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SMASH (CO)PRODUCTS AND SKEW PAIRINGS

J. N. Alonso Alvarez, J. M. Fernández Vilaboa and
R. González Rodríguez

Abstract

Let τ be an invertible skew pairing on (B, H), where B and H are Hopf algebras in a symmetric monoidal category $\mathcal{C}$ with (co)equalizers. Assume that H is quasitriangular. Then we obtain a new algebra structure such that B is a Hopf algebra in the braided category ${}^H_H\mathcal{YD}$ and there exists a Hopf algebra isomorphism $w\colon B \infty H \to B \bowtie_\tau H$ in $\mathcal{C}$, where $B \infty H$ is a Hopf algebra with (co)algebra structure the smash (co)product and $B \bowtie_\tau H$ is the Hopf algebra defined by Doi and Takeuchi.

1. Introduction

The smash product algebra and the smash coproduct coalgebra are well know in the theory of Hopf algebras. If B and H are Hopf algebras Radford [**10**] found necessary and sufficient conditions for the smash product algebra structure and the smash coproduct coalgebra structure on $B \otimes H$ to afford $B \otimes H$ a Hopf algebra structure. Also, in [**10**], Radford describes completely the structure of Hopf algebras with a projection. Assume B and H Hopf algebras and $j\colon B \to H$ anf $f\colon H \to B$ Hopf algebra morphisms such that $j \circ f = \mathrm{id}_H$. Then, B decomposes as a tensor product $D \infty H$ where D is a Hopf algebra in the braided monoidal category of Yetter-Drinfeld modules and ∞ denotes the smash (co)product. Afterwards, Majid in [**8**] and Bespalov in [**2**] obtain a braided interpretation of Radford's theorem. In [**1**] this result is proved in the context of braided categories, using the notions of H-Cleft comodule (module) algebras (coalgebras). On the other hand, Doi and Takeuchi in [**6**] studied the double crossproducts $B \bowtie_\tau H$. These are determined by an skew pairing $\tau\colon B \otimes H \to K$ where B and H bialgebras. When B and H are Hopf algebras and τ is a convolution invertible skew pairing, $(B, \varphi_B =$

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$(\tau\otimes B\otimes\tau^{-1})\circ(B\otimes H\otimes\delta_B\otimes H)\circ\delta_{B\otimes H}\circ c_{H,B})$ is a left H-module coalgebra and $(H,\psi_H=((\tau\circ c_{H,B})\otimes H\otimes(\tau^{-1}\circ c_{H,B}))\circ(H\otimes B\otimes\delta_H\otimes B)\circ\delta_{H\otimes B})$ is a right B-module coalgebra. The construction of $B\bowtie_\tau H$ is an example of Majid's double crossproduct $B\bowtie H$ [**9**] because the Majid double crossproduct is defined as coalgebra $B\bowtie H=B\otimes H$ and the product is equal to the one defined in [**6**] for $B\bowtie_\tau H$. In [**6**], it is shown that, if H is quasitriangular, there exists a Hopf algebra projection $g\colon B\bowtie_\tau H\to H$ and therefore, using the Radford's theorem, it is possible to find $D\in{}^H_H\mathcal{YD}$ such that $D\propto H$ is isomorphic with $B\bowtie_\tau H$. In this paper we prove that the Hopf algebra $B\bowtie_\tau H$, defined by Doi and Takeuchi, is an H-Cleft comodule algebra and an H-Cleft module coalgebra and then using Theorem 3.2 of [**1**] we show that the object D such that $D\propto H\approx B\bowtie_\tau H$ is B with a modified Hopf algebra structure. As a consequence, we obtain that (B,φ_B) is an H-module algebra too. Analogously, we have a similar result for the dual case studied by Caenepeel, Dăscălescu, Militaru and Panaite in [**3**]. The proof can be derived easily from the one developed in Theorem 4.1.

2. Preliminaries

We assume the reader is familiar with the machinery of monoidal categories. Details may be found in [**7**]. By $(\mathcal{C},\otimes,c,K)$ we denote a strict symmetric monoidal category with (co)equalizers where c is the natural isomorphism of symmetry and K is the base object. For every object A in $\mathcal{C}$, id_A denotes the identity morphism.

