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ON POISSON-DIRICHLET PROBLEMS WITH POLYNOMIAL DATA

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Abstract

In this note we provide a probabilistic proof that Poisson and/or Dirichlet problems in an ellipsoid in $\mathbb{R}^d$, that have polynomial data, also have polynomial solutions. Our proofs use basic stochastic calculus. The existing proofs are based on famous lemma by E. Fisher which we do not use, and present a simple martingale proof of it as well.

1. Introduction and preliminaries

During the last few years there has been some work on polynomial solutions to problems of Dirichlet and/or Poisson types, in which the domain is the ellipsoid $\mathcal{E} = \{x \in \mathbb{R}^d \mid \sum_{i=1}^d (x_i/r_i)^2 < 1\}$, the boundary of which will be denoted by $\partial\mathcal{E}$. See for example [**1**], [**2**] and [**4**]. What is curious here is the fact that, the passage from data to solution, which is achieved by integration against appropriate Green or Poisson kernels preserves the polynomial character of the data. We shall begin by recalling some relevant results by the above mentioned authors.

It is proved in [**1**, Theorem 1.7] by Axler and Ramey, using basic properties of the derivation operator that

$$\mathcal{P}_m = \mathcal{H}_m + |x|^2\mathcal{P}_{m-2} \tag{1}$$

where $\mathcal{P}_m$ denotes the vector space of polynomials in d variables, homogeneous of degree m, and $\mathcal{H}_m$ is the subspace of harmonic polynomials of degree m. Furthermore, defining $c_0 = 1$ and $c_m = \prod_{j=0}^{m-1}(2-d-2j)$, it is also proved there that the family of mappings $\Lambda_m\colon \mathcal{P}_m \to \mathcal{H}_m$, defined

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by

$$\Lambda_m(p) = \frac{1}{c_m}|x|^{d-2-2m}[p(D)]|x|^{2-d} \tag{2}$$

is a projection operator, and in [**1**, Corollary 2.2] they prove that: *The solution to the Dirichlet problem on the sphere, with boundary data* $p|_{\partial\mathcal{E}}$ *is given by*

$$p_m + p_{m-2} + \cdots + p_{m-2k}$$

where $p_m := \Lambda_m(p)$ *and* $k := [\frac{m}{2}]$.

In [**2**] and [**4**], the authors use variations on a theme based on the following lemma by E. Fisher to obtain their results. It goes as follows

Lemma 1.1. *Let* $\mathcal{P}$ *denote the algebra of polynomials in* d *variables, and put* $b(x) = 1 - \sum_{k=1}^{d}(x_k/r_k)^2$. *Define* $L\colon \mathcal{P} \to \mathcal{P}$ *by* $L(p) := \Delta(bp)$ *where* Δ *is the standard Laplacian operator. Then* L *is a degree preserving, linear bijection of* $\mathcal{P}$ *into itself.*

If we denote by $\mathcal{P}^m$ the vector space of polynomials of degree at most m, the content of [**2**, Theorem 1.1] is that: *For any* $p \in \mathcal{P}^m$ *for* $m > 0$ *there exists* $u \in \mathcal{P}^m$ *such that*

$$\Delta u(x) = 0 \quad x \in \mathcal{E} \text{ and } u|_{\partial\mathcal{E}}(x) = p(x). \tag{3}$$

Herzog [**4**] obtains the complementary result as Theorem 1 as well, and it asserts that: *Given* $p \in \mathcal{P}^m$ *and* $q \in \mathcal{P}^{m+2}$ *then there exists* $u \in \mathcal{P}^{m+2}$ *such that*

$$\Delta u(x) = q(x) \quad x \in \mathcal{E} \text{ and } u|_{\partial\mathcal{E}}(x) = p(x). \tag{4}$$

We shall devote the rest of this section to establish some notation and to recall some basics about d-dimensional Brownian motion, and in Section 2 we shall apply the basics of stochastic calculus to present a probabilistic proof of (4). We shall use the standard notation for multi-indices: $\mathbf{n} = (n_1, \ldots, n_d)$ and all components are non negative integers, if x is a d-dimensional vector, then $x^{\mathbf{n}} = \prod x_j^{n_j}$ and by $|\mathbf{n}| = \sum n_j$ we shall denote the length of $\mathbf{n}$. The "unit" indices $\mathbf{e}_j$ have all components equal to zero except the j-th which equals 1. And we shall use $\langle x, y\rangle$ for the Euclidean scalar product of d-vectors x and y.

