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TRACE AND DETERMINANT IN JORDAN-BANACH ALGEBRAS

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Abstract

Using an appropriate definition of the multiplicity of a spectral value, we introduce a new definition of the trace and determinant of elements with finite spectrum in Jordan-Banach algebras. We first extend a result obtained by J. Zemánek in the associative case, on the connectedness of projections which are close to each other spectrally (Theorem 2.3). Secondly we show that the rank of the Riesz projection associated to a finite-rank element a and a finite subset of its spectrum is equal to the sum of the multiplicities of these spectral values (Theorem 2.6). Then we turn to the study of properties such as linearity and continuity of the trace and multiplicativity of the determinant.

1. Introduction

Determinants of infinite matrices were investigated for the first time by the astronomer G. W. Hill in his studies on lunar theory and his ideas were put into a rigorous form by Henri Poincaré in 1886. Ten years later H. von Koch refined and generalized Poincaré's results. In 1903, I. Fredholm developed a determinant theory for integral operators. Unlike von Koch, Fredholm studied eigenvalues and looked at the analyticity of $\mathrm{Det}(I + \lambda M)$. Fredholm's determinant theory is certainly one of the milestones in the history of functional analysis. In the early fifties A. F. Ruston, T. Lezański and A. Grothendieck almost simultaneously defined determinants for nuclear or integral operators on a Banach space. In the seventies, A. Pietsch developed an axiomatic approach to the determinant of elements of certain operator ideals. In 1978, J. Puhl [**17**] studied the trace on the socle and nuclear elements

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of a semisimple Banach algebra, basing his difficult arguments on the standard trace defined for finite-rank operators. In [**5**] it was shown that the trace and determinant on the socle of a Banach algebra can be developed in a purely spectral and analytic way, that is to say internally, without using operators on the algebra.

The essential ingredient in all the arguments is the fact that the spectrum is an analytic multifunction. So this point of view gives us the possibility to extend almost all the results of [**5**] to more general situations where the spectrum is also analytic, for instance the case of Jordan-Banach algebras. For a finite-dimensional Jordan algebra A the trace is defined on A using the trace of the left multiplication operator L_x defined on A by $L_x a = xa$. In this paper we shall define the trace and determinant for elements of the socle of an arbitrary Jordan-Banach algebra in a purely spectral and internal way. At this point it is worth mentioning that purely algebraic proofs of some of our results have been given by O. Loos in [**12**].

We recall that a complex Jordan algebra A is non-associative and the product satisfies the identities $ab = ba$ and $(ab)a^2 = a(ba^2)$ for all a, b in A (see [**21**] for the theory of Jordan algebras). A unital Jordan-Banach algebra is a Jordan algebra with a complete norm satisfying $\|xy\| \le \|x\|\|y\|$, for $x, y \in A$ and $\|1\| = 1$. Equivalently, we could define a unital Jordan-Banach algebra as a Banach space A over $\mathbb{C}$ with a continuous quadratic map U from A to the bounded linear operators on A, written as $x \to U_x$ and a distinguished element $1 \in A$ satisfying in particular the following identities:

$$
\begin{aligned}
U_1 &= \mathrm{Id}, \\
U(U_x y) &= U_x U_y U_x, \\
U_x V_{y,x} &= V_{x,y} U_x,
\end{aligned}
$$

where $V_{x,y} z = U_{x+z} y - U_x y - U_z y$ is trilinear in x, y, z and $U_{x,x} = 2U_x$. The U operators can be recovered by $U_x y = 2x(xy) - x^2 y$. The linear Jordan product xy is defined by $2xy = U_{x,y} 1$, and powers inductively by $x^0 = 1$, $x^{n+1} = xx^n$. An element $x \in A$ is said to be invertible if there exists $y \in A$ such that $xy = 1$ and $x^2 y = x$. This is equivalent to the operator U_x having inverse U_y. For an element x of a Jordan-Banach algebra A, the spectrum of x is by definition the non-empty set of complex numbers λ such that $\lambda - x$ is not invertible.

The main tool in all the arguments is the following theorem which is an immediate consequence of the fact that $\lambda \longrightarrow \mathrm{Sp}(f(\lambda))$ is an analytic multifunction, where f is an analytic function from a domain $D \subset \mathbb{C}$ into a Jordan-Banach algebra, and of Theorem 7.1.7 of [**1**]. In the following we use the symbol $\#S$ to denote the cardinality of the set S.

