective groups and generalizations

A *nice composition series* for the valuated group Y is an ascending chain of nice subgroups $\{X_i : i \leq \delta\}$ such that

(a) $X_0 = \{0\}$, $X_\delta = Y$;

(b) for all $i < \delta$, $X_{i+1}/X_i \cong \mathbf{Z}_p$;

(c) for all limit ordinals $\lambda \leq \delta$, $X_\lambda = \cup_{i<\lambda} X_i$.

It is reasonably easy to verify that if Y has a nice composition series, then so does $Y/Y(\alpha)$ for every ordinal α, and that if Y is Honda (so in particular, if Y is countable), then it has a nice composition series. It is well known that a reduced group G using the height valuation has a nice composition series iff it is totally projective (see, for example, [**11**, Theorem 81.9]). The following (essentially well-known) observation follows as a direct consequence.

Proposition 2.1. *Suppose G is a nice-elongation of A by N. If N has a nice composition series as a valuated group and A is totally projective, then G is totally projective.*

Proof: If $\{X_i\}_{i\leq\delta}$ is a nice composition series for N, it can readily be checked that each X_i is also nice in G. If $\{Z_j\}_{j\leq\epsilon}$ is a nice composition series for $A = G/N$, and for $j < \mu$ we let Z'_j be the subgroup of G containing N defined by the equation $Z'_j/N = Z_j$, then each Z'_j will also be nice in G. It readily follows that $\{X_i\}_{i\leq\delta} \cup \{Z'_j\}_{j\leq\epsilon}$ is a composition series for G, so G is totally projective. □

We include the following (also well-known) observation for future reference.

Corollary 2.2. *Suppose G is a reduced group which is a nice-$\aleph_0$-elongation of A by N. Then A is totally projective iff G is totally projective.*

Proof: Sufficiency follows immediately from Proposition 2.1, and necessity follows directly from Proposition 1.1 of [**6**] (the second statement does not require the niceness of N in G). □

Recall that if $\lambda \leq \omega_1$, then G is a C_λ-group if for every $\alpha < \lambda$, one (and so each) p^α-high subgroup H of G is a dsc-group. If λ is a limit, this is equivalent to requiring that $G/p^\alpha G$ is a dsc-group for all $\alpha < \lambda$, and if λ is isolated, this is equivalent to requiring that one (and so each) $p^{\lambda-1}$-high subgroup H of G is a dsc-group (see, for example, the discussion in the first two paragraphs of Section 1 in [**17**]).

Theorem 2.3. *Suppose $\lambda \leq \omega_1$, G is a nice-elongation of A by N and N has a nice composition series (using the height function on G). If A is a C_λ-group then G is a C_λ-group.*

Proof: Assume A is a C_λ-group. First, suppose λ is a limit. Now, for all $\alpha < \lambda$ there is a nice short-exact sequence

$$0 \to N/N(\alpha) \to G/p^\alpha G \to A/p^\alpha A \to 0.$$

Since $A/p^\alpha A$ is a dsc-group, it follows from Proposition 2.1 that $G/p^\alpha G$ must be a dsc-group. Since this holds for all $\alpha < \lambda$, it follows that G is a C_λ-group, as required.

Next, suppose $\lambda = \gamma + 1$ is isolated; so, in particular, λ is countable. If λ is finite, the result is trivial (since then any group is a C_λ-group), so suppose λ is infinite. We therefore have a nice short-exact sequence:

$$0 \to N/N(\lambda) \to G/p^\lambda G \to A/p^\lambda A \to 0.$$

Since a p^γ-high subgroup of G maps to a p^γ-high subgroup of $G/p^\lambda G$, G is a C_λ-group iff $G/p^\lambda G$ is a C_λ-group. Without loss of generality, then, replace N, G and A by $N/N(\lambda)$, $G/p^\lambda G$ and $A/p^\lambda A$, so that we may assume all these groups have length at most λ.

Let Y be a p^γ-high subgroup of A, then since γ is infinite, we have that A/Y is divisible and

$$(p^\gamma A)[p] = p^\gamma A \to (A/Y)[p]$$

is an isomorphism (see, for example, [**14**, Theorem 92]).

