ALGEBRAIC WEBS INVARIANT UNDER ENDOMORPHISMS

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Abstract

We classify noninvertible, holomorphic selfmaps of the projective plane that preserve an algebraic web. In doing so, we obtain interesting examples of critically finite maps.

Introduction

In this paper we classify holomorphic selfmaps of the complex projective plane $\mathbf{P}^2$ that are integrable in the quite specific sense that they preserve an algebraic web.

Recall that an *algebraic web* is given by a reduced curve $C \subset \check{\mathbf{P}}^2$, where $\check{\mathbf{P}}^2$ is the dual projective plane consisting of lines in $\mathbf{P}^2$. We say that the web is irreducible if C is an irreducible curve. The web is *invariant* for a holomorphic mapping $f \colon \mathbf{P}^2 \to \mathbf{P}^2$ if every line in $\mathbf{P}^2$ belonging to C is mapped to another such line. See Sections 1 and 3 for details. We will assume f is noninvertible.

Theorem A. *If C is irreducible, it is of one of the following types:*

  (i) *a line;*
 (ii) *a smooth conic;*
(iii) *a smooth cubic;*
(iv) *a nodal cubic.*

The maps in (iii) and (iv) are always critically finite.

Theorem B. *If C is reducible, it is of one of the following types:*

  (i) *the union of two lines;*
 (ii) *the union of three lines in general position;*
(iii) *the union of a conic and a line in general position.*

The maps appearing in Theorems A and B are quite rare among all holomorphic selfmaps, but nevertheless interesting. Indeed, they provide concrete examples of critically finite mappings. See [**FS1**], [**FS2**], [**U1**], [**U2**], [**U3**], [**J1**], [**R**], [**K**] for examples and dynamics of critically finite maps, and [**S**] for a survey of iterations of rational maps on projective spaces.

There are several other notions of integrability for selfmaps of $\mathbf{P}^2$. In [**DJ**] we classified invariant pencils of curves. Much more generally, Favre and Pereira [**FP**] have classified invariant foliations for rational maps. The case of birational maps was studied earlier by Cantat and Favre [**CF**]. Related work includes the classification of totally invariant curves for holomorphic endomorphisms of $\mathbf{P}^2$ [**FS2**], [**CL**], [**D**], [**SSU**] and for birational maps of surfaces [**DJS**]; see also [**BD**].

This note is organized as follows. After some background in Sections 1 through 3 we describe in Section 4 the mappings appearing in Theorem A. The proof takes place in Section 5 and Section 6 treats the case of a reducible web.

1. Algebraic webs and plane geometry

We start by reviewing some elementary facts of plane geometry. Let $\mathbf{P}^2$ denote the complex projective plane. The support of a divisor D on $\mathbf{P}^2$ is denoted by $|D|$. Let $\check{\mathbf{P}}^2$ the dual projective plane, that is, the set of complex lines in $\mathbf{P}^2$. Then $\check{\mathbf{P}}^2$ is itself isomorphic to the projective plane. Let $\mathcal{D}$ denote the involutive duality between lines (resp. points) in $\mathbf{P}^2$ and points (resp. lines) in $\check{\mathbf{P}}^2$. Given a point $p \in \mathbf{P}^2$, $\mathcal{D}p \subset \check{\mathbf{P}}^2$ is the line of lines passing through p. Given a line $L \subset \check{\mathbf{P}}^2$, $\mathcal{D}L = \bigcap_{l \in L} l$.

Consider a reduced, irreducible curve $B \subset \mathbf{P}^2$ (resp. $B \subset \check{\mathbf{P}}^2$) of degree > 1. The *dual curve* $\check{B} \subset \check{\mathbf{P}}^2$ (resp $\check{B} \subset \mathbf{P}^2$) is the curve of tangents to the local branches to B. If $\psi \colon A \to B$ is a normalization map, then $\check{\psi} \colon A \to \check{B}$ defined by $\check{\psi}(a) = \mathcal{D}T_a C$, is also a normalization map. Here $T_a C \subset \mathbf{P}^2$ (resp. $T_a C \subset \check{\mathbf{P}}^2$) is the tangent line to the irreducible curve germ $\psi(A, a)$ at $\psi(a)$. The double dual $\check{\check{B}}$ is isomorphic to B.