Theorem 1.1. *Let A be a Jordan-Banach algebra with nonzero socle. Suppose that f is an analytic function from a domain D of $\mathbb{C}$ into the socle of A. Then there exist an integer N and a closed discrete subset F of D such that*

$$\#(\mathrm{Sp}\, f(\lambda)\backslash\{0\}) = N \quad \textit{for} \quad \lambda \in D\backslash F$$

$$\#(\mathrm{Sp}\, f(\lambda)\backslash\{0\}) < N \quad \textit{for} \quad \lambda \in F.$$

2. The multiplicity of a spectral value

Throughout this paper we shall assume that the Jordan-Banach algebra A is semisimple with an identity. Here *semisimple* means that the Jacobson radical vanishes. We recall that an idempotent p is *minimal* if $U_pA = \mathbb{C}p$, in which case U_pA is a minimal inner ideal. By definition the *socle* of A, denoted by $\mathrm{Soc}\, A$, is the sum of all minimal inner ideals of A. By Osborn-Racine theorem [**16**] the socle of A is an ideal and it is the sum of simple ideals generated by minimal projections. By a theorem of Aupetit-Baribeau [**4**] the socle of a *scattered Jordan-Banach algebra* is not reduced to zero. It is easy to see that the elements of the socle are algebraic, so they have finite spectrum. Consequently by Newburgh's theorem the spectrum function is continuous on the socle (see [**1**, Corollary 3.4.5], and [**15**]).

For a semisimple Jordan-Banach algebra, it is known that the socle coincides with the largest von Neumann regular ideal, as shown by A. Fernández López and A. Rodríguez Palacios in [**8**]. Given a semisimple Jordan-Banach algebra, A. Fernández López proved that its socle is an algebraic ideal and conversely if I is any algebraic ideal then every element of I can be written as the sum of an element of the socle and a nilpotent element. This result was improved by M. Benslimane, O. Jaa and A. Kaïdi [**6**] who proved that every element of a spectrally finite ideal can be written as the sum of an element of the socle and an element whose square is zero. Using recent results of O. Loos [**11**] and A. Rodríguez Palacios [**18**], it can be proved that a spectrum-finite ideal of A is included in the socle.

The *rank* of $a \in A$ is defined in [**2**] as the maximum of the number of points of $\operatorname{Sp} U_x a \backslash \{0\}$, when $x \in A$, if it exists, otherwise the rank is infinite. It is also proved that the set of finite-rank elements coincides with the socle. This rank is equal to the algebraic rank defined by O. Loos [**10**], as this was shown in [**9**]. We recall that $a \in \operatorname{Soc} A$ is said to be a *maximal-finite-rank* element if $\#(\operatorname{Sp} a \backslash \{0\}) = \operatorname{rank}(a)$. Let $a \in A$ be a finite-rank element. By the definition of the rank, the set $E(a) = \{x : \#(\operatorname{Sp} U_x a \backslash \{0\}) = \operatorname{rank}(a)\}$ is non-empty. If $x \in A$ we have $\#(\operatorname{Sp} U_x a \backslash \{0\}) \le \operatorname{rank}(U_x a) \le \operatorname{rank}(a)$ ([**2**, Theorem 3.5]), consequently $U_x a$ is a maximal-finite-rank element when x belongs to $E(a)$. In the associative case, maximal finite-rank elements have a simple structure, they are linear combinations of orthogonal projections ([**5**, Theorem 2.8]). What can be said in the Jordan-Banach case?

Theorem 2.1 (Density of maximal finite-rank elements)**.** *Let a be a finite-rank element of A. Then $E(a)$ is a dense open subset of A. Consequently the set of maximal finite-rank elements is dense in the socle.*

Proof: The proof is almost identical to the proof of Theorem 2.2 in [**5**], replacing everywhere xa by $U_x a$. □

Theorem 2.2. *Let a be a finite-rank element of A. Let Γ be an oriented regular contour, not intersecting* $\operatorname{Sp} a$*, and denote by $\triangle_0$, $\triangle_1$ respectively its interior and its exterior. By upper semicontinuity of the spectrum there exists an open ball B in A, centred at the identity, such that* $\operatorname{Sp} U_x a \cap \Gamma = \emptyset$ *for $x \in B$. Then for $x, y \in B \cap E(a)$ we have*

$$\#(\operatorname{Sp} U_x a \cap \triangle_i) = \#(\operatorname{Sp} U_y a \cap \triangle_i)$$

for $i = 0, 1$.