We refer the reader to either [**3**], [**5**] or [**6**] for the basic notations and properties of the Brownian motion process and stochastic calculus. Here we only recall that it consists of a collection $\mathbf{B} = \{\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, (B(t))_{t\geq 0}, (P^x)_{x\in\mathbb{R}^d}\}$; where for each $t \geq 0$, $B(t) = (B_1(t), \ldots, B_d(t))$, and the $B_j(t)$ are coordinate maps, defined on $\Omega := \{\omega\colon [0,\infty) \to \mathbf{R} \mid$

ω continuous} by $B_j(t)(\omega) = \omega(t)$. These coordinate maps turn out to define both a Markov process and a vector valued martingale, such that the components $B_j(t)$ are P^x-independent for each $x \in \mathbb{R}^d$. Below we shall also make use of the fact that $P^x(B(0) \in A) = 1$ or 0 depending on whether $x \in A$ or not. As usual E^x denotes integration with respect to P^x.

A standard result which we will be making use of repeatedly, is contained in

Theorem 1.1. *Let us set* $T := \inf\{t > 0 \mid B(t) \notin \mathcal{E}\}$. *Then,*

(a) $\sup\{E^x[T] \mid x \in \mathcal{E}\} < \infty$, *and*
(b) *the unique solution to* (4) *can be represented by*

$$u(x) = E^x[p(B(T))] + E^x\left[\int_0^T q(B(t))\,dt\right]. \tag{5}$$

All we have to do to reobtain the desired results is to prove that the representations appearing in (5) are actually polynomials. This will drop out as consequence of some simple lemmas presented in the next section.

After this we present a very short martingale proof of Fisher's Lemma.

2. The results

Let us begin with

Lemma 2.1. (a) *The functions* x_i *and* $x_i x_j$ *with* $i \neq j$ *are harmonic polynomials.*
(b) *There exists a quadratic form* Q *and a constant* k *such that* $x_j^2 + \langle x, Qx\rangle + k$ *is harmonic in* $\mathcal{E}$ *and has value* x_j^2 *on* $\partial\mathcal{E}$.
(c) *There are infinitely many polynomials* $q(x)$ *such that* $\Delta q(x) = 1$.

Proof: Case (a) is obvious. To get (b) set $u(x) = x_j^2 + \langle x, Qx\rangle + k$, where Q and k are to be determined, and note that $\Delta u(x) = 2(1+\Sigma Q_{ii})$, and that $u|_{\partial\mathcal{E}}(x) = x_j^2$ implies that $\langle x, Qx\rangle + k = 0$ on $\partial\mathcal{E}$. Note now that we can choose Q to be diagonal with elements $Q_{jj} = k/r_j^2$ and that we can choose k so that $(1 + k\Sigma r_j^{-2}) = 0$. To obtain (c) let R be any matrix with $tr(R) = 1$ and put $q(x) = \frac{1}{2}\langle x, Rx\rangle + \langle b, x\rangle + a$ where b is a d-vector and a is any real number. Check that $\Delta q(x) = 1$. □

Comment. We leave it up to the reader to verify that R, b and a can be chosen so that $q(x)$ vanishes on $\partial\mathcal{E}$.