Let X be the subgroup of G containing N such that $X/N = Y$. Note that

$$G/X \cong (G/N)/(X/N) = A/Y$$

is divisible and $(p^\gamma G)[p] \to (G/X)[p]$ can be factored as follows:

$$(p^\gamma G)[p] = p^\gamma G \to p^\gamma A = (p^\gamma A)[p] \cong (A/Y)[p] \cong (G/X)[p]$$

and since these maps are all surjective, it follows that X is p^λ-pure in G (see, for example, [**14**, Theorem 91]). This means that X is isotype in G, so that

$$0 \to N \to X \to Y \to 0$$

is a nice sequence. Since A is a C_λ-group, Y is a dsc-group. By Proposition 2.1 this means that X is also a dsc-group. Next, observe that $G[p]$ is isometric to a valuated direct sum $X[p] \oplus W$ where $W \subseteq p^\gamma G[p]$. This implies that if X' is a p^γ-high subgroup of X, then X' is also p^γ-high in G. Since X' will be isotype in X, it follows that X' is a dsc-group (see, for example, [**14**, Theorem 104]). However, this implies that G is a C_λ-group, as required. □

Corollary 2.4. *Suppose $\lambda \leq \omega_1$ and G is a nice-$\aleph_0$-elongation of A. Then G is a C_λ-group iff A is a C_λ-group.*

Proof: If A is a C_λ-group, then Theorem 2.3 implies that G is as well; and conversely, if A is a C_λ-group, then Theorem 3.5 of [**6**] implies that G is as well (the second implication does not require the niceness of N in G). □

Remark 1. If $\lambda > \omega$, then Example 1.5 shows that in Corollary 2.4, if N is only assumed to be Honda, as opposed to countable, then A may be a C_λ-group, while G is not.

For the remainder of this section n will denote a fixed positive integer; G is said to be an *n-Σ-group* if $G[p^n]$ is the ascending union of a sequence of subgroups $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$ such that for all $i < \omega$, $X_i(i) = X_i \cap p^i G = (p^\omega G)[p^n]$. Utilizing Proposition 3.2 of [**6**], a group is an n-Σ-group iff it is a $C_{\omega+n}$-group. Therefore, these last two results can be reformulated as follows:

Corollary 2.5. *Suppose G is a nice-elongation of A by N.*

(a) *If N has a nice composition series (using the height function on G) and A is an n-Σ-group, then G is an n-Σ-group.*
(b) *If N is countable, then G is an n-Σ-group iff A is an n-Σ-group.*

A group G is said to be *pillared* if $G/p^\omega G$ is Σ-cyclic. Since this is true iff for all $n < \omega$, $G/p^{\omega+n}G$ is a dsc-group, it follows that a group is pillared iff it is a $C_{\omega\cdot 2}$-group. Letting $\lambda = \omega \cdot 2$, then, we have the following:

Corollary 2.6. *Suppose G is a nice-elongation of A by N.*

(a) *If N has a nice composition series (using the height function on G) and A is pillared, then G is pillared.*

(b) *If N is countable, then G is pillared iff A is pillared.*

A group G is said to be $p^{\omega+n}$*-projective* if there is a subgroup $P \subseteq G[p^n]$ such that G/P is Σ-cyclic (e.g., [**21**]). By a classical result of Fuchs (see [**13**]), if G_1 and G_2 are $p^{\omega+n}$-projective, then G_1 and G_2 are isomorphic iff $G_1[p^n]$ and $G_2[p^n]$ are isometric. On the other hand if P is any valuated group with $p^n P = \{0\}$ which is separable in the sense that $P(\omega) = \{0\}$, then there is a separable group G containing P such that the valuation on P agrees with the height function on G, G is separable and G/P is Σ-cyclic; note that this G will be $p^{\omega+n}$-projective.

More generally, imitating [**3**], a group G is called $p^{\omega+n}$*-totally projective* if $p^\omega G$ is totally projective and $G/p^\omega G$ is $p^{\omega+n}$-projective. Clearly both $p^{\omega+n}$-projective groups and totally projective groups are themselves $p^{\omega+n}$-totally projective. By the same token, we define a group G to be $p^{\omega+n}$*-summable* if $p^\omega G$ is summable and $G/p^\omega G$ is $p^{\omega+n}$-projective. Observe that $p^{\omega+n}$-projective groups are $p^{\omega+n}$-summable, whereas summable groups need not be $p^{\omega+n}$-summable since there exists a summable group with unbounded torsion-complete first Ulm factor and it is known that torsion-complete $p^{\omega+n}$-projective groups are bounded (see, for instance, [**16**]).

Proposition 2.7. *Suppose G is a nice-$\aleph_0$-elongation of A by N. We then have:*

(a) *G is $p^{\omega+n}$-totally projective iff A is $p^{\omega+n}$-totally projective.*

(b) *If $p^{\omega_1} G = \{0\}$, then G is $p^{\omega+n}$-summable iff A is $p^{\omega+n}$-summable.*

Proof: Observe that $p^\omega G$ is a nice-$\aleph_0$-elongation of $p^\omega A$ by $N(\omega) = N \cap p^\omega G$, and $G/p^\omega G$ is a nice-$\aleph_0$-elongation of $A/p^\omega A$ by $N/N(\omega)$. By Theorem 4.2 of [**6**], $G/p^\omega G$ is $p^{\omega+n}$-projective iff $A/p^\omega A$ is $p^{\omega+n}$-projective.

