Publ. Mat. **54** (2010), 73–81

f-POLYNOMIALS, h-POLYNOMIALS AND l^2-EULER CHARACTERISTICS

Dan Boros

Abstract

We introduce a many-variable version of the f-polynomial and h-polynomial associated to a finite simplicial complex. In this context the h-polynomial is actually a rational function. We establish connections with the l^2-Euler characteristic of right-angled buildings. When L is a triangulation of a sphere we obtain a new formula for the l^2-Euler characteristic.

1. f-polynomials and h-polynomials

Let us recall the definitions of the f-polynomial and h-polynomial. Suppose L is a finite simplicial complex of dimension $m-1$, that f_i is the number of i-simplices in L and that $f_{-1}=1$. The f-vector of L is the m-tuple $(f_{-1}, f_0, \dots, f_{m-1})$ and the h-vector $(h_0, \dots, h_m)$ is defined by the equation:

$$\sum_{i=0}^{m} f_{i-1}(t-1)^{m-i} = \sum_{i=0}^{m} h_i t^{m-i}.$$

The f-polynomial $f(t) = f_L(t)$ and the h-polynomial $h(t) = h_L(t)$ are defined by:

$$f(t) = \sum_{i=0}^{m} f_{i-1} t^i$$

$$h(t) = \sum_{i=0}^{m} h_i t^i.$$

Replacing t by $t-1$ in the equation defining the h-vector and multiplying each side by t^m we see that the relation between the f-polynomial

2000 *Mathematics Subject Classification.* 57M15, 20F55, 52B11.
Key words. h-polynomial, l^2-Euler characteristic.

and h-polynomial can be written as:

$$h(t) = (1-t)^m f\left(\frac{t}{1-t}\right)$$

or, by replacing t by $-t$ as:

$$h(-t) = (1+t)^m f\left(\frac{-t}{1+t}\right).$$

It is $h(-t)$ that is of interest in l^2-topology.

For example, the f-polynomial and h-polynomial for a triangulation of a 1-sphere as a 4-gon are:

$$f(t) = 1 + 4t + 4t^2$$

$$h(t) = t^2 + 2t + 1.$$

We now proceed to define the f-polynomial and h-polynomial in several variables. Given a finite simplicial complex L as above, denote by $\mathcal{S}(L)$ the set of simplices in L together with the empty set $\emptyset$. Let $v_1, v_2, \ldots, v_n$ be the vertices of L and $\mathbf{t} = (t_1, t_2, \ldots, t_n)$. If $\sigma \in \mathcal{S}(L)$ let $I(\sigma) = \{i | v_i \in \sigma\}$.

We define the monomials:

$$\mathbf{t}_\sigma = \prod_{i \in I(\sigma)} t_i \text{ and } \mathbf{t}_\emptyset = 1.$$

Similarly, we define:

$$(\mathbf{1}+\mathbf{t})_\sigma = \prod_{i \in I(\sigma)} (1 + t_i)$$

$$\left(\frac{-\mathbf{t}}{\mathbf{1}+\mathbf{t}}\right)_\sigma = \prod_{i \in I(\sigma)} \frac{-t_i}{1+t_i}.$$

In this several variables context the correct version of the f-polynomial $f(\mathbf{t}) = f_L(\mathbf{t})$ is defined by:

$$f(\mathbf{t}) = \sum_{\sigma \in \mathcal{S}(L)} \mathbf{t}_\sigma$$

while the "h-polynomial" $H(\mathbf{t}) = H_L(\mathbf{t})$ is defined by:

$$H(\mathbf{t}) = f\left(\frac{-\mathbf{t}}{\mathbf{1}+\mathbf{t}}\right).$$

For example, if L is a 4-gon then $\mathbf{t} = (t_1, t_2, t_3, t_4)$ and

$$f(\mathbf{t}) = 1 + t_1 + t_2 + t_3 + t_4 + t_1t_2 + t_2t_3 + t_3t_4 + t_4t_1$$

while

$$H(t_1, t_2, t_3, t_4) = f\left(\frac{-t_1}{1+t_1}, \frac{-t_2}{1+t_2}, \frac{-t_3}{1+t_3}, \frac{-t_4}{1+t_4}\right).$$

When $t_1 = t_2 = \cdots = t_m = t$ we obtain the one variable version of the f-polynomial but

$$H(t) = \frac{h(-t)}{(1+t)^m}.$$

This version of the "h-polynomial" is useful for applications in l^2-topology. In combinatorics, $H(-\mathbf{t})$ could be of interest.