We shall need to compare the degree and singularities of a curve with those of its dual. To this end, define the ramification divisor of $\psi \colon A \to B$ to be $R_\psi = \sum_{a \in A} (m_\psi(a) - 1)a$. Here $m_\psi(a)$ is the largest integer k such

that $\psi^* \mathfrak{m}_{B,\psi(a)} \subset \mathfrak{m}_{A,a}^k$, where $\mathfrak{m}$ denote the maximal ideals. We then have the following Plücker-type formula:

$$(1.1) \qquad 2 \deg B - \deg \check{B} - \deg R_\psi = \chi(A) = 2 \deg \check{B} - \deg B - \deg R_{\check{\psi}},$$

where $\chi(A)$ is the topological Euler characteristic of A. This is proved using the Riemann-Hurwitz formula as in [**GH**, pp. 277–280]; see also [**Ho**, p. 289].

A *web* W on $\mathbf{P}^2$ of degree δ is locally defined by an unordered set of δ holomorphic (possibly singular) foliations. In particular, through a general point $p \in \mathbf{P}^2$ passes exactly δ leaves, and these intersect transversely at p. Globally, the leaves may exhibit complicated behavior and even be dense in $\mathbf{P}^2$. See e.g. [**Pe**], [**Pi1**], [**GS**] for general facts on webs.

We shall only consider the particular case of an *algebraic web* on $\mathbf{P}^2$. By definition, this is a web $W = W_C$ given by a reduced curve $C \subset \check{\mathbf{P}}^2$ of degree $\delta > 1$. The leaves of W_C are exactly the lines in $\mathbf{P}^2$ corresponding to the points on C. Through a generic point $p \in \mathbf{P}^2$ passes exactly δ distinct lines of the web. See Figure 1 for a picture of the algebraic web associated to a conic C.
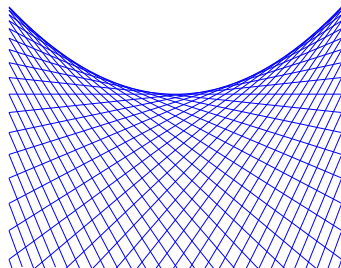

Figure 1. The algebraic web associated to a conic.

Assume that C is irreducible and $\delta = \deg C > 1$. The normalization map $\psi\colon A \to C$ then induces a symmetric rational map

$$\pi\colon A \times A \dashrightarrow \mathbf{P}^2$$

defined by $\pi(a_1, a_2) = \mathcal{D}L(\psi(a_1), \psi(a_2))$, where $L(c_1, c_2) \subset \check{\mathbf{P}}^2$ is the line passing through (distinct) points $c_1, c_2 \in \check{\mathbf{P}}^2$. Let us record some facts that are easily established by direct computation. The indeterminacy locus I_π of π is exactly the set of pairs (a, a) with $a \in R_\psi$ and pairs (a, b) with $a \neq b$ but $\psi(a) = \psi(b)$. In particular, C is smooth if and only if π is holomorphic.

We have $\pi(\Delta) = \check{C}$, where $\Delta \subset A \times A$ denotes the diagonal and $\check{C} \subset \mathbf{P}^2$ the dual curve. More precisely, $\pi(a,a) = \check{\psi}(a)$. Note that π has topological degree $\delta(\delta - 1)$. Further, π is locally biholomorphic outside $\Delta \cup I_\pi$.

2. Selfmaps of curves and of the plane

Consider a smooth algebraic curve A of genus g and a surjective holomorphic map $\phi\colon A \to A$ of topological degree $d > 1$. The canonical divisor class K_A has degree $2g - 2$. The Riemann-Hurwitz formula asserts $K_A = \phi^* K_A + R_\phi$, where R_ϕ is the ramification divisor. Taking degrees, we find $0 \le \deg R_\phi = (d-1)(2-2g)$. As $d > 1$, we have $g = 0$ or $g = 1$, that is, A is a rational or elliptic curve.

A subset $E \subset A$ is *totally invariant* if $\phi^{-1}(E) \subset E$. For finite subsets E, this in fact implies $\phi^{-1}(E) = E = \phi(E)$. When $A = \mathbf{C}/\Lambda$ is an elliptic curve, ϕ lifts to an affine map $\tilde{\phi}\colon \mathbf{C} \to \mathbf{C}$ and one easily sees that there is no finite totally invariant set. When A is rational, it is not hard to prove that a totally invariant set contains at most two points, see e.g. [**CG**, Theorem 1.5, p. 56].