An algebra in $\mathcal{C}$ is a triple $A=(A,\eta_A,\mu_A)$ where A is an object in $\mathcal{C}$ and $\eta_A\colon K\to A$, $\mu_A\colon A\otimes A\to A$ are morphisms in $\mathcal{C}$ such that $\mu_A\circ(A\otimes\eta_A)=\mathrm{id}_A=\mu_A\circ(\eta_A\otimes A)$, $\mu_A\circ(A\otimes\mu_A)=\mu_A\circ(\mu_A\otimes A)$. Given two algebras $A=(A,\eta_A,\mu_A)$ and $B=(B,\eta_B,\mu_B)$, $f\colon A\to B$ is an algebra morphism if $\mu_B\circ(f\otimes f)=f\circ\mu_A$, $f\circ\eta_A=\eta_B$. Also, if A, B are algebras in $\mathcal{C}$, the product algebra is $AB=(A\otimes B,\eta_{A\otimes B}=\eta_A\otimes\eta_B,\mu_{A\otimes B}=(\mu_A\otimes\mu_B)\circ(A\otimes c_{B,A}\otimes B))$.

A coalgebra in $\mathcal{C}$ is a triple $D=(D,\varepsilon_D,\delta_D)$ where D is an object in $\mathcal{C}$ and $\varepsilon_D\colon D\to K$, $\delta_D\colon D\to D\otimes D$ are morphisms in $\mathcal{C}$ such that $(\varepsilon_D\otimes D)\circ\delta_D=\mathrm{id}_D=(D\otimes\varepsilon_D)\circ\delta_D$, $(\delta_D\otimes D)\circ\delta_D=(D\otimes\delta_D)\circ\delta_D$. If $D=(D,\varepsilon_D,\delta_D)$ and $E=(E,\varepsilon_E,\delta_E)$ are coalgebras, $f\colon D\to E$ is a coalgebra morphism if $(f\otimes f)\circ\delta_D=\delta_E\circ f$, $\varepsilon_E\circ f=\varepsilon_D$. When D, E are coalgebras in $\mathcal{C}$, the product coalgebra is $DE=(D\otimes E,\varepsilon_{D\otimes E}=\varepsilon_D\otimes\varepsilon_E,\delta_{D\otimes E}=(D\otimes c_{D,E}\otimes E)\circ(\delta_D\otimes\delta_E))$.

Let D be a coalgebra and let A be an algebra. By $\mathrm{Reg}(D,A)$ we denote the set of invertible morphisms $f\colon D\to A$ in $\mathcal{C}$ respect to the convolution operation $f * g = \mu_A \circ (f \otimes g) \circ \delta_D$. $\mathrm{Reg}(D, A)$ is a monoid where the unit element is $\varepsilon_D \otimes \eta_A$.

Let A be an algebra. (M, ψ_M) is a right A-module if M is an object in $(\mathcal{C}, c)$ and $\psi_M : M \otimes A \to M$ is a morphism in $\mathcal{C}$ satisfying $\psi_M \circ (M \otimes \eta_A) = \mathrm{id}_M$, $\psi_M \circ (\psi_M \otimes A) = \psi_M \circ (M \otimes \mu_A)$. Given two right A-modules (M, ψ_M) and (N, ψ_N), $f : M \to N$ is a morphism of right A-modules if $\psi_N \circ (f \otimes A) = f \circ \psi_M$. We denote the category of right A-modules by $\mathcal{C}_A$. In an analogous way we define the left A-modules and we denote this category by ${}_A\mathcal{C}$.

Let D be a coalgebra. (M, ρ_M) is a right D-comodule if M is an object in $(\mathcal{C}, c)$ and $\rho_M : M \to M \otimes D$ is a morphism in $\mathcal{C}$ satisfying $(M \otimes \varepsilon_D) \circ \rho_M = \mathrm{id}_M$, $(\rho_M \otimes D) \circ \rho_M = (M \otimes \delta_D) \circ \rho_M$. Given two right D-comodules (M, ρ_M) and (N, ρ_N), $f : M \to N$ is a morphism of right D-comodules if $\rho_N \circ f = (f \otimes D) \circ \rho_M$. We denote the category of right D-comodules by $\mathcal{C}^D$. Analogously, ${}^D\mathcal{C}$ denotes the category of left D-comodules.

A bialgebra in $\mathcal{C}$ is an object H with algebra and coalgebra structures, and such that ε_H and δ_H are algebra morphisms (equivalently η_H and μ_H are coalgebra morphisms). We say that H is a Hopf algebra if there exists a morphism $\lambda_H : H \to H$ which is convolution inverse to the identical map. We call λ_H the antipode of H.

If H and B are bialgebras $f : H \to B$ is a morphism of bialgebras if f is a morphism of algebras and coalgebras. Moreover, in this case if H and B are Hopf algebras, is not difficult to see that $\lambda_B \circ f = f \circ \lambda_H$ and then we will say that f is a morphism of Hopf algebras.