2. h-polynomials and l^2-Euler characteristics

Associated to a finite simplicial complex L we have a right-angled Coxeter system (W_L, S) and two other simplicial complexes, K_L and the Davis complex Σ_L. The right-angled Coxeter group associated to L, denoted W_L, is defined as follows. The set of generators S is the vertex set of L and the edges of L give relations: $s^2 = 1$ and $(st)^2 = 1$, whenever $\{s, t\}$ spans an edge in L. Note that W_L depends only on the 1-skeleton of L. K_L is the cone on the barycentric subdivision of L. Equivalently, K_L can be viewed as the geometric realization of $\mathcal{S}(L)$, where $\mathcal{S}(L)$ denotes the poset of vertex sets of simplices of L (including the empty simplex). For each $s \in S$, K_s is the closed star of the vertex corresponding to s in the barycentric subdivision of L. Equivalently, K_s can be viewed as the geometric realization of the poset $\mathcal{S}(L)_{\geq s}$. Σ_L is constructed by pasting together copies of K_L along the mirrors K_s, one copy of K_L for each element of W_L. For more details on this construction we refer the reader to Chapter 5 of [**3**]. Given $\mathbf{t} = (t_i)_{i \in S}$, where t_i are positive integers, one can define a group G_L and the Davis realization $\Sigma(\mathbf{t}, L)$ of a right-angled building of thickness $\mathbf{t}$ (whose apartments are copies of Σ_L). The group G_L is defined as follows: Suppose we are given a family of groups $(P_i)_{i \in S}$ such that each P_i is a cyclic group of order $t_i + 1$. G_L is defined as the graph product of the $(P_i)_{i \in S}$ with respect to L. Note that G_L depends only on the 1-skeleton of L and the thickness $\mathbf{t}$. $\Sigma(\mathbf{t}, L)$ is obtained by pasting together copies of K_L, one for each element of G_L, using the same construction used for defining Σ_L . Explicitly, given $x \in K_L$, let $S(x)$ be the set of $s \in S$ such that x belongs to K_s. Then $\Sigma(\mathbf{t}, L)$ is the quotient of $G_L \times K_L$ by the equivalence relation $\sim$, where $(g, x) \sim (g', x')$ if and only if $x = x'$ and g, g' belong to the same coset of $G_{S(x)}$. In this definition, $\Sigma(\mathbf{t}, L)$ is a simplicial complex. There is one orbit of simplices for each simplex σ in K_L.

Let L be a finite simplicial complex and let G the group associated to $\Sigma(\mathbf{t}, L)$. One can define the l^2-Euler characteristic by

$$\chi_{\mathbf{t}}(L) = \sum_{\sigma} \frac{(-1)^{\dim(\sigma)}}{|G_\sigma|}$$

where the sum runs over all orbits of simplices in $\Sigma(\mathbf{t}, L)$ and G_σ denotes the stabilizer of the simplex σ. Equivalently, the sum runs over all simplices σ of K_L.

Lemma 2.1. *Let X be an index set with n elements. Then:*

$$\sum_{A\subset X} (-1)^{n-|A|} \frac{1}{(\mathbf{1}+\mathbf{t})_{X-A}} = \left(\frac{-\mathbf{t}}{\mathbf{1}+\mathbf{t}}\right)_X .$$

Proof: Dividing each side by

$$(\mathbf{1}+\mathbf{t})_X$$

in the formula from Lemma 3.1 we get

$$\sum_{A\subset X} (-1)^{n-|A|} \frac{1}{(\mathbf{1}+\mathbf{t})_{X-A}} = \left(\frac{-\mathbf{t}}{\mathbf{1}+\mathbf{t}}\right)_X . \qquad \square$$