A holomorphic map $f\colon \mathbf{P}^2 \to \mathbf{P}^2$ can be written in homogeneous coordinates as $f[x : y : z] = [P(x,y,z) : Q(x,y,z) : R(x,y,z)]$, where P, Q and R are homogeneous polynomials on $\mathbf{C}^3$ of the same degree $d \ge 1$, and $\{P = Q = R = 0\} = \{0\}$. The number d is the *algebraic degree* of f; we shall assume $d > 1$. The topological degree of f is d^2. The ramification divisor R_f of f has degree $3(d-1)$. It is known [**FS2**], [**CL**], [**D**], [**SSU**] that if $C \subset \mathbf{P}^2$ is a (reduced, but possibly reducible) curve such that $f^{-1}(C) \subset C$, then $f^{-1}(C) = C = f(C)$ and C is the union of at most three lines. Any such line $L \subset C$ occurs with multiplicity $d-1$ in R_f.

3. Invariant webs

Consider the algebraic web W_C associated to a curve $C \subset \check{\mathbf{P}}^2$ of degree δ. Assume for now that C is irreducible.

Let $f\colon \mathbf{P}^2 \to \mathbf{P}^2$ be a holomorphic mapping of algebraic degree $d \ge 2$. We say that the web is *invariant* under f if the image under f of any line in the web is again a line in the web. There is then an induced selfmap $g\colon C \to C$ defined by $g = \mathcal{D} \circ f \circ \mathcal{D}$. The web is *totally invariant* if the preimage of any line in the web is a union of lines in the web.

Proposition 3.1. *A web is invariant if and only if it is totally invariant. Moreover, if the web is invariant, then:*

  (i) *the induced map $g\colon C \to C$ is regular, of topological degree d;*
  (ii) *for any $c \in C$, we have $f_* \mathcal{D}c = d\mathcal{D}g(c)$ as divisors on $\mathbf{P}^2$.*

Proof: Clearly, the web is invariant if it is totally invariant. Also, (ii) is clear, since f_* multiplies the degree of any effective divisor by d.

For $k \geq 1$, let $\mathcal{A}_k \simeq \mathbf{P}^{\frac{k(k+3)}{2}}$ denote the space of effective divisors of degree k on $\mathbf{P}^2$. Then $\mathcal{A}_1 = \check{\mathbf{P}}^2$. Let $\rho_d \colon \mathcal{A}_1 \to \mathcal{A}_d$ be the Veronese map given by multiplication by d: $\rho_d(L) = dL$ and $f_* \colon \mathcal{A}_1 \to \mathcal{A}_d$ the pushforward map induced by f. Then f_* is holomorphic, of topological degree d^2, and ρ_d is a holomorphic embedding. Suppose the web associated to C is invariant. Then $f_* = \sigma_d \circ g$, so g is holomorphic. Now $(f_*)^* \mathcal{O}_{\mathcal{A}_d}(1) = d^2 \mathcal{O}_{\mathcal{A}_1}(1)$ and $\rho_d^* \mathcal{O}_{\mathcal{A}_d}(1) \simeq d \mathcal{O}_{\mathcal{A}_1}(1)$. Restricting to C implies $g^* \mathcal{O}_C(1) = d \mathcal{O}_C(1)$, so g has topological degree d. In particular, the preimage of every line in the web is the union of (at most) d lines in the web, so the web is totally invariant. $\qquad\square$

The induced selfmap $g \colon C \to C$ preserves collinearity: if $c_1, c_2, c_3 \in C$ are collinear in $\check{\mathbf{P}}^2$, then so are $g(c_1)$, $g(c_2)$, $g(c_3)$. Indeed, $\bigcap_i \mathcal{D}g(c_i) = f(\bigcap_i \mathcal{D}c_i)$. Conversely, if $g \colon C \to C$ is a surjective holomorphic map preserving collinearity, then there is a unique holomorphic mapping $f \colon \mathbf{P}^2 \to \mathbf{P}^2$ satisfying $g = \mathcal{D} \circ f \circ \mathcal{D}$.