Therefore, (a) follows from Corollary 2.2 applied to $p^\omega G$ and $p^\omega A$, and (b) follows from Corollary 1.3 applied to the same. □

3. CPS groups and ω_1-separable groups

Proposition 3.1. *Suppose A is a separable group.*

(a) *If A is Σ-cyclic, then it has the cps-elongation property.*

(b) *If A has the cps-elongation property then it has the ω-cps-elongation property.*

Proof: In (a), if A is Σ-cyclic and G is an $\aleph_0$-elongation of A by X, then there is clearly a countable pure subgroup Y of G containing X such that Y/X is a summand of A. Since $G/Y \cong A/(Y/X)$ will be Σ-cyclic, it follows that $G \cong Y \oplus S$, for some Σ-cyclic group S, so that G is cps. It follows that A has the cps-elongation property.

The implication in (b) is obvious. □

We now translate a notion that has been of considerable importance in the study of (torsion-free) free groups (see [**10**]) into the language of valuated vector spaces. We will say the valuated vector space V is $\aleph_1$-*coseparable* if it is separable (i.e., $V(\omega) = \{0\}$), and for all subspaces W of V with V/W countable there is a closed subspace U of V such that $U \subseteq W$ and V/U is countable —note that this implies that the quotient valuated vector space V/U is separable and hence free, so that V is isometric to $U \oplus F$ where F is countable and free. Note also that if V/W is unbounded (i.e., $(V/W)(m) \neq \{0\}$ for all $m < \omega$), then the same will be true of F.

We pause to review the following standard construction (see, for example, [**14**, Theorem 106]).

Lemma 3.2. *Suppose A is a separable group, $D \subseteq A[p]$ is a dense subsocle and X is a group such that $A[p]/D \cong X/pX$. Then there is an ω-elongation G of A by X such that $D = [G[p] + p^\omega G]/p^\omega G \subseteq A[p]$.*

Proof: Let Y be a pure and dense subgroup of A such that $Y[p] = D$, so that A/Y is divisible; and let Z be a divisible hull for X, so that Z/N is also divisible. The isomorphism $A[p]/D \cong X/pX$ implies that there is an isomorphism $\phi\colon A/Y \to Z/X$, and we can let $G = \{(a, z) \in A \oplus Z : f(a+Y) = z+X\}$. The assignment $x \to (0, x)$ gives an injection $X \to G$, and the assignment $(a, z) \to a$ gives a surjection $G \to A$, and it can be checked that these conditions will imply the conclusions. □

Theorem 3.3. *The following conditions are equivalent (in ZFC):*

(a) *Every separable group A with the ω-cps-elongation property is Σ-cyclic.*

(b) *Every separable $p^{\omega+1}$-projective group A with the ω-cps-elongation property is Σ-cyclic.*

(c) *Every $\aleph_1$-coseparable valuated vector space is free (as a valuated vector space).*

Proof: It is clear that (a) implies (b). Suppose next that (b) holds and that V is $\aleph_1$-coseparable; we want to show that V must be free. Let A be a separable group containing V as a subgroup where A/V is Σ-cyclic, so that A is $p^{\omega+1}$-projective. Since there is a valuated injection of the valuated quotient space $A[p]/V$ into $(A/V)[p]$ and the latter is Honda, so is $A[p]/V$, and hence free as a valuated vector space. It follows that $A[p]$ is isometric to $V \oplus F$ where F is a free valuated vector space. In fact, after possibly adding to A a summand which is an infinite Σ-cyclic group, we may assume that every Ulm invariant of F is infinite.

We now show that this A has the ω-cps-elongation property. To this end, let G be an ω-$\aleph_0$-elongation of A, so that $A = G/p^\omega G$; we need to show G is cps. We begin by letting $W' = [G[p] + p^\omega G]/p^\omega G \subseteq A[p]$. It can easily be checked that $\phi(x + p^\omega G) = px + p^{\omega+1}G$ gives a well-defined homomorphism $A[p] \to p^\omega G/p^{\omega+1}G$ whose kernel is W'. It follows that $A[p]/W'$ embeds in the countable group $p^\omega G/p^{\omega+1}G$, and hence $A[p]/W'$ is also countable.

Now let $W = W' \cap V$. Since V/W embeds in $A[p]/W'$, it follows that V/W is countable. Let U be a closed subspace of V contained in W such that V/U is countable. Finally, let P be a subgroup of $G[p]$ such that $G[p] \to W'$ maps P isomorphically onto U; since the homomorphism $G[p] \to W' \subseteq A[p]$ preserves all finite heights, it follows that P also maps isometrically onto U.