Proof: By Theorem 2.1, $B \cap E(a)$ is non-empty. So let $x, y \in B \cap E(a)$ and let D be the convex domain of $\mathbb{C}$ containing 0 and 1 such that $\lambda \in D$ is equivalent to $\lambda x + (1-\lambda) y \in B$. Taking $f(\lambda) = U_{\lambda x + (1-\lambda) y}(a)$ which has finite-rank, and applying the Localisation Principle ([**1**, Theorem 7.1.5]) to the analytic multifunctions $\lambda \longrightarrow \operatorname{Sp} f(\lambda) \cap \triangle_i$, by the Scarcity Theorem we conclude that there exist two integers N_0, N_1 and two closed discrete subsets F_0, F_1 of D such that

$$\begin{cases} \#(\operatorname{Sp} f(\lambda) \cap \triangle_i) = N_i & \text{for } \lambda \in D \backslash F_i \\ \#(\operatorname{Sp} f(\lambda) \cap \triangle_i) < N_i & \text{for } \lambda \in F_i. \end{cases} \tag{1}$$

To prove the theorem we have to show that $0, 1 \notin F_0 \cup F_1$. Suppose for instance that $0 \in F_0 \cup F_1$. For $\lambda \notin F_0 \cup F_1$ we have

$$\# \operatorname{Sp} U_y a = \#(\operatorname{Sp} U_y a \cap \triangle_0) + \#(\operatorname{Sp} U_y a \cap \triangle_1) < N_0 + N_1 = \# \operatorname{Sp} f(\lambda).$$

If $0 \in \operatorname{Sp} U_y a$ the relation (1) implies $m + 1 = \# \operatorname{Sp} U_y a < N_0 + N_1 = \# \operatorname{Sp} f(\lambda)$, from the definition of the rank, where $m = \operatorname{rank}(a)$, so this is absurd. If $0 \notin \operatorname{Sp} U_y a$ by upper semicontinuity of the spectrum we can choose $\lambda \notin F_0 \cup F_1$ such that $0 \notin \operatorname{Sp} f(\lambda)$. Relation (1) implies

$$m = \# \operatorname{Sp} U_y a < N_0 + N_1 = \# \operatorname{Sp} f(\lambda) = m$$

and this is also absurd. By a similar argument the case $1 \in F_0 \cup F_1$ also gives a contradiction, so the theorem is proved. □

The number $\#(\operatorname{Sp}(U_x a) \cap \triangle_0)$ which is independent of x by Theorem 2.2, is denoted by $m(\Gamma, a)$ and is called the *multiplicity* of a *associated to* Γ. It is independent of Γ if the corresponding spectral points of a in $\triangle_0$, $\triangle_1$ do not change. For α in $\operatorname{Sp} a$ we define $m(\alpha, a)$, the multiplicity of a at α, as $m(\Gamma, a)$ where Γ is a small circle centred at α and isolating α from the rest of the spectrum. We have $m(\alpha, a) \geq 1$. It is not difficult to see that

$$m(\Gamma, a) = \sum_{\alpha \in \operatorname{Sp} a \cap \triangle_0} m(\alpha, a).$$

If a is a maximal finite-rank element then necessarily we must have $m(\alpha, a) = 1$ for every $\alpha \in \operatorname{Sp} a$, because $1 \in B \cap E(a)$.

In particular if we take for Γ a contour surrounding all the spectrum of a we obtain

$$\sum_{\alpha \in \operatorname{Sp} a} m(\alpha, a) = \begin{cases} 1 + \operatorname{rank}(a) & \text{if } 0 \in \operatorname{Sp} a, \\ \operatorname{rank}(a) & \text{if } 0 \notin \operatorname{Sp} a. \end{cases} \tag{2}$$

In fact we shall see that the multiplicity of a at $\alpha \neq 0$ is equal to the rank of the Riesz projection associated to a and α.

Let $\alpha \in \mathbb{C}$ and Γ be a small curve isolating α from the rest of the spectrum of a. By definition the Riesz projection is

$$p(\alpha, a) = \frac{1}{2\pi i} \int_\Gamma (\lambda - a)^{-1} \, d\lambda. \tag{3}$$

The Holomorphic Functional Calculus implies that the $p(\alpha, a)$ are orthogonal projections for different α and their sum is one.

For $\alpha \neq 0$, the identity $(\lambda - a)^{-1} = \frac{1}{\lambda} + \frac{1}{\lambda} a (\lambda - a)^{-1}$ implies that

$$p(\alpha, a) = \frac{a}{2\pi i} \int_\Gamma \frac{1}{\lambda} (\lambda - a)^{-1} \, d\lambda, \tag{4}$$

where Γ is a small circle isolating α from the rest of $\operatorname{Sp} a \cup \{0\}$.

In the case of Banach algebras, J. Zemánek proved in [**20**] that two projections p, q such that $||p-q|| < 1$ are analytically connected. In particular they have the same rank by Lemma 2.5 of [**5**]. We now extend this result to Jordan-Banach algebras.