Clearly, $g \colon C \to C$ lifts uniquely through the normalization map $\psi \colon A \to C$ to a holomorphic selfmap $\phi \colon A \to A$ of topological degree $d > 1$. In particular, A is rational or elliptic. Moreover, $f \colon \mathbf{P}^2 \to \mathbf{P}^2$ lifts through the rational map $\pi \colon A \times A \dashrightarrow \mathbf{P}^2$ to the selfmap $A \times A \xrightarrow{(\phi, \phi)} A \times A$. This implies $f(\check{C}) = \check{C}$.

For the proof of Theorem A we need to compare the ramification divisors $R_f \subset \mathbf{P}^2$ and $R_\phi \subset A$ of f and ϕ, respectively. In general, R_f has degree $3(d-1)$. Since f preserves the algebraic web associated to $C \subset \check{\mathbf{P}}^2$, we can write $R_f = R_f^C + R_f^\sigma$, where R_f^C is the part of R_f supported on the lines of the web, and R_f^σ is the "sectional" part of R_f.

Lemma 3.2. *If $a \in A$ and $\psi(a) \notin \operatorname{sing} C$, then the multiplicity of the point a in R_ϕ equals the multiplicity of the line $\mathcal{D}\psi(a)$ in R_f^C.*

Proof: If $a, b \in A$, $\psi(a), \psi(b) \notin \operatorname{sing} C$ and $\psi(a) \neq \psi(b)$, then $\pi \colon A \times A \dashrightarrow \mathbf{P}^2$ is a local biholomorphism at (a, b). Picking b generic gives the result. $\qquad\square$

4. Examples

We now go through the examples appearing in Theorem A and briefly discuss their dynamics.

4.1. A line. The case when C is a line corresponds to a pencil of lines through a point $p \in \mathbf{P}^2$. In other words, $C = \mathcal{D}p$. We have $\deg R_f^C = 2(d-1)$ and $\deg R_f^\sigma = (d-1)$.

Pick homogeneous coordinates $[x : y : z]$ on $\mathbf{P}^2$ such that $p = [0 : 0 : 1]$. Then f preserves C if and only if it takes the form $f[x : y : z] = [P(x,y) : Q(x,y) : R(x,y,z)]$.

Holomorphic selfmaps of $\check{\mathbf{P}}^2$ preserving a pencil of curves were classified in [**DJ**]. Their dynamics is studied in [**J2**], [**J3**].

4.2. A conic. Suppose $C \subset \check{\mathbf{P}}^2$ is a smooth conic so that $A \simeq C \simeq \mathbf{P}^1$. Pick any holomorphic selfmap $\phi \colon \mathbf{P}^1 \to \mathbf{P}^1$ of degree $d > 1$. The map $\pi \colon \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^2$ is then holomorphic, of topological degree 2, and $\mathbf{P}^1 \times \mathbf{P}^1 \xrightarrow{(\phi,\phi)} \mathbf{P}^1 \times \mathbf{P}^1$ induces a holomorphic selfmap $f \colon \mathbf{P}^2 \to \mathbf{P}^2$ of algebraic degree d. Any holomorphic selfmap f of $\mathbf{P}^2$ preserving a conic $C \subset \check{\mathbf{P}}^2$ is of this form. We have $\deg R_f^C = 2(d-1)$ and $\deg R_f^\sigma = (d-1)$. The dual curve $\check{C} \subset \mathbf{P}^2$ is an invariant smooth conic and $f^*\check{C} = \check{C} + 2|R_f^\sigma|$, see Lemma 5.2. Selfmaps as above were first introduced in a dynamic setting by Ueda [**U1**]. They can be used to provide simple examples of critically finite maps, in particular maps whose Julia set is all of $\mathbf{P}^2$ [**U1**, Proposition 4.1].

4.3. A smooth cubic. Let $C \subset \check{\mathbf{P}}^2$ be a smooth cubic. Then $C \simeq \mathbf{C}/\Lambda$ is an elliptic curve and admits a group law: three points in C are collinear if and only if their sum is zero. Choose the origin of the group law $(C, +, 0)$ at any flex of C and consider any holomorphic selfmap $g \colon C \to C$. Then g preserves collinearity if and only if its translation factor is a flex of C. In this case, g induces a selfmap $f \colon \mathbf{P}^2 \to \mathbf{P}^2$ preserving the web associated to C, and any such f is of this form.