From this construction we can conclude that $[p^\omega G \oplus P]/p^\omega G = U$. Finally, let $Q = p^\omega G \oplus P \subseteq G$, so that $G/Q \cong (G/p^\omega G)/([p^\omega G \oplus P]/p^\omega G) = A/U$.

Claim. *The decomposition $Q = p^\omega G \oplus P \subseteq G$ is valuated, G/Q is Σ-cyclic and, for all $m < \omega$, the relative Ulm function $f_{G,Q}(m)$ is infinite.*

Since $p^\omega G \cap P = \{0\}$, the first statement readily follows. Since $G/Q \cong A/U$, we must show that the latter is Σ-cyclic. Observe that V/U is a free, separable valuated vector space, so there is a valuated embedding $V/U \to J$, where J is a Σ-cyclic group using the height valuation. Since V is nice in A and A/V is Σ-cyclic, it follows that this embedding extends to a homomorphism $f\colon A \to J$, for which the kernel of $f\big|_V$ is U. It follows that the kernel of the obvious map $A \to (A/V) \oplus J$ is U, and since $(A/V) \oplus J$ is Σ-cyclic, so is A/U.

Observe that if $m < \omega$, then $f_{G,Q}(m) = f_{A,U}(m)$. Since F has infinite Ulm invariants, $U \subseteq V$, and $A[p]$ is isometric to $V \oplus F$, the claim follows.

Let C be a countable group whose Ulm function is defined by

$$f_C(\alpha) = \begin{cases} f_G(\alpha), & \text{when } \omega \le \alpha \le \infty; \\ 1, & \text{when } \alpha < \omega. \end{cases}$$

Note this implies that $p^\omega C$ and $p^\omega G$ are isomorphic, even when they are not reduced. Let $H = C \oplus A$, so that $Q' = p^\omega C \oplus U$ is nice in H, and $H/Q' \cong (C/p^\omega C) \oplus (A/U)$ is Σ-cyclic. In addition, Q′ and Q are both isometric to $p^\omega G \oplus U$ and so they are isometric to each other. Next, it is readily checked that if $\alpha \geq \omega$, $f_{G,Q}(\alpha) = 0 = f_{H,Q'}(\alpha)$ and if $m < \omega$,

$$f_{H,Q'}(m) = f_{G,Q}(m) + 1 = f_{G,Q}(m).$$

Therefore, by the fundamental result of Hill on extending isometries on nice valuated subgroups of groups (see, for example, [**11**, Theorem 83.4]), it follows that there is an isomorphism $G \to H = C \oplus A$. It therefore follows that G is cps, so that A has the ω-cps-elongation property. By hypothesis, then, A must be Σ-cyclic, and this implies that $A[p]$ is free as a valuated vector space, and since V is a subspace of $A[p]$, it is also free, as required.

We now need to show that (c) implies (a), so suppose every $\aleph_1$-coseparable valuated vector space is free and A is a separable group with the ω-cps-elongation property, and let $V = A[p]$; it follows trivially that V is separable. Let W be a subspace of V such that V/W is countable. Note that if $\overline{W}$ is the p-adic closure of W in V, then $\overline{W}/W = (V/W)(\omega)$. Since V/W is countable, there is a subspace W' of V containing W such that V/W is isometric to the valuated direct sum $(W'/W) \oplus (\overline{W}/W)$. Note that W is closed (and hence, nice) in W' and W'/W is countable (and hence, free), so that $W' = W \oplus F$ for some countable, free subspace F.

Note that W' is dense in V. By Lemma 3.2 there is an ω-elongation G of A such that $p^\omega G$ is countable, $G/p^\omega G = A$ and $[G[p]+p^\omega G]/p^\omega G = W'$.

By hypothesis, $G = C \oplus S$, where C is countable and S is separable. First, note that we can identify $A[p] = V$ with $(C/p^\omega C)[p] \oplus S[p]$, so that we can view $S[p]$ as a subgroup of $W' \subseteq A[p] = V$. We now let $U = S[p] \cap W$. Since both $V/S[p]$ and V/W are countable, and V/U embeds in $(V/S[p]) \oplus (V/W)$, it follows that V/U is countable.

To show that U is closed, let u_i for $i < \omega$ be a sequence in U which converges in the p-adic topology to $z \in V$. Since $A[p] = (C/p^\omega C)[p] \oplus S[p]$, $S[p]$ is closed in V, so that $z \in S[p] \subseteq W'$. In the decomposition $W' = W \oplus F$, each $u_i \in W$, so that $z \in W$, as well. It follows that $z \in S[p] \cap W = U$.