Theorem 2.3. *Let p, q be two projections of A such that $\rho(p-q) < 1$, where ρ denotes the spectral radius. Then there exists $s \in A$ such that $s^2 = 1$, $q = U_s p$ and $p = U_s q$. In particular* $\operatorname{rank}(p) = \operatorname{rank}(q)$.

Proof: a) Let $\mathcal{L}$ be the non-closed subalgebra generated by $1, p, q$. By the Shirshov-Cohn Theorem [**21**] this algebra has a faithful representation as a subalgebra of a special Jordan algebra B^+, where B is an associative algebra with identity. We denote by $x \bullet y$ the associative product. Then taking $a = p+q-1$ we have $p \bullet a = p \bullet q = a \bullet q$. Moreover $a^2 = 1-(p-q)^2$ and $(p-q)^2 \bullet p = (p+q-p\bullet q - q \bullet p) \bullet p = p - p \bullet q \bullet p = p \bullet (p-q)^2$ and $(p-q)^2 \bullet q = q - q \bullet p \bullet q = q \bullet (p-q)^2$. Consequently we have $p \bullet y = y \bullet p$ and $q \bullet y = y \bullet q$ for all y in the Jordan algebra $\mathbb{C}((p-q)^2)$ generated by 1 and $(p-q)^2$. Let $t = U_y a = y \bullet a \bullet y = y^2 \bullet a$ for $y \in \mathbb{C}((p-q)^2)$. Then we have $t^2 = y^2 \bullet a^2 \bullet y^2 = U_{y^2}(1-(p-q)^2)$. Now we have $p \bullet t = p \bullet y \bullet a \bullet y = y \bullet p \bullet a \bullet y = y \bullet p \bullet y \bullet a = y^2 \bullet p \bullet a = y^2 \bullet a \bullet q = y \bullet a \bullet y \bullet q = t \bullet q$, similarly $t \bullet p = q \bullet t$. Finally we have $U_t p = t \bullet p \bullet t = \frac{1}{2}(t^2 \bullet q + q \bullet t^2)$. All of this proves that the following Jordan identities are true in A:

$$\begin{cases} t^2 = U_{y^2}(1-(p-q)^2) \\ U_t p = q t^2 \end{cases} \tag{5}$$

for $t = U_y a$, $y \in \mathbb{C}((p-q)^2)$. By continuity these identities certainly extend for y in the closure of $\mathbb{C}((p-q)^2)$.

b) Now suppose that $\rho(p-q) < 1$. By the Holomorphic Functional Calculus we can define $(1-(p-q)^2)^{-\frac{1}{4}}$ by a convergent series containing only terms with even powers of $p-q$. Consequently this element is in the closure of $\mathbb{C}((p-q)^2)$. Taking $y = (1-(p-q)^2)^{-\frac{1}{4}}$ and $s = U_y a$ we conclude from (5) that $s^2 = 1$, $U_s p = q$ and $p = U_{s^{-1}} q = U_s q$. Consequently p and q have same rank by [**2**, Theorem 3.5]. $\square$

Corollary 2.4. *Let p, q be two projections of A. If p, q are in the same connected component of the set of projections there exists a sequence $s_1, \ldots, s_n$ of elements of A such that $s_i^2 = 1$ $(i = 1, \ldots, n)$, $p = U_{s_1} \ldots U_{s_n} q$ and $q = U_{s_n} \ldots U_{s_1} p$. In particular p and q have the same rank.*

Proof: Since p and q lie in the same connected component, we know by a standard topological result (see for example [**7**, Proposition 19.2]) that there exist a chain of points $r_1, \dots, r_{n+1}$ such that $p = r_1$, $q = r_{n+1}$ and $\|r_i - r_{i+1}\| < 1$ $(i = 1, \dots, n)$. By n successive applications of Theorem 2.3, we obtain the result. $\square$

Modifying slightly the argument of the proof of Corollary 3.12 of [**2**] we obtain the following

Lemma 2.5. *Let a be a finite-rank element. Then*

$$\operatorname{rank}(a^2) = \max_{x \in A} \#(\operatorname{Sp} U_a x \backslash \{0\}) \leq \operatorname{rank}(a).$$

Consequently the quadratic ideal $U_a A$ contains at most $\operatorname{rank}(a)$ *orthogonal non-zero projections. If p is a projection then*

$$\operatorname{rank}(p) = \max_{x \in A} \#(\operatorname{Sp} U_p x \backslash \{0\}).$$

Proof: By the Shifting Principle [**21**, p. 315] we have

$$\operatorname{Sp} U_a x^2 \backslash \{0\} = \operatorname{Sp} U_x a^2 \backslash \{0\}, \text{ for } a, x \in A.$$

Consequently, it follows from the definition of the rank that,

$$\operatorname{rank}(a^2) = \max_{x \in A} \#(\operatorname{Sp} U_a x^2 \backslash \{0\}).$$