We have $R_f^C = \emptyset$ and $\deg R_f^\sigma = 3(d-1)$. The dual curve $\check{C} \subset \mathbf{P}^2$ is an invariant sextic with nine cusps and $f^*\check{C} = \check{C} + 2|R_f^\sigma|$, see Lemma 5.2.

In particular, f is always critically finite, $f(|R_f|) = \check{C} = f(\check{C})$. It follows from [**U2**, Theorem 5.9] that the Julia set of f is all of $\mathbf{P}^2$.

4.4. A nodal cubic. Let $C \subset \check{\mathbf{P}}^2$ be a nodal cubic, with the node at c_*. There is a geometrically defined multiplicative group law on $C^* := C \backslash \{c^*\}$, given as follows: $c_1 c_2 c_3 = e$ in the group if and only if c_1, c_2 and c_3 are collinear in $\check{\mathbf{P}}^2$. Here the unit element e can be chosen as any of the three flexes of C. Concretely, the unique normalization map $\mathbf{P}^1 \simeq A \xrightarrow{\psi} C$ such that $\psi(0) = \psi(\infty) = c_*$ and $\psi(1) = e$ restricts to a group homomorphism $\psi \colon \mathbf{C}^* \to C^*$. We then see that $g \colon C \to C$ preserves collinearity if and

only if $\phi\colon A \to A$ takes the form $\phi(a) = \tau a^{\pm d}$, where $\tau^3 = 1$. In fact, by changing the flex representing e, we obtain $\tau = 1$.

We have $\deg R_f^C = (d-1)$ and $\deg R_f^\sigma = 2(d-1)$. The dual curve $\check{C} \subset \mathbf{P}^2$ is an invariant quartic with three cusps and one bitangent. We have $f^*\check{C} = \check{C} + 2|R_f^\sigma|$, see Lemma 5.2. In particular, f is critically finite.

In suitable coordinates on $\check{\mathbf{P}}^2$ and $\mathbf{P}^2$, we have $C = \{u^3 + v^3 = uvw\}$, $\psi(a) = [-a^2 : a : a^3 - 1]$, $\check{\psi}(a) = [2a^3 + 1 : 2a + a^4 : a^2]$ and $\pi(a,b) = [ab(a+b) + 1 : a + b + a^2b^2 : ab]$. The selfmap of $\mathbf{P}^2$ associated to $\phi(a) = a^d$ is then a polynomial mapping $f_d\colon \mathbf{C}^2 \to \mathbf{C}^2$ of the form $f_d(x,y) = (A_d(x,y,1), A_d(y,x,1))$, where $A_d(x+y+z, xy+yz+zx, xyz) = x^d+y^d+z^d$. For example, $f_2(x,y) = (x^2 - 2y, y^2 - 2x)$.

5. Proof of Theorem A

Without loss of generality, assume that $\delta := \deg C \geq 3$ and that C is not a smooth cubic. We shall prove, in a self-contained way, that C is a nodal cubic. An alternative approach, suggested by the referee, is to use a result in web theory to first show that $\deg C \leq 3$: see Remark 6.1.

We need two results, the proofs of which are given below. Let $m_c(C)$ denote the multiplicity of the curve C at a point c.

Lemma 5.1. *The singular locus* $\operatorname{sing} C$ *is totally invariant for* $g\colon C \to C$. *Moreover, if* $c \in \operatorname{sing} C$, *then* $m_c(C) = \delta - 1$. *As a consequence, the set* $A_s := \psi^{-1}(\operatorname{sing} C)$ *is totally invariant for* $\phi\colon A \to A$.

For the second result, recall the notation R_f^C and R_f^σ for the fiber and sectional parts of the ramification locus of f, respectively.

Lemma 5.2. *We have* $\deg \check{C} \leq \frac{2}{d-1} \deg R_f^\sigma$ *with equality if and only if* $f^*\check{C} = \check{C} + 2|R_f^\sigma|$ *as divisors.*

Now let us prove Theorem A. The map $\phi\colon A \to A$ has topological degree $d > 1$, so A has to be a rational or elliptic curve. Since we have assumed $\delta \geq 3$, C is singular. Lemma 5.1 implies that $A_s = \psi^{-1}(\operatorname{sing} C)$ is a totally invariant set for $\phi\colon A \to A$. This is impossible if A is elliptic, so C must be rational. Moreover, A_s consists of one or two points. Each such point corresponds to a line in $\mathbf{P}^2$ that is totally invariant for f, and hence contributes to R_f^C as a divisor of degree $d - 1$.