Note that the arguments in the last two paragraphs imply that V is $\aleph_1$-coseparable. It follows, therefore, that $V = A[p]$ is free, so that A is Σ-cyclic, as required. □

Theorem 3.4. *The following hold:*

(a) *Assuming (MA + ¬CH), there is a separable group A which has the cps-elongation property (and hence it also has the ω-cps-elongation property), but is not Σ-cyclic.*

(b) *Assuming ($V = L$), if A is a separable group which has the ω-cps-elongation property (in particular, if A has the cps-elongation property), then A is Σ-cyclic.*

Before proceeding, note that the last result implies the following.

Corollary 3.5. *The following statements are independent of ZFC:*

(a) *Every separable group with the cps-elongation property is Σ-cyclic.*

(b) *Every separable group with the ω-cps-elongation property is Σ-cyclic.*

(c) *Every $\aleph_1$-coseparable valuated vector space is free.*

Proof: By Theorem 3.4, both (a) and (b) fail in a model of MA + ¬CH and both are true in $V = L$. By Theorem 3.3, (b) is equivalent to (c). □

Recall that a group A is *ω_1-separable* iff it is separable and every countable subgroup of A is contained in a countable summand of A. Using a (by now) standard construction, it can be shown in ZFC that there are ω_1-separable groups of cardinality $\aleph_1$ which are not Σ-cyclic.

Proof of Theorem 3.4: Regarding (a), suppose A is an ω_1-separable group of cardinality $\aleph_1$ which is not Σ-cyclic. If G is an $\aleph_0$-elongation of A by X, then let P be a countable pure subgroup of G containing X. Since $A = G/X$ is ω_1-separable, we can find a countable subgroup C of G containing P such that C/X is a summand of $G/X = A$.

We claim that C will be pure in G: Note that C/X is clearly pure in G/X, which implies that $C/P \cong (C/X)/(P/X)$ is pure in $G/P \cong (G/X)/(P/X)$. Since P is pure in G, it follows that C is pure in G, as required.

If we let $A = A' \oplus (C/X)$, then the countability of C/X and the fact that A is ω_1-separable readily implies that A' is also ω_1-separable. In the presence of MA + ¬CH, by Theorem 3.1 of [**18**], we can conclude that $\mathrm{Pext}(A', C) = \{0\}$. However, since C is pure in G and $G/C \cong (G/X)/(C/X) \cong A'$, we can conclude that $G \cong A' \oplus C$, as required.

Regarding (b), note that if G is an ω-elongation of A by $\mathbb{Z}_p$, then since A has the ω-cps-elongation property, it follows that $G \cong C \oplus S$, where C is countable and S is separable. If $H_{\omega+1}$ is the generalized Prüfer group of length $\omega + 1$, there is clearly a homomorphism $G \to C \to H_{\omega+1}$ which is non-zero on $p^\omega G = p^\omega C$. In the presence of $V = L$, by Theorem 2.2 of [**20**], A must be Σ-cyclic. □

Although by Proposition 7.2 of [**4**], A being ω_1-separable implies that A/C is ω_1-separable whenever C is a countable nice subgroup of A, we shall see that the converse implication does not hold in some versions of set theory. First, we pause for the following assertion which answers Problem 7.3 from [**4**].

Proposition 3.6. *Suppose A is a separable group and $f\colon G \to A$ is an ω_1-bijective homomorphism. If G is ω_1-separable, then A is ω_1-separable.*

Proof: Let I be the image of f and K be the kernel of f. Since I is a subgroup of A, K is a countable and nice subgroup of G, so by Proposition 7.2 of [**4**], $I \cong G/K$ is ω_1-separable. Now, if C is a countable subgroup of A, then after possibly expanding C, we may assume that $A = I + C$. Let C' be a countable subgroup of I containing $I \cap C$ such that $I = I' \oplus C'$; the existence of such a C' follows from the fact that I is ω_1-separable. Then $C \subseteq C + C'$ is countable and one easily checks that $A = I' \oplus (C' + C)$. It follows that A is ω_1-separable, as desired. □

The next result shows that the converse of Proposition 3.6 is undecidable in ZFC for groups of cardinality $\aleph_1$. It settles Problem 7.2 from [**4**].

Theorem 3.7. (a) *($MA + \neg CH$) If the separable group G is an $\aleph_0$-elongation of the ω_1-separable group A and $|A| = \aleph_1$, then G is also ω_1-separable.*

(b) *($V = L$) There is a separable group G which is an $\aleph_0$-elongation of an ω_1-separable group A with $|A| = \aleph_1$ such that G is not ω_1-separable.*

Proof: Regarding (a), a separable group H is *weakly ω_1-separable* if for every countable subgroup C of H, the p-adic closure $\overline{C}$ is also countable. By [**19**], in the presence of $\mathrm{MA} + \neg\mathrm{CH}$, for groups of cardinality $\aleph_1$, the classes of weakly ω_1-separable and ω_1-separable groups coincide. Finally, in view of Corollary 5.2 of [**6**] we have that G is weakly ω_1-separable iff A is weakly ω_1-separable.