Let H be a Hopf algebra in $\mathcal{C}$. Let M be an algebra (a coalgebra) such that (M, ψ_M) is in $\mathcal{C}_H$ (resp. (M, φ_M) is in ${}_H\mathcal{C}$). We say that M is a right (resp. left) H-module (co)algebra if η_M and μ_M (ε_M and δ_M) are morphisms of right (resp. letf) H-modules. If (M, ρ_M) is in $\mathcal{C}^H$ (resp. (M, r_M) is in ${}^H\mathcal{C}$). We say that M is a right (resp. left) H-comodule (co)algebra if η_M and μ_M (ε_M and δ_M) are morphisms of right (resp. letf) H-comodules.

We say that a right H-module coalgebra (M, ψ_M) is H-Cleft if there exists a cointegral g, i.e., an H-module morphism in $\mathrm{Reg}(M, H)$. Replacing g by $((\varepsilon_H \circ g^{-1}) \otimes g) \circ \delta_M$ we may assume that $\varepsilon_H \circ g = \varepsilon_M$. In this case, we will say that g is a total cointegral. Analogously, a right H-comodule algebra (M, ρ_M) is H-Cleft if there exists an integral f, i.e., an H-comodule morphism in $\mathrm{Reg}(H, M)$. Replacing f by $\mu_M \circ ((f^{-1} \circ \eta_H) \otimes f)$ we may assume that $f \circ \eta_H = \eta_M$. In this case, we will say that f is a total integral.

Let H be a bialgebra in $\mathcal{C}$, $(A;\varphi_A)$ be a left H-module algebra. We define

$$\eta_{A\sharp H} = \eta_A \otimes \eta_H,$$
$$\mu_{A\sharp H} = (\mu_A \otimes \mu_H) \circ (A \otimes \varphi_A \otimes H \otimes H) \circ (A \otimes H \otimes c_{H,A} \otimes H) \circ (A \otimes \delta_H \otimes A \otimes H).$$

It is well know that $A \sharp H = (A \otimes H, \eta_{A\sharp H}, \mu_{A\sharp H})$ is an algebra in $\mathcal{C}$, called the smash product.

On the other hand, if (A, r_A) is a left H-comodule coalgebra, we have that $A \propto H = (A \otimes H, \varepsilon_{A\propto H}, \delta_{A\propto H})$, where $\varepsilon_{A\propto H} = \varepsilon_A \otimes \varepsilon_H$ and

$$\delta_{A\propto H} = (A\otimes\mu_H\otimes A\otimes H)\circ(A\otimes H\otimes c_{A,H}\otimes H)\circ(A\otimes r_A\otimes H\otimes H)\circ(\delta_A\otimes\delta_H),$$

is a coalgebra, called the smash coproduct.

3. Two-cocycles and skew pairings

Let H be a bialgebra in $\mathcal{C}$. A morphism $\sigma \in \mathrm{Reg}(H \otimes H, K)$ is a two cocycle if

$$\sigma \circ (H \otimes \mu_{{}_\sigma H}) = \sigma \circ (\mu_{{}_\sigma H} \otimes H)$$

where $\mu_{{}_\sigma H} = (\sigma \otimes \mu_H) \circ \delta_{H\otimes H}$.

It is well know (see Theorem 1.6.a of [**5**]) that if σ is a two cocycle we have the next equalities:

a) $(\sigma \circ (\eta_H \otimes \eta_H)) * (\sigma^{-1} \circ (\eta_H \otimes \eta_H)) = \mathrm{id}_K$,

b) $\sigma \circ (H \otimes \eta_H) = \varepsilon_H = \sigma \circ (\eta_H \otimes H)$.

Moreover, if H is a Hopf algebra $\sigma \circ (H \otimes \lambda_H) \in \mathrm{Reg}(H \otimes H, K)$ has inverse $\sigma^{-1} \circ (\lambda_H \otimes H)$.