Theorem 2.2. *Let L be a finite simplicial complex and $\Sigma(\mathbf{t}, L)$ the Davis realization of a right-angled building of thickness $\mathbf{t}$. Then:*

$$\chi_{\mathbf{t}}(L) = H(\mathbf{t}).$$

Proof: To prove this formula we divide K_L into subcomplexes (not disjoint), namely the geometric realization of $\mathcal{S}(L)_{\geq T}$, where $T \in \mathcal{S}(L)$ and denoted K_T. Each such subcomplex groups together simplices in K_L that have the same isotropy group. Let B_T be the subset consisting of the union of open simplices in one such subcomplex. The B_T are disjoint. Let C_T be the orbit of B_T under G_T. Then the orbits of C_T partition $\Sigma(t, L)$. If we take the signed sum of the simplices in C_T, divided by the order of their isotropy subgroup, we get the following alternative formula for the l^2-Euler characteristic (compare [**2**]).

$$\chi_{\mathbf{t}}(L) = \sum_{T\in\mathcal{S}(L)} \frac{1-\chi(\partial K_T)}{|G_T|}$$

where ∂K_T denotes the boundary complex of K_T, $\chi(\partial K_T)$ denotes the ordinary Euler characteristic of ∂K_T and $|G_T|$ denotes the order of the finite isotropy group G_T. Note that there is a one-to-one correspondence where each T corresponds to a simplex σ in L.

On the other hand, in the formula that defines $H(\mathbf{t})$, the sum runs over all simplices of L. Each term

$$\left(\frac{-\mathbf{t}}{\mathbf{1}+\mathbf{t}}\right)_\sigma = \prod_{i\in I(\sigma)} \frac{-t_i}{1+t_i}$$

can be written as

$$\sum_{A\subset I(\sigma)} (-1)^{I(\sigma)-|A|} \frac{1}{(\mathbf{1}+\mathbf{t})_{I(\sigma)-A}}$$

according to the previous lemma. The sum runs over all subsets of $I(\sigma)$ including the empty set.

We sum now over all simplices σ in L and grouping terms with the same denominator it is easily seen that $H(\mathbf{t})$ coincides with the alternative formula for $\chi_{\mathbf{t}}(L)$ written above. □

3. Convex polytopes and l^2-Euler characteristics

We now restrict our attention to triangulations of spheres which are dual to simple polytopes. We prove another formula for the l^2-Euler characteristic but here we take a dual approach. Let P^n be an n-dimensional simple convex polytope (an n-dimensional convex polytope is simple if the number of codimension-one faces meeting at each vertex is n; equivalently, P^n is simple if the dual of its boundary complex is an $(n-1)$-dimensional simplicial complex). For more on convex polytopes we refer the reader to [**1**]. The f- and h-polynomials associated to a finite simplicial complex, as well as their many-variable analogues, were defined in Section 1. Similarly, for an n-dimensional simple convex polytope the associated f- and h-polynomial are defined as those associated to the dual of its boundary complex.

Let P^n be an n-dimensional simple convex polytope. Let $F_1, F_2, \dots, F_m$ be the codimension-one faces of P^n (also called facets). Let $\mathcal{F}$ denote the set of all faces of P_n and $\mathbf{t} = (t_1, t_2, \dots, t_m)$. If $F \in \mathcal{F}$, then $I(F) = \{i | F \subset F_i\}$. We define:

$$\mathbf{t}_F = \prod_{i\in I(F)} t_i \text{ and } \mathbf{t}_P = 1.$$

Similarly, we define:

$$\left(\frac{-\mathbf{t}}{\mathbf{1}+\mathbf{t}}\right)_F = \prod_{i\in I(F)} \frac{-t_i}{1+t_i}.$$

We now introduce a different formula for the h-polynomial in several variables. The vertices and edges of a polytope form in an obvious way a nonoriented graph. Following [**1**, pp. 93–96] we introduce an orientation on the edges of P^n using an admissible vector. A vector $w \in \mathbb{R}^n$ is called admissible for P^n if $\langle x, w\rangle \neq \langle y, w\rangle$ for any two vertices x and y of P^n. Geometrically, this means that no hyperplane in $\mathbb{R}^n$ with w as a normal contains more than one vertex of P^n. It is shown in [**1**, Theorem 15.1, p. 93] that the set of admissible vectors is dense in $\mathbb{R}^n$. Any vector w which is admissible for P^n induces an orientation of the edges of P^n according to the following rule: An edge determined by vertices x and y is oriented towards x (and away from y) if