Regarding (b), Megibben [**19**, Theorem 3.2] found in the presence of $V = L$ an ω_1-separable group A of cardinality $\aleph_1$ containing a pure and dense subgroup G' which is not ω_1-separable such that A/G' is countable. If C is a countable subgroup of A such that $G' + C = A$, then let $G = G' \oplus C$ (note this is an *external* direct sum). Since G' is not ω_1-separable, the countability of C easily implies that G is not ω_1-separable. If we then consider the sum map $G = G' \oplus C \to G' + C = A$, defined by $(g, c) \mapsto g + c$, we have our result. □

4. Groups with nice bases

Following [**2**], recall that a nice basis for a group G is an ascending sequence of nice subgroups $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$ whose union is all of G such that each X_m is Σ-cyclic.

Proposition 4.1. *Let G be a nice-$\aleph_0$-elongation of A by N such that $N \cap p^\omega G = \{0\}$. If A has a nice basis, then G has a nice basis.*

Proof: Write $G/N = \cup_{i<\omega}(G_i/N) = (\cup_{i<\omega}G_i)/N$, where $N \leq G_i \subseteq G_{i+1} \leq G$ and all G_i/N are nice in G/N and are Σ-cyclic groups. Therefore, $G = \cup_{i<\omega}G_i$ with $p^\omega G_i \subseteq N$. Hence $p^\omega G_i \subseteq N \cap p^\omega G = \{0\}$ and we conclude that all G_i are separable. By Corollary 2.2, they are Σ-cyclic groups. Moreover, utilizing Lemma 79.3 of [**11**], we derive that these G_i are also nice in G, as needed. □

The last result can also be deduced from [**4**, Proposition 9.1]; nevertheless we have given another, more smooth, proof.

Example 4.2. In Proposition 4.1, the restriction $N(\omega) = N \cap p^\omega G = \{0\}$ is necessary.

Suppose A is any separable thick group and let G be any group such that $p^\omega G$ is countable, reduced inseparable (thus it is not Σ-cyclic) with $G/p^\omega G \cong A$. Applying Theorem 1.3 of [**7**], G does not has a nice basis (see also Proposition 4.5 below). □

Example 4.3. In Proposition 4.1, the niceness of N in G is also necessary.

Let G be a reduced group without a nice basis with a countable basic subgroup N (for example, if $G/p^\omega G$ is torsion-complete with a countable basic subgroup and $p^\omega G$ is not Σ-cyclic, it follows from [**7**] that G does not have a nice basis). Clearly, $N(\omega) = \{0\}$ is satisfied and $A = G/N$ is divisible, hence it does have a nice basis owing to [**2**]. □

We now investigate the hypothesis that $N(\omega) = \{0\}$ in the last result by contrasting it with the case where G is an ω-$\aleph_0$-elongation of A by N; i.e., we are contrasting the situation in which $N \cap p^\omega G = \{0\}$ with the situation in which $N = p^\omega G$. We begin by considering an apparently unrelated notion.

A separable valuated vector space V (i.e., $V(\omega) = \{0\}$) will be called *essentially finitely indecomposable* (or efi for short) if (1) it is *unbounded* (in the sense that $V(m) \neq \{0\}$ for all $m < \omega$); and (2) there is no valuated decomposition $V = W \oplus F$ where F is an unbounded free valuated vector space. For example, if V is complete in the p-adic topology (i.e., if it is isometric to the socle of a torsion-complete group), then V is efi. The following topological characterization utilizes ideas from [**7**].

Proposition 4.4. *An unbounded separable valuated vector space V is efi iff V cannot be expressed as the ascending union of a sequence of closed subspaces $C_0 \subseteq C_1 \subseteq C_2 \subseteq \cdots$ such that V/C_m is unbounded for every $m < \omega$.*

Proof: If there is a decomposition $V = W \oplus F$ and we let $F = \oplus_{i<\omega} B_i$ where $B_i(i) \neq \{0\}$, then we may simply let $C_m = W \oplus (\oplus_{i<m} B_i)$. Therefore, $V/C_m \cong \oplus_{m \leq i} B_i$, so that C_m is closed and V/C_m is unbounded.