Let $x \in A$ be fixed and $|\lambda| > \rho(x)$ (the spectral radius of x) then $x - \lambda = y^2$, by the Holomorphic Functional Calculus. Consequently it follows that $\#(\operatorname{Sp} U_a(x - \lambda) \backslash \{0\}) \leq \operatorname{rank}(a^2)$, for $|\lambda| > \rho(x)$. So by the Scarcity Theorem this is true for every λ, in particular for $\lambda = 0$. All of this implies that $\operatorname{rank}(a^2) = \max_{x \in A} \#(\operatorname{Sp} U_a x \backslash \{0\})$. The second inequality is a consequence of Corollary 3.6 of [**2**]. Suppose now that $U_a A$ contains $r + 1$ orthogonal projections $p_1, \dots, p_{r+1}$ (where $r = \operatorname{rank}(a)$). Then $b = p_1 + 2p_2 + \cdots + (r + 1)p_{r+1} \in U_a A$ and $\{1, 2, \dots, r + 1\} \subset \operatorname{Sp} b$ but this contradicts the second inequality. The last part is obvious because $p = p^2$. $\square$

We now prove the fundamental result of this paper.

Theorem 2.6. *Let a be a finite-rank element and $\alpha_1, \dots, \alpha_n$ some non-zero elements of its spectrum. If p denotes the Riesz projection associated to a and $\{\alpha_1, \dots, \alpha_n\}$ (which is the sum of the Riesz projections associated to a and the different α_i) then we have*

$$\operatorname{rank}(p) = m(\alpha_1, a) + \cdots + m(\alpha_n, a).$$

Proof: a) From Theorem 2.2 and the definition of multiplicity there exists $\epsilon_0 > 0$ such that

$$m(\alpha_1, a) + \cdots + m(\alpha_n, a) = \max_{||x-1||<\epsilon} \#(\operatorname{Sp} U_x a \cap \triangle_0)$$

for $0 < \epsilon < \epsilon_0$, where $\triangle_0$ is a domain limited by a regular contour Γ separating $\alpha_1, \ldots, \alpha_n$ from the rest of the spectrum of a and 0. Consequently by the Holomorphic Functional Calculus we have

$$m(\alpha_1, a) + \cdots + m(\alpha_n, a) = \max_{||x-1||<\epsilon} \#(\operatorname{Sp} qU_x a \backslash \{0\})$$

where q denotes the Riesz projection associated to $U_x a$ and Γ, that is the sum of the Riesz projections associated to $U_x a$ and the different $\alpha_1, \ldots, \alpha_n$. The proof of this is almost identical to the proof of Theorem 3.3.4 in [**1**]. Consequently

$$m(\alpha_1, a) + \cdots + m(\alpha_n, a) \leq \max_{||x-1||<\epsilon} (\operatorname{rank} qU_x a) \leq \operatorname{rank}(q),$$

by [**2**, Corollary 3.6]. If we take $\epsilon > 0$ small enough, then we can suppose that $\|p - q\| < 1$, so by Theorem 2.3, we have

$$m(\alpha_1, a) + \cdots + m(\alpha_n, a) \leq \operatorname{rank}(p). \tag{6}$$

b) We now prove a small preliminary result. If b has finite-rank, $\alpha \neq 0$ is in the spectrum of b and if q is the Riesz projection associated to b and α, then $U_q A \subset U_b A$. This comes immediately from the fact that we have $q = \frac{b}{2\pi i} \int_\Gamma \frac{1}{\lambda}(\lambda - b)^{-1}\, d\lambda$, that is $q = bu$ where u is in a maximal associative Jordan algebra containing 1 and b, so $q = q^2 = (bu)(bu) = b^2 u^2 = U_b u^2 \in U_b A$ consequently, because $U_b A$ is a quadratic ideal, we have $U_q A \subset U_b A$.

c) We now prove the converse inequality of (6). By Theorem 2.2 we may choose x arbitrarily near to 1 such that

$$\begin{cases} m(\alpha_1, a) + \cdots + m(\alpha_n, a) = \#(\operatorname{Sp} U_x a \cap \triangle_0) \\ \#(\operatorname{Sp} U_x a \backslash \{0\}) = \operatorname{rank}(U_x a) = \operatorname{rank}(a). \end{cases}$$

We set $r = \operatorname{rank}(a)$, $m = m(\alpha_1, a) + \cdots + m(\alpha_n, a)$, $b = U_x a$. We know that $\operatorname{Sp} b$ contains m points in $\triangle_0$ and $r - m$ nonzero points outside. Denote by $q_1, \ldots, q_{r-m}$ the $r-m$ Riesz projections associated to b and to the $r - m$ non-zero points outside of $\triangle_0$. These projections are orthogonal and we have $U_{q_i} A \subset U_b A$ by part b). Moreover these q_i are orthogonal to q, the Riesz projection associated to b and Γ. Suppose $\operatorname{rank}(q) > m$, then by Lemma 2.5 there exists $y \in A$ such that $\#(\operatorname{Sp} U_q y \backslash \{0\}) > m$.