Proposition 3.1. *Let H be a bialgebra in $\mathcal{C}$ and let σ be a two cocycle in* $\mathrm{Reg}(H \otimes H, K)$*. Then:*

a) *The triple $(H, \eta_H, \mu_{{}_\sigma H})$ is an algebra in $\mathcal{C}$.*

b) *The triple $(H, \eta_H, \mu_{H_{\sigma^{-1}}})$ is an algebra in $\mathcal{C}$, where $\mu_{H_{\sigma^{-1}}} = (\mu_H \otimes \sigma^{-1}) \circ \delta_{H\otimes H}$.*

Proof: We show b). The proof a) is analogous and we leave the details for the reader. Trivially, $\mu_{H_{\sigma^{-1}}} \circ (\eta_H \otimes H) = \mathrm{id}_H = \mu_{H_{\sigma^{-1}}} \circ (H \otimes \eta_H)$. Finally,

$$\begin{aligned}
&\mu_{H_{\sigma^{-1}}} \circ (\mu_{H_{\sigma^{-1}}} \otimes H)\\
&= \mu_H \circ (H \otimes \mu_H) \circ (H \otimes H \otimes H \otimes (\sigma^{-1} \circ (\mu_{H_{\sigma^{-1}}} \otimes H)) \circ \delta_{H\otimes H\otimes H}\\
&= \mu_H \circ (H \otimes \mu_H) \circ (H \otimes H \otimes H \otimes (\sigma^{-1} \circ (H \otimes \mu_{H_{\sigma^{-1}}})) \circ \delta_{H\otimes H\otimes H}\\
&= \mu_{H_{\sigma^{-1}}} \circ (H \otimes \mu_{H_{\sigma^{-1}}}).
\end{aligned}$$

□

Proposition 3.2. *Let H be a Hopf algebra in $\mathcal{C}$ and let $\sigma \in \mathrm{Reg}(H \otimes H, K)$ be a two cocycle. Then*

$$H^{\sigma} = (H, \eta_H, \mu_{({}_{\sigma}H)_{\sigma^{-1}}}, \varepsilon_H, \delta_H, \lambda_{H^{\sigma}})$$

is a Hopf algebra in $\mathcal{C}$ where $\lambda_{H^{\sigma}} = (f \otimes \lambda_H \otimes f^{-1}) \circ (H \otimes \delta_H) \circ \delta_H$ and $f = \sigma \circ (H \otimes \lambda_H) \circ \delta_H$ is a morphism in $\mathrm{Reg}(H, K)$ with inverse $f^{-1} = \sigma^{-1} \circ (\lambda_H \otimes H) \circ \delta_H$.

Proof: See 1.6(b) of [**5**]. □

Let B and H be bialgebras in $\mathcal{C}$. A morphism $\tau\colon B \otimes H \to K$ is called a skew pairing on (B, H) if:

a) $\tau \circ (\mu_B \otimes H) = (\tau \otimes \tau) \circ (B \otimes c_{B,H} \otimes H) \circ (B \otimes B \otimes \delta_H)$,
b) $\tau \circ (B \otimes \mu_H) = (\tau \otimes \tau) \circ (B \otimes c_{B,H} \otimes H) \circ (\delta_B \otimes c_{H,H})$.

As a direct consequence of a) and b), if τ is a convolution invertible skew pairing on (B, H), then $\tau \circ (\eta_B \otimes H) = \varepsilon_H$ and $\tau \circ (B \otimes \eta_H) = \varepsilon_B$.

Let τ be a convolution invertible skew pairing on (B, H). Then the morphism $\sigma_{\tau} = \varepsilon_B \otimes (\tau \circ c_{H,B}) \otimes \varepsilon_H$ is a two cocycle in $\mathrm{Reg}(B \otimes H \otimes B \otimes H, K)$, with inverse $\sigma_{\tau}^{-1} = \varepsilon_B \otimes (\tau^{-1} \circ c_{H,B}) \otimes \varepsilon_H$, called the two cocycle associated with τ (1.5 of [**6**]).

Let H, B be Hopf algebras in $\mathcal{C}$ and let τ be a convolution invertible skew pairing on (B, H). Then (B, φ_B) is a left H-module coalgebra and (H, ψ_H) is a right B-module coalgebra, where

$$\varphi_B = (\tau \otimes B \otimes \tau^{-1}) \circ (B \otimes H \otimes \delta_B \otimes H) \circ \delta_{B \otimes H} \circ c_{H,B},$$

$$\psi_H = ((\tau \circ c_{H,B}) \otimes H \otimes (\tau^{-1} \circ c_{H,B})) \circ (H \otimes B \otimes \delta_H \otimes B) \circ \delta_{H \otimes B}.$$

The object $B \bowtie_{\tau} H = (B \otimes H, \eta_{B \otimes H}, \mu_{B \bowtie_{\tau} H}, \varepsilon_{B \otimes H}, \delta_{B \otimes H}, \lambda_{B \bowtie_{\tau} H})$ is a Hopf algebra in $\mathcal{C}$ where

$$\mu_{B \bowtie_{\tau} H} = (\mu_B \otimes \mu_H) \circ (B \otimes \varphi_B \otimes \psi_H \otimes H) \circ (B \otimes \delta_{H \otimes B} \otimes H),$$

$$\lambda_{B \bowtie_{\tau} H} = (\varphi_B \otimes \psi_H) \circ \delta_{H \otimes B} \circ (\lambda_H \otimes \lambda_B) \circ c_{B,H}.$$

Moreover, if σ_{τ} is the two cocycle associated with τ then the Hopf algebras $(B \otimes H)^{\sigma_{\tau}}$ and $B \bowtie_{\tau} H$ are the same [**6**] and [**4**].