Conversely, if we have the collection of closed subgroup C_m satisfying the above, then let $x_m \in V - C_m$ be chosen so that $v(x_m) \geq m$ and $\langle x_m \rangle$ is a valuated summand of V/C_m; denote the valuated projection $V/C_m \to \langle x_m \rangle$ by π_m. If $f\colon V \to \prod_{m<\omega} \langle x_m \rangle$ is given by $f(x) = (\pi_m(x + C_m))_{m<\omega}$, then it is easy to verify that f is a valuated homomorphism whose image is unbounded and is actually contained in the free valuated vector space $\oplus_{m<\omega} \langle x_m \rangle$. If we let W be the kernel of f, then $V = W \oplus F$, where F is an unbounded free valuated vector space. □

Proposition 4.5. *If A is any unbounded separable group that has a subsocle $P \subseteq A[p]$ which is efi, then there is a group G which is an ω-$\aleph_0$-elongation of A, such that G does not have a nice basis.*

Proof: Let N be any reduced countable group which is not Σ-cyclic, i.e., $p^\omega N \neq \{0\}$. Let D be a dense subspace of $A[p]$ such that $A[p]/D$ is countably infinite and $D + P(i) = A[p]$ for all $i < \omega$ [start with a dense subspace D' of P such that P/D' is countably infinite, and if W is a subspace of $A[p]$ such that $P \oplus W = A[p]$, then we can let $D = D' + W$].

By Lemma 3.2 there is a group G which is an ω-elongation of A by N such that $D = [G[p] + p^\omega G]/p^\omega G \subseteq A[p]$. Suppose now that G actually does have a nice basis $M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$. If for each $i < \omega$ we let $L_i = [M_i + N]/N$, then L_i is closed in the p-adic topology on A and their union is all of A. It follows that P is the union of the closed subspaces $C_i = P \cap L_i$. Since P if efi, it follows from Proposition 4.4 that, for some i, $P(i) \subseteq P \cap C_i \subseteq L_i$. We therefore have $A[p] \subseteq D + L_i$.

If $x \in N$, then find $y \in G$ such that $py = x$. Since $y + N \in A[p]$ there is a $w \in G[p]$ and $z \in M_i$ such that $y + N = (w + z) + N$. So there is a $u \in N$ such that $y = w + z + u$. This implies that $x = py = pw + pz + pu = pz + pu$. Note that $pz \in M_i(\omega)$, $pu \in pN$, so that we can conclude $N = M_i(\omega) + pN$. This, in turn, implies that $N/M_i(\omega)$ is divisible. Since M_i is nice in G, $M_i(\omega)$ is nice in N. Since N is reduced and $N/M_i(\omega)$ is divisible, we must have that $N = M_i(\omega) \subseteq M_i$. This is a contradiction, because M_i being a part of a nice basis is Σ-cyclic, though we assumed N was not. □

Example 4.6. There are $p^{\omega+1}$-projective groups A satisfying Proposition 4.5.

If P is any separable valuated vector space which is efi, then there is a separable group A containing P such that the valuation on P agrees with the height function on A and A/P is Σ-cyclic. □

Remark 2. If we start with the group A from Example 4.6 and construct the group G as in Proposition 4.5, then since G does not have a nice basis, it follows from [**2**] that it is not $p^{\omega+1}$-projective. Therefore, in Theorem 4.2 of [**6**], the condition of separability is necessary and cannot be eliminated.

Following [**7**], the separable group A is said to have the *nice basis extension property* if for all ω-elongations G of A, if $p^\omega G$ has a nice basis, then so does G. By Corollary 2.13 of [**7**], the Continuum Hypothesis (CH) implies that when A has a countable basic subgroup, then A has the nice basis extension property iff it is Σ-cyclic, and it was asked whether all groups with this property are necessarily Σ-cyclic. The following is related to this question by restricting to the case of ω-$\aleph_0$-elongations.

Example 4.7. In the presence of MA + ¬CH, there is a separable group A which is not Σ-cyclic, with the property that every ω-$\aleph_0$-elongation G of A has a nice basis. This group may be chosen to be $p^{\omega+1}$-projective.

Let A be as in Theorem 3.4(a). If G is an ω-elongation of A, then since A has the ω-cps-elongation property, $G \cong C \oplus S$, where C is countable and S is separable. Since C is countable, it is totally projective, and hence it has a nice basis by [**2**]. Since S is separable, it clearly has a nice basis, as well. Therefore, in view of [**2**], G has a nice basis, as required. The last part follows from Theorem 3.3. □

The following completely answers Problem 4 of [**7**].

Example 4.8. There are groups G_1 and G_2 with $p^\omega G_1 \cong p^\omega G_2$ and $G_1/p^\omega G_1 \cong G_2/p^\omega G_2$ such that G_1 has a nice basis but G_2 does not have a nice basis.

Suppose A' is a separable group with a subsocle P which is efi, and B is an unbounded Σ-cyclic group. So if $A = A' \oplus B$, then since P is also a subsocle of A, by Proposition 4.5, there is an ω-$\aleph_0$-elongation G_2 of A by a countable group $N = p^\omega G_2$ which does not have a nice basis. Since N is countable and B is unbounded, there a dsc-group H such that $p^\omega H \cong N$ and $H/p^\omega H \cong B$. If we let $G_1 = H \oplus A'$, then H has a nice basis by [**2**] (since it is a dsc-group and hence totally projective), A' has a nice basis (since it is separable), so that G_1 has a nice basis in virtue of [**2**], as required. □

Question 1. Is the converse of Proposition 3.1(b) valid in ZFC? That is, if A is a separable group with the ω-cps-elongation property, does it follow in ZFC that A has the cps-elongation property?