By the Holomorphic Functional Calculus applied to $U_q y$ we can construct $m+1$ orthogonal non-zero projections $p_1, \ldots, p_{m+1}$ which are in $U_q A$, consequently in $U_b A$, by part b). Every p_i is orthogonal to every q_j because $p_i \in U_q A$, $q_j \in U_{q_j} A$, q is orthogonal to q_j, so we apply Proposition 6 of [**4**]. Hence $U_b A$ contains $r+1$ nonzero orthogonal projections $q_1, \ldots, q_{r-m}, p_1, \ldots, p_{m+1}$, but this violates Lemma 2.5 applied to b. So $\operatorname{rank}(q) \leq m$. But for x near to 1, p and q satisfy the conditions of Theorem 2.3, that is $\operatorname{rank}(p) = \operatorname{rank}(q)$ □

Corollary 2.7. *If p is a non-zero projection of the socle and $p \neq 1$ then $m(0,p) = 1$ and $m(1,p) = \operatorname{rank}(p)$.*

Proof: We have $1+\operatorname{rank}(p) = m(0,p)+m(1,p)$ and we apply Theorem 2.6 to $a = p$, $\alpha_1 = 1$, noticing that the Riesz projection associated to p and 1 is p itself because

$$\begin{aligned} \frac{p}{2\pi i}\int_\Gamma (\lambda - p)^{-1}\frac{d\lambda}{\lambda} &= \frac{p}{2\pi i}\int_\Gamma \left(\frac{p}{\lambda - 1} + \frac{1-p}{\lambda}\right)\frac{d\lambda}{\lambda} \\ &= \frac{p}{2\pi i}\int_\Gamma \frac{d\lambda}{\lambda(\lambda-1)} \\ &= p. \end{aligned}$$ □

3. The trace and the determinant

If $a \in \operatorname{Soc} A$ we define the *trace* of a by

$$\operatorname{Tr}(a) = \sum_{\lambda \in \operatorname{Sp} a} \lambda m(\lambda, a), \tag{7}$$

and the *determinant* of $1+a$ by

$$\operatorname{Det}(1+a) = \prod_{\lambda \in \operatorname{Sp} a} (1+\lambda)^{m(\lambda,a)}. \tag{8}$$

It is obvious that $\operatorname{Det}(1+a) \neq 0$ is equivalent to $1+a$ invertible. From (7) and (8) it is clear that we have

$$|\operatorname{Tr}(a)| \leq \rho(a)\operatorname{rank}(a), \tag{9}$$

and for $\rho(a) < 1$ we have

$$(1-\rho(a))^{\operatorname{rank}(a)} \leq |\operatorname{Det}(1+a)| \leq (1+\rho(a))^{\operatorname{rank}(a)}, \tag{10}$$

where ρ denotes the spectral radius.

Theorem 3.1. *Let f be an analytic function from a domain D of $\mathbb{C}$ into the socle of A. Then $\operatorname{Tr}(f(\lambda))$ and $\operatorname{Det}(1+f(\lambda))$ are holomorphic on D.*

Proof: The proof is identical to the proof of Theorem 3.1 of [**5**] except that Lemma 2.5 and Theorem 2.6 of [**5**] must be replaced by Theorem 2.3 and Theorem 2.6. □

Using the previous arguments with some results from [**5**] we obtain:

Theorem 3.2. *We have the following properties of the trace:*

1) *If $a, b \in \operatorname{Soc} A$, then $\operatorname{Tr}(a+b) = \operatorname{Tr}(a) + \operatorname{Tr}(b)$.*
2) *The trace is continuous on $\mathcal{F}_n$, the set of elements with rank less or equal to n.*
3) *If $a \in \operatorname{Soc} A$, then $\rho(a) = \limsup_{k\to\infty} |\operatorname{Tr}(a^k)|^{\frac{1}{k}}$.*
4) *If $a \in \operatorname{Soc} A$, $\phi(x) = \operatorname{Tr}(ax)$ is a bounded linear functional on A.*

Proof: 1) By Theorem 3.1 we know that $h(\lambda) = \operatorname{Tr}(a+\lambda b)$ is entire. Consequently,

$$\lim_{\lambda\to\infty} \frac{h(\lambda)}{\lambda} = \lim_{\lambda\to\infty} \operatorname{Tr}\left(\frac{a}{\lambda} + b\right) = \lim_{\mu\to 0} \operatorname{Tr}(\mu a + b) = \operatorname{Tr} b.$$

By Liouville's theorem, $h(\lambda) = \lambda \operatorname{Tr}(b) + c$, with $c = h(0) = \operatorname{Tr} a$.