Case 1: $\#A_s = 1$. The point in A_s contributes a line of multiplicity $d - 1$ to R_f^C. By Lemma 3.2, the critical points of ϕ in $A \setminus A_s$ contribute lines of

total degree $d-1$ in R_f^C. Thus $\deg R_f^\sigma = d-1$. Lemma 5.2 gives $\check{\delta} \leq 2$, so that $\check{C}$, and hence C, is a conic. This contradicts the assumption $\delta \geq 3$.

Case 2: $\#A_s = 2$. Then $\phi\colon A \to A$ has no critical points outside A_s, so $\deg R_f^\sigma = d-1$ or $2(d-1)$, depending on whether $\#\operatorname{sing} C = 2$ or $\#\operatorname{sing} C = 1$. If $\deg R_f^\sigma = d-1$, Lemma 5.2 implies $\check{\delta} \leq 2$, so that $\check{C}$, and hence C is a smooth conic, contradicting $\delta \geq 3$. Hence suppose $\#\operatorname{sing} C = 1$ and $\deg R_f^\sigma = 2(d-1)$. Write $A_s = \{a, b\} = |R_\psi|$. Then $\psi(a) = \psi(b) = \operatorname{sing} C$. By Lemma 5.1, $\deg R_\psi = (m_\psi(a) - 1) + (m_\psi(b) - 1) = \delta - 3$. It follows from (1.1) applied to $B = C$ that $\delta = \check{\delta} - 1$. Now Lemma 5.2 shows that $\check{\delta} \leq 4$. We have assumed $\delta \geq 3$, hence $\delta = 3$. Since $\#\operatorname{sing} C = 1$ and $\#A_s = 2$, C is a nodal cubic.

Proof of Lemma 5.1: We may assume that $\delta \geq 3$ or else the irreducible curve C is smooth, and there is nothing to prove.

Note that for any $c \in C$, a generic line $l \subset \check{\mathbf{P}}^2$ through c intersects C in exactly $\delta + 1 - m_c(C)$ points. Dually, through a generic point on $\mathcal{D}c \subset \mathbf{P}^2$ passes exactly $\delta + 1 - m_c(C)$ distinct lines of the web. In particular, $c \in \operatorname{reg} C$ if and only if a generic point on the line $\mathcal{D}c \subset \mathbf{P}^2$ belongs to δ distinct lines of the web.

Now consider $c \in C$ and $c' := g(c) \in C$. Let $p \in \mathcal{D}c \subset \mathbf{P}^2$ be a generic point and write $p' := f(p)$. Assume that $c' \in \operatorname{sing} C$. We will show that $c \in \operatorname{sing} C$. This will prove that $\operatorname{sing} C$ is totally invariant.

First assume $\mathcal{D}c \not\subset |R_f|$. Then f is a local biholomorphism at p, so p and p' belong to the same number of lines of the web. Thus $c \in \operatorname{sing} C$.

Now assume $\mathcal{D}c \subset |R_f|$. Then the kernel of the differential Df_p is one-dimensional. Thus there is at most one line L in $\mathbf{P}^2$ through p such that the curve germ $f(L, p)$ is transverse to the line $\mathcal{D}c'$ at p'. This implies that $\delta + 1 - m_c(C) \leq 2$, that is, $m_c(C) \geq \delta - 1$. The reverse inequality always holds (since C is irreducible and not a line), and so $m_c(C) = \delta - 1$. In particular $c \in \operatorname{sing} C$.

Thus $\operatorname{sing} C$ is totally invariant. This implies that $\mathcal{D}c \subset |R_f|$ for every $c \in \operatorname{sing} C$. The above argument then shows $m_c(C) = \delta - 1$ for every $c \in \operatorname{sing} C$. $\square$

Proof of Lemma 5.2: The dual curve $\check{C}$ is the set of points in $\mathbf{P}^2$ belonging to $< \delta$ lines of the web. It is therefore clear that $f^{-1}\check{C} \subset \check{C} \cup |R_f|$. In fact, we have $f^{-1}\check{C} \subset \check{C} \cup |R_f^\sigma|$ since no line in the web is mapped into the irreducible curve $\check{C}$.