$$\langle x, w\rangle \leq \langle y, w\rangle.$$

For each vertex $v \in P^n$, denote by $F_v^{\text{in}} \in \mathcal{F}$ the face determined by the inward-pointing edges at v, and by $F_v^{\text{out}} \in \mathcal{F}$ the face determined by the outward-pointing edges at v. We have $I(v) = \{i | v \in F_i\}$. Moreover:

$$I(F_v^{\text{in}}) = \{i | F_v^{\text{in}} \subset F_i\} \text{ and } I(F_v^{\text{out}}) = I(v) - I(F_v^{\text{in}}).$$

Using an admissible vector w we now define:

$$H_w(\mathbf{t}) = \sum_v \frac{(-\mathbf{t})_{F_v^{\text{out}}}}{(\mathbf{1}+\mathbf{t})_v} = \sum_v \frac{\prod_{i\in I(F_v^{\text{out}})}(-t_i)}{\prod_{i\in I(v)}(1+t_i)}.$$

For example, if P^2 is a 4-gon, then

$$H_w(\mathbf{t}) = \frac{(-t_1)(-t_2)}{(1+t_1)(1+t_2)} + \frac{(-t_2)}{(1+t_2)(1+t_3)} + \frac{(-t_1)}{(1+t_1)(1+t_4)} + \frac{1}{(1+t_3)(1+t_4)}$$

which can be simplified to

$$H_w(t_1,t_2,t_3,t_4) = \frac{(1-t_1t_3)(1-t_2t_4)}{(1+t_1)(1+t_2)(1+t_3)(1+t_4)}.$$

To prove the next theorem we need the following combinatorial lemma.

Lemma 3.1. *Let X be an index set with n elements. Then:*

$$\sum_{A\subset X} (-1)^{n-|A|}(\mathbf{1}+\mathbf{t})_A = (-\mathbf{t})_X.$$

Proof: The following identity is well-known:

$$(\mathbf{x}-\mathbf{u})_X = \sum_{A\subset X} x^{n-|A|}(-\mathbf{u})_A$$

where $\mathbf{x} = (x,\dots,x)$. Let $\mathbf{u} = \mathbf{1}+\mathbf{t}$ where $\mathbf{1} = (1,\dots,1)$ and evaluate the above identity when $x = 1$. □

Theorem 3.2. *Let P^n be an n-dimensional simple convex polytope and denote by L the dual of its boundary complex. Suppose L is a finite simplicial complex and w is an admissible vector for P^n. Then*

$$\chi_{\mathbf{t}}(L) = H_w(\mathbf{t}).$$

Proof: To prove this formula we use a different "cell" structure on $\Sigma(t, L)$ obtained with the help of an admissible vector. We refer the reader to [**4**] for more details concerning this construction. There is one orbit of (open) "cells" for each vertex $v \in P^n$. The dimension of C_v is $\dim(F_v^{\text{in}})$. C_v is constructed as follows. Let $\widehat{F_v^{\text{in}}}$ denote the union of the relative interiors of those faces $F' \in \mathcal{F}$ which are contained in F_v^{in} and contain v. The "cell" C_v consists of $\widehat{F_v^{\text{in}}}$ and all its translates under $G_{F_v^{\text{in}}}$. To prove our formula we have to show that the contribution c_v of C_v to the l^2-Euler characteristic is exactly

$$\frac{(-\mathbf{t})_{F_v^{\text{out}}}}{(\mathbf{1}+\mathbf{t})_v}.$$

For a face F we denote by G_F its stabilizer. $|G_F|$ denotes the order of the finite group G_F. We have:

$$c_v = \sum_{F' \subset \widehat{F_v^{\text{in}}}} \frac{(-1)^{\dim(F')}}{|G_{F'}|} = \sum_{F' \subset \widehat{F_v^{\text{in}}}} \frac{(-1)^{\dim(F')} \frac{|G_v|}{|G_{F'}|}}{|G_v|}$$

$$= \sum_{F' \subset \widehat{F_v^{\text{in}}}} \frac{(-1)^{\dim(F')} |G_{I(v)-I(F')}|}{|G_v|}$$

where $|G_{I(v)-I(F')}|$ denotes $\prod_{i \in I(v)-I(F')}(t_i + 1)$. Since $|G_v| = (\mathbf{1}+\mathbf{t})_v$ the proof is complete if

$$\sum_{F' \subset \widehat{F_v^{\text{in}}}} (-1)^{\dim(F')} |G_{I(v)-I(F')}| = (-\mathbf{t})_{F_v^{\text{out}}}.$$