2) If $a, b \in \mathcal{F}_n$ then

$$|\operatorname{Tr} b - \operatorname{Tr} a| \le \operatorname{rank}(b-a)\rho(b-a) \le 2n\|b-a\|.$$

3) Exactly the same as in [**5**, Theorem 3.5].

4) Follows from 1) and 2) because $ax \in \mathcal{F}_{2n}$ if $n = \operatorname{rank}(a)$. □

Corollary 3.3. *Let $p_1, \dots, p_n$ be orthogonal finite-rank projections and let $\alpha_1, \dots, \alpha_n$ be non-zero complex numbers. Then*

$$\operatorname{rank}(\alpha_1 p_1 + \cdots + \alpha_n p_n) = \operatorname{rank}(p_1) + \cdots + \operatorname{rank}(p_n).$$

Proof: We may suppose that all the projections are non-zero. By [**2**, Theorem 3.9], we have

$$\operatorname{rank}(\alpha_1 p_1 + \cdots + \alpha_n p_n) \le \operatorname{rank}(p_1) + \cdots + \operatorname{rank}(p_n).$$

Moreover

$$\begin{aligned} \operatorname{rank}(p_1 + \cdots + p_n) &= \operatorname{rank}\left((\alpha_1 p_1 + \cdots + \alpha_n p_n)\left(\frac{p_1}{\alpha_1} + \cdots + \frac{p_n}{\alpha_n}\right)\right) \\ &\le \operatorname{rank}(\alpha_1 p_1 + \cdots + \alpha_n p_n), \end{aligned}$$

by [**2**, Corollary 3.6] and the fact that the algebra generated by the p_i is associative. So without loss of generality we may suppose that $\alpha_1 = \cdots = \alpha_n = 1$. In this case we set $p = p_1 + \cdots + p_n$ which is a non-zero projection. By Corollary 2.7 and Theorem 3.2 1) we have

$$\begin{aligned}\operatorname{rank}(p) = m(1,p) = \operatorname{Tr}(p) &= \operatorname{Tr}(p_1) + \cdots + \operatorname{Tr}(p_n)\\ &= \operatorname{rank}(p_1) + \cdots + \operatorname{rank}(p_n). \qquad \square\end{aligned}$$

The following lemma is essential in the proof of the next theorem. We include a detailed proof here for the sake of completeness.

Lemma 3.4. *Let $f(\lambda,\mu)$ be a complex-valued function of two complex variables which is separately entire in λ, μ and such that $f(\lambda,\mu) \neq 0$ for all $\lambda, \mu \in \mathbb{C}$. Suppose moreover that there exists two positive constants A, B such that*

$$|f(\lambda,\mu)| \leq e^{A|\lambda|+B|\mu|}.$$

Then there exist two complex constants α, β such that

$$f(\lambda,\mu) = f(0,0)e^{\alpha\lambda+\beta\mu}.$$

Proof: Since the complex plane is simply connected, by [**19**, Theorem 13.11], there exists a function $\Phi(\lambda,\mu)$ separately entire in λ, μ such that $\exp(\Phi(\lambda,\mu)) = f(\lambda,\mu)$. Then we have $\Re\Phi(\lambda,\mu) \leq A|\lambda| + \beta|\mu|$. So if we fix μ and apply Liouville's theorem for the real part we conclude that $\Phi(\lambda,\mu) = \lambda f_1(\mu) + f_2(\mu)$ for every λ. Taking two different values of λ and solving the system of two equations in f_1, f_2 we conclude that f_1, f_2 are entire in μ. Fixing μ again and taking λ real and going to $+\infty$ we conclude that $\Re f_1(\mu) \leq A$ for arbitrary μ, so again we conclude that $f_1(\mu)$ is a constant α. A similar argument with λ fixed proves that $\Phi(\lambda,\mu) = \mu g_1(\lambda) + g_2(\lambda)$, where $g_1(\lambda)$ is a constant β. Finally, we have $-\lambda\alpha + g_2(\lambda) = -\beta\mu + f_2(\mu)$ for every λ, μ, hence this quantity must be a constant and we get the result. $\square$

Theorem 3.5. *Let $a, b \in \operatorname{Soc} A$, then $e^a - 1$, $e^b - 1$, $e^{a+b} - 1 \in \operatorname{Soc} A$ and we have the following properties:*