Write

$$f^*\check{C} = a\check{C} + \sum_j m_j X_j,$$

where $a \geq 0$, $m_j > 1$ and X_j are irreducible components of $|R_f^\sigma|$. Write $\lambda_j = \deg X_j$. Then $(d-a)\check{\delta} = \sum m_j \lambda_j$ and

$$\deg R_f^\sigma \geq \max\{a-1,0\}\check{\delta} + \sum (m_j - 1)\lambda_j$$

$$\geq \max\{a-1,0\}\check{\delta} + \frac{1}{2}\sum m_j \lambda_j$$

$$= \frac{1}{2}(2\max\{a-1,0\}\check{\delta} + (d-a)\check{\delta}) \geq \frac{1}{2}(d-1)\check{\delta}.$$

Equality holds if and only if $a = 1$, $m_j = 2$ for all j and $|R_f^\sigma| = \sum X_j$. $\square$

6. The reducible case

We now prove Theorem B. Let C_j be the irreducible components of C and write $\delta_j = \deg C_j$. Replacing f by an iterate, we may assume that the web associated to each C_j is (totally) invariant for f. Write $R_f = R_f^{C_j} + R_f^{\sigma_j}$ for any j.

First assume $\delta_j > 1$ for some j, say $j = 1$. From the analysis above we have $f(|R_f^{\sigma_1}|) = \check{C}_1$. For $i > 1$, this implies $\delta_i = 1$ and $R_f^{\sigma_1} = R_f^{\sigma_i}$, hence also $R_f^{C_1} = R_f^{C_i}$. Taking degrees and consulting Section 4 we see that C_1 must be a conic. It has to intersect each line C_i, $i > 1$, in two distinct points c_1, c_2. Then $R_f^{C_1} = (d-1)(\mathcal{D}c_1 + \mathcal{D}c_2)$. In particular, the line C_i is unique. Thus $C = C_1 \cup C_2$, where C_1 is a conic, C_2 is a line and $C_1 \cap C_2 = \{c_1, c_2\}$. As f preserves the web W_{C_1}, it comes from a selfmap of $C_1 \simeq \mathbf{P}^1$ for which c_1 and c_2 are totally invariant. Conversely, if g is such a selfmap, the associated map $f \colon \mathbf{P}^2 \to \mathbf{P}^2$ leaves the lines $\mathcal{D}c_1$ and $\mathcal{D}c_2$ totally invariant. One can then check that f preserves the linear pencil of lines through $\mathcal{D}c_1 \cap \mathcal{D}c_2$, that is, the web W_{C_1}.

Now suppose all the irreducible components of C are lines. We cannot have three concurrent lines, as then f would admit a totally invariant line l such that the restriction of f to l would have three totally invariant points. We also cannot have four lines, as f then would admit five totally invariant lines.

If C is a union of two lines, f can be written in suitable coordinates as a polynomial product map $f(x,y) = (p(x), q(y))$, where $\deg p = \deg q = d$. If C is a union of three lines, then $f(x,y) = (x^d, y^d)$.

Remark 6.1. As the referee points out, one can use known results in web geometry to directly show that $\deg C \leq 3$ in Theorems A and B. Indeed, it is known that if $(W, 0)$ and $(W', 0)$ are germs of linear webs of degree ≥ 4 on $\mathbf{C}^2$ (i.e. the leaves are lines and through a general point passes at least four leaves) and $\Phi\colon (\mathbf{C}^2, 0) \to (\mathbf{C}^2, 0)$ is a local biholomorphism mapping leaves to leaves, then Φ extends to an automorphism of $\mathbf{P}^2 \supset \mathbf{C}^2$. If $\deg C \geq 4$, one can get a contradiction by taking Φ as the germ of $f\colon \mathbf{P}^2 \to \mathbf{P}^2$ at a generic point. The result above goes back to the Hamburg school of web geometry: see [**BB**, Section 42] or [**He**, Corollaire 2, p. 535] for a precise modern reference. See also [**Pi2**] for related results.

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