Written explicitly, the above formula coincides with the identity proved in the previous lemma. Summing over vertices of P^n, the proof is completed. $\square$

Remark 3.3. It follows from the previous theorem that $H_w(\mathbf{t})$ does not depend on the choice of the admissible vector w.

The next corollary is a reciprocity statement.

Corollary 3.4. *Let P^n be an n-dimensional simple convex polytope and $H(\mathbf{t})$ be the "h-polynomial" associated to P^n. Then:*

$$H(\mathbf{t}) = (-1)^{n+1} H(\mathbf{t}^{-1})$$

where $\mathbf{t}^{-1} = (t_1^{-1}, t_2^{-1}, \dots, t_m^{-1})$.

Proof: If $w \in \mathbb{R}^n$ is an admissible vector for P^n then $-w \in \mathbb{R}^n$ is also an admissible vector for P^n. A simple calculation shows that:

$$H_{-w}(\mathbf{t}) = (-1)^{n+1} H_w(\mathbf{t}^{-1}).$$

The result follows since by the previous theorem we have:

$$H_w(\mathbf{t}) = H_{-w}(\mathbf{t}).$$ □

Remark 3.5. Let L be a finite simplicial complex, P^n its dual simple polytope and w an admissible vector for P^n. Then the l^2-Euler characteristic, the growth series of the associated Coxeter group W, the associated f-polynomial and "h-polynomial" in several variables are related as follows:

$$\chi_{\mathbf{t}}(L) = \frac{1}{W(\mathbf{t})} = f\left(\frac{-\mathbf{t}}{\mathbf{1}+\mathbf{t}}\right) = H_w(\mathbf{t}) = H(\mathbf{t}).$$

The first equality is proven in [**6**] and the rest follow from the previous theorems.

Remark 3.6. Writing the corollary above in one variable we get:

$$\chi_t(L) = \frac{1}{W(t)} = f\left(\frac{-t}{1+t}\right) = \frac{h_w(-t)}{(1+t)^n} = \frac{h(-t)}{(1+t)^n}$$

where $h_w(t) = \sum_v t^{\mathrm{ind}(v)}$ and $\mathrm{ind}(v)$ denotes the index of the vertex v with respect to w in P.

Acknowledgement. The ideas developed in this paper were inspired by the program initiated by Davis and Okun in [**5**]. I would like to thank Professor M. W. Davis and the referee for their helpful comments and suggestions on this work.

References

[1] A. BRØNDSTED, *"An introduction to convex polytopes"*, Graduate Texts in Mathematics **90**, Springer-Verlag, New York-Berlin, 1983.

[2] R. CHARNEY AND M. DAVIS, Reciprocity of growth functions of Coxeter groups, *Geom. Dedicata* **39(3)** (1991), 373–378.

[3] M. W. DAVIS, *"The geometry and topology of Coxeter groups"*, London Mathematical Society Monographs Series **32**, Princeton University Press, Princeton, NJ, 2008.

[4] M. W. DAVIS AND T. JANUSZKIEWICZ, Convex polytopes, Coxeter orbifolds and torus actions, *Duke Math. J.* **62(2)** (1991), 417–451.
[5] M. W. DAVIS AND B. OKUN, Vanishing theorems and conjectures for the ℓ^2-homology of right-angled Coxeter groups, *Geom. Topol.* **5** (2001), 7–74 (electronic).
[6] J. DYMARA, Thin buildings, *Geom. Topol.* **10** (2006), 667–694 (electronic).

Department of Mathematical Sciences
Otterbein College
One Otterbein College
Westerville, Ohio 43081
USA
E-mail address: dboros@otterbein.edu

Primera versió rebuda el 14 d'agost de 2008,
darrera versió rebuda el 30 de setembre de 2008.