Corollary 2.1. *Denote by* $q(0,x)$ *the solution to* $\Delta u(x) = 1$ *that vanishes on* $\partial\mathcal{E}$. *Then*

$$q(0,x) = E^x[T]. \tag{6}$$

Proof: This is a direct application of Theorem 1.1. □

Lemma 2.2. *Let* $q(\mathbf{e}_i, x) := c(i)x_i(\Sigma(x_j/r_j)^2 - 1)$. *For* $i = 1, \dots, d$ *the constant* $c(i)$ *can be chosen so that*

$$q(\mathbf{e}_i, x) = E^x\left[\int_0^T B_i(t)\,dt\right]. \tag{7}$$

Proof: This amounts to verifying that $c(i)$ can be chosen so that $\Delta q(\mathbf{e}_i, x) = x_i$. That $q(\mathbf{e}_i, x)$ vanishes on $\partial\mathcal{E}$ is clear. □

Lemma 2.3. *For any* $n \geq 0$ *and any* $x \in \mathbb{R}^d$

$$E^x\left[\int_0^T B_1^n(s)\,ds\right] = \frac{2}{(n+2)(n+1)}\{E^x[B_1^{n+2}(T)] - x_1^{n+2}\}. \tag{8}$$

Proof: This is just a standard application of Îto's formula for $B_1^{n+2}(T)$ and the property of zero expectation of stochastic integrals. □

Comment. The case $n = 0$ is related to Corollary 2.1 above.

Lemma 2.4 (Transference Lemma)**.** *For any multi-index* $\mathbf{n}$, *with* $|\mathbf{n}| \geq 2$,

$$E^x\left[\int_0^T B^{\mathbf{n}}(t)\,dt\right] = \frac{2}{(n_1+2)(n_1+1)}\left\{E^x[B^{\mathbf{n}+2\mathbf{e}_1}(T)] - x^{\mathbf{n}+2\mathbf{e}_1} - \sum_{j=2}^d n_j(n_j-1)E^x\left[\int_0^T B^{\mathbf{n}+2\mathbf{e}_1-2\mathbf{e}_j}(t)\,dt\right]\right\}. \tag{9}$$

Remarks. Recall that $B(t)^{\mathbf{n}} = \Pi B_j(t)^{n_j}$. Observe as well that $x^{\mathbf{n}+2\mathbf{e}_1-2\mathbf{e}_j}$ is a monomial of degree $|\mathbf{n}|$ in which two powers have been transferred from the x_j-th variable to the x_1-th variable. Hence the name of the lemma. Clearly, if originally j-th component of $\mathbf{n}$ satisfies $n_j \leq 1$, then the comment does not apply and the corresponding them does not apply in Îto's formula.

Proof: From now on we shall throw all terms leading to stochastic integrals with respect to Brownian motion under the generic $dM(t)$ without further ado. Note now that an application of Îto's formula yields

$$dB^{\mathbf{n}+2\mathbf{e}_1}(t) = dM(t) + \frac{1}{2}(n_1+2)(n_1+1)B^{\mathbf{n}}(t)\,dt + \frac{1}{2}\Sigma_{j=2}^{d} n_j(n_j-1)B^{\mathbf{n}+2\mathbf{e}_1-2\mathbf{e}_j}(t)\,dt$$

from which the desired relationship drops out after integration between 0 and T and taking expected values with respect to P^x. □

The previous lemmas constitute the basis for the inductive argument in

Theorem 2.1. *Let* $\mathbf{n}$ *be any multi-index with* $|\mathbf{n}| \geq 2$. *Then*

(a) *For any multi-index* $\mathbf{n}$ *there exist harmonic polynomials* $p(\mathbf{n}, x)$ *of degree* $|\mathbf{n}|$ *such that*

$$E^x[B^{\mathbf{n}}(T)] = p(\mathbf{n}, x). \tag{10}$$

(b) *For any multi-index* $\mathbf{n}$ *there exist harmonic polynomials* $q(\mathbf{n}, x)$ *of degree* $|\mathbf{n}| + 2$ *such that*

$$E^x\left[\int_0^T B^{\mathbf{n}}(t)\,dt\right] = q(\mathbf{n}, x). \tag{11}$$

Comment. Observe that (10) and (11) satisfy, respectively a Dirichlet and a Poisson problem, and so do both terms in (12) below. All that is actually needed to verify is that they are polynomials.