1) $\operatorname{Det} e^{a+b} = \operatorname{Det}(e^a)\operatorname{Det}(e^b)$.
2) $\operatorname{Det}(e^a) = e^{\operatorname{Tr}(a)}$.
3) $\operatorname{Det} U_{1+a}(1+b) = (\operatorname{Det}(1+a))^2 \operatorname{Det}(1+b)$.
4) $\operatorname{Det}((1+a)(1+b)) = \operatorname{Det}(1+a)\operatorname{Det}(1+b)$ *if 1, a, b generate an associative subalgebra.*

Proof: The first relations follow from the expansion of the exponential function in power series and the fact that $\operatorname{Soc} A$ is an ideal. Precisely, we have $e^a - 1 = a(1 + \frac{a}{2!} + \frac{a^2}{3!} + \dots) \in \operatorname{Soc} A$ when $a \in \operatorname{Soc} A$.

1) Let $f(\lambda, \mu) = \operatorname{Det}(e^{\lambda a + \mu b})$ then

$$|f(\lambda, \mu)| \leq \|e^{\lambda a + \mu b}\|^N$$

where N is a constant bounding the rank of $e^{\lambda a + \mu b} - 1$ by [**2**, Theorem 3.4]. Then

$$|f(\lambda, \mu)| \leq e^{N\|a\||\lambda| + N\|b\||\mu|}.$$

Moreover $f(\lambda, \mu) \neq 0$ since $e^{\lambda a + \mu b}$ is invertible. From Lemma 3.4 we know that $f(\lambda, \mu) = e^{c_1 \lambda + c_2 \mu}$ so $e^{c_1} = \operatorname{Det} e^a$ and $e^{c_2} = \operatorname{Det} e^b$.

2) We have

$$\operatorname{Det}(e^{\lambda a}) = \prod_{\alpha \in \operatorname{Sp} a} e^{\lambda \alpha m(e^{\lambda \alpha} - 1, e^{\lambda a} - 1)} = \prod_{\alpha \in \operatorname{Sp} a} e^{\lambda \alpha m(\frac{e^{\lambda \alpha} - 1}{\lambda}, \frac{e^{\lambda a} - 1}{\lambda})}$$

for $\lambda \neq 0$. The argument of 1) shows that $\phi(\lambda) = \operatorname{Det}(e^{\lambda a}) = e^{\lambda c}$, so

$$c = \sum_{\alpha \in \operatorname{Sp} a} \alpha m \left(\frac{e^{\lambda \alpha} - 1}{\lambda}, \frac{e^{\lambda a} - 1}{\lambda} \right).$$

Then $\operatorname{Tr}(\frac{e^{\lambda a} - 1}{\lambda}) \to c$ when $\lambda \to 0$. But by Theorem 3.2, $\operatorname{Tr}(\frac{e^{\lambda a} - 1}{\lambda})$ has an analytic continuation at 0 as $\operatorname{Tr} a$. Thus $c = \operatorname{Tr} a$.

3) Put $f(\lambda, \mu) = \operatorname{Det} U_{e^{\lambda a}} e^{\mu b}$ with $\lambda, \mu \in \mathbb{C}$. This function is entire and never vanishes, so $f(\lambda, \mu) = e^{\alpha \lambda + \beta \mu}$. Simple calculations show that $f(\lambda, 0) = e^{\alpha \lambda} = (\operatorname{Det} e^{\lambda a})^2$, and $f(0, \mu) = e^{\beta \mu} = \operatorname{Det} e^{\mu b}$, so we have $f(\lambda, \mu) = (\operatorname{Det} e^{\lambda a})^2 \operatorname{Det} e^{\mu b}$. We can suppose $1 + a$ and $1 + b$ invertible because otherwise the formula is trivially true. Since $1 + a$ and $1 + b$ are invertible and of finite spectrum, by the Holomorphic Functional Calculus there exist $x, y \in A$ such that $1 + a = e^x$, $1 + b = e^y$, and the result follows by taking $\lambda = \mu = 1$.

4) Follows from the associative case [**5**]. $\square$

Remark 3.6. Property 4) is not true in general, even for a finite-dimensional Jordan-Banach algebra. To see this, one has just to consider the Jordan algebra of matrices $M_2(\mathbb{C})$ with the usual Jordan product $x \circ y = \frac{1}{2}(xy + yx)$. Now take the following two matrices

$$a = \begin{pmatrix} -1, & 1 \\ 1, & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 0, & 0 \\ 0, & -2 \end{pmatrix},$$

for which we have

$$(1+a) \circ (1+b) = 0$$

and

$$\mathrm{Det}(1+a) = \mathrm{Det}(1+b) = -1.$$

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