Question 2. Suppose A is a separable group with the property that every ω-$\aleph_0$-elongation G of A has a nice basis. In $V = L$, does it follow that A must be Σ-cyclic? (Note that by Example 4.7, this does not hold in a model of MA + ¬CH.)

Question 3. Suppose A is a separable group with the property that every ω-elongation G by a totally projective group $X = p^\omega G$ has a nice basis. Can we conclude that A is Σ-cyclic? (If this holds, then all such G will also be totally projective.)

5. Almost totally projective groups

A group G is *almost totally projective* if it has a collection of nice subgroups $\mathcal{N}$ with the properties:

(0) $\{0\} \in \mathcal{N}$.

(1) If $\{H_i\}_{i\in I}$ is an ascending chain of subgroups in $\mathcal{N}$, then $\cup_{i\in I} H_i \in \mathcal{N}$.

(2) If $C \subseteq G$ is countable, then there is a countable $M \in \mathcal{N}$ such that $C \subseteq M$.

We pause to review the following.

Proposition 5.1 ([**15**], [**5**]). *If K is a subgroup of the reduced group H such that H/K is countable and K is almost totally projective, then H is almost totally projective.*

Proposition 5.2 ([**5**]). *If K is a countable and nice subgroup of the reduced group H such that H/K is almost totally projective, then H is almost totally projective.*

More directly, we can state the following (see [**6**] for totally projective groups).

Proposition 5.3. *Suppose A is reduced and G is an $\aleph_0$-elongation of A by X. If G is almost totally projective, then A is almost totally projective.*

Proof: Let $\mathcal{N}$ satisfy (0), (1), (2) above, and let $\mathcal{N}' = \{0\} \cup \{N/X : N \in \mathcal{N} \text{ and } X \subseteq N\}$. First, note that if $N/X \in \mathcal{N}'$, then N is nice in G, so that N/X is nice in A. Next, that $\mathcal{N}'$ satisfies (0) is immediate. As for (1), if $\{H_i/X\}_{i\in I}$ is an ascending chain of subgroups in $\mathcal{N}'$, then $\{H_i\}_{i\in I}$ is an ascending chain of subgroups in $\mathcal{N}$. It follows that $\cup_{i\in I} H_i \in \mathcal{N}$, so that $\cup_{i\in I}(H_i/X) \in \mathcal{N}'$, proving that (1) holds for $\mathcal{N}'$. Finally, if C/X is countable for some $C \leq G$, then so is C and hence there is a countable $M \in \mathcal{N}$ such that $C \subseteq M$. Thus $C/X \subseteq M/X \in \mathcal{N}'$, whence (2) follows as required. □

Example 5.4. The converse of Proposition 5.3 need not hold.

By Example 2.3 of [**6**], there is a separable non Σ-cyclic group G with a countable but not nice subgroup X such that $A = G/X$ is a dsc-group, hence is almost totally projective, with $p^\omega A$ uncountable. If this G were, in fact, almost totally projective, then X would be contained in a countable closed subgroup $N \in \mathcal{N}$. If $\overline{X}$ is the p-adic closure of X in G, then $p^\omega A = \overline{X}/X \subseteq N/X$ would be countable, contrary to hypothesis. □

We have the following consequence of the above.

Corollary 5.5. *Let G be a reduced group which is a nice-$\aleph_0$-elongation of A. Then G is almost totally projective iff A is almost totally projective.*

Proof: Sufficiency follows from Proposition 5.2 and necessity follows from Proposition 5.3. □

Observe that similar results were proved by Dieudonné [**9**] for Σ-cyclic groups and by Balof-Keef [**1**] for almost Σ-cyclic groups.

We conclude with the following result.

Theorem 5.6. *Suppose G and A are reduced groups and $f\colon G \to A$ is an ω_1-bijective homomorphism. If G is almost totally projective, then A is almost totally projective.*

Proof: Let I be the image of f and X be the kernel of f. Note that G is an $\aleph_0$-elongation of I by X, so by Proposition 5.3, I is almost totally projective. Since A/I is also countable, by Proposition 5.1, we can conclude that A is almost totally projective, as required. □

The above discussion leads naturally to the following.

Question 4. If $f\colon G \to A$ is an ω_1-bijective homomorphism and A is totally projective, what we can say about G?

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Primera versió rebuda el 24 de març de 2009,
darrera versió rebuda el 21 de setembre de 2009.