Proof: We shall proceed by induction on the degree of the polynomial. As commented above, the conclusions of the theorem hold for $|\mathbf{n}| \leq 2$. Assume that they hold for some $\mathbf{n}$ with $|\mathbf{n}| \geq 2$. Then, for any $j = 1, \ldots, d$, there exist two polynomials $p(\mathbf{n}+\mathbf{e}_j, x)$ and $q(\mathbf{n}+\mathbf{e}_j, x)$ of degrees $|\mathbf{n}|+1$ and $|\mathbf{m}|+3$ respectively, such that

$$E^x[B^{\mathbf{m}+\mathbf{e}_j}(T)] = p(\mathbf{m}+\mathbf{e}_j, x) \qquad \text{and}$$

$$E^x\left[\int_0^T B^{\mathbf{m}+\mathbf{e}_j}(t)\,dt\right] = q(\mathbf{m}+\mathbf{e}_j, x). \tag{12}$$

By Îto's formula and the optional sampling theorem

$$E^x[B^{\mathbf{n}+\mathbf{e}_j}(T)] = x^{\mathbf{n}+\mathbf{e}_j} + \frac{1}{2}\Sigma n_i'(n_i'+1)E^x\left[\int_0^T B^{\mathbf{n}+\mathbf{e}_j-2\mathbf{e}_i}(t)\,dt\right]$$

where $n'_i = n_i$ if $i \neq j$ and $n'_j = n_j + 1$. Taking into account the inductive hypotheses this becomes

$$E^x[B^{\mathbf{n}+\mathbf{e}_j}(T)] = x^{\mathbf{n}+\mathbf{e}_j} + \frac{1}{2}\Sigma n'_i(n'_i+1)q(\mathbf{n}+\mathbf{e}_j - 2\mathbf{e}_i, x) \tag{13}$$
$$\equiv p(\mathbf{n}+\mathbf{e}_j, x).$$

We thus display $E^x[B^{\mathbf{n}+\mathbf{e}_j}(T)]$ as a polynomial of degree $|\mathbf{n}|+1$, which at the boundary coincides with $x^{\mathbf{n}+\mathbf{e}_j}$.

Consider now $E^x[\int_0^T B^{\mathbf{m}+\mathbf{e}_j}(t)\,dt]$. By the transference lemma, this can be written as a sum of terms like that considered in the first part of the proof, plus a term like that in the right hand side of (8), which again by the first part of the proof is a polynomial in x of the right degree. Thus the right hand sides of both terms in (13) are polynomials. Since the left hand sides solve either Dirichlet or Poisson problems, we conclude that these admit polynomial solutions. From this to the general case corresponding to (5) it just suffices to invoque the linearity of the problems. □

3. Fisher's Lemma

We present a simple martingale proof of Lemma 1.1. With the notations introduced above, we have

Proof: Let $p(x)$ be such that $L(p) = \Delta(pb)(x) = 0$, we shall see that $p(x) = 0$. We apply Îto's formula as follows: For all $x \in \mathcal{E}$ the hypotheses imply that

$$E^x[p(B(T))b(B(T))] = p(x)b(x) + E^x\left[\int_0^T \frac{1}{2}\Delta(Pb)(B(t)\,dt\right] = p(x)b(x).$$

Thus, from $E^x[p(B(T))b(B(T))] = p(x)b(x)$, and the fact that $b(x)=0$ on $\partial\mathcal{E}$, it follows that $p(x)b(x) = 0$ inside $\mathcal{E}$, and since $b(x) > 0$ there, this implies that $p(x) = 0$ everywhere. Thus L is injective. That L is degree preserving is trivial to verify. □

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