4. The Hopf algebras $B \bowtie_{\tau} H$ and $B \infty H$

A quasitriangular Hopf algebra in $\mathcal{C}$ is a pair $(H, \mathcal{R})$ where H is a Hopf algebra in $\mathcal{C}$ and $\mathcal{R}$ is a morphism in $\mathrm{Reg}(K, H \otimes H)$ such that:

a) $(\delta_H \otimes H) \circ \mathcal{R} = (H \otimes H \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \circ (\mathcal{R} \otimes \mathcal{R})$.
b) $(H \otimes \delta_H) \circ \mathcal{R} = (\mu_H \otimes c_{H,H}) \circ (H \otimes c_{H,H} \otimes H) \circ (\mathcal{R} \otimes \mathcal{R})$.

c) $\mu_{H\otimes H} \circ ((c_{H,H} \circ \delta_H) \otimes \mathcal{R}) = \mu_{H\otimes H} \circ (\mathcal{R} \otimes \delta_H)$.

If $(H, \mathcal{R})$ is a quasitriangular Hopf algebra in $\mathcal{C}$, it is not difficult to prove that the morphism $\mathcal{R}$ obeys:

1) $(\varepsilon_H \otimes H) \circ \mathcal{R} = \eta_H = (H \otimes \varepsilon_H) \circ \mathcal{R}$

2) $$(\mu_H \otimes \mu_H \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes c_{H,H} \otimes H) \circ (\mathcal{R} \otimes \mathcal{R} \otimes \mathcal{R}) = (\mu_H \otimes H \otimes (\mu_H \circ c_{H,H})) \circ (H \otimes H \otimes c_{H,H} \otimes H) \circ (H \otimes c_{H,H} \otimes (\mu_H \circ c_{H,H}) \otimes H) \circ (\mathcal{R} \otimes \mathcal{R} \otimes \mathcal{R})$$

3) $\mathcal{R}^{-1} = (\lambda_H \otimes H) \circ \mathcal{R}$

4) $(\lambda_H \otimes \lambda_H) \circ \mathcal{R} = \mathcal{R}$

5) $(H \otimes \lambda_H) \circ \mathcal{R}^{-1} = \mathcal{R}$

6) $$(\mu_{H\otimes H} \otimes H) \circ (H \otimes H \otimes \delta_H \otimes H) \circ (\mathcal{R} \otimes \mathcal{R}) = (H \otimes \mu_{H\otimes H}) \circ (H \otimes \mathcal{R} \otimes \delta_H) \circ \mathcal{R}.$$

Let H be a Hopf algebra in $(\mathcal{C}, c)$. Let (M, φ_M) be in ${}_H\mathcal{C}$ and (M, r_M) be in ${}^H\mathcal{C}$. We say that (M, φ_M, r_M) is in ${}_H^H\mathcal{YD}$ if φ_M and r_M satisfies the compatibility condition:

$$(\mu_H \otimes M) \circ (H \otimes c_{M,H}) \circ ((r_M \circ \varphi_M) \otimes H) \circ (H \otimes c_{H,M}) \circ (\delta_H \otimes M) = (\mu_H \otimes \varphi_M) \circ (H \otimes c_{H,H} \otimes M) \circ (\delta_H \otimes r_M).$$

If B is a Hopf algebra in ${}_H^H\mathcal{YD}$ with antipode λ_B then

$$B \infty H = (B \otimes H, \eta_{B\sharp H}, \mu_{B\sharp H}, \varepsilon_{B\propto H}, \delta_{B\propto H})$$

is a Hopf algebra in $\mathcal{C}$ with antipode

$$\lambda_{B\infty H} = (\varphi_B \otimes H) \circ (H \otimes c_{H,B}) \circ (\delta_H \otimes B) \circ (\lambda_H \otimes \lambda_B) \circ (\mu_H \otimes B) \circ (H \otimes c_{B,H}) \circ (r_B \otimes H)$$

(see [**2**] for more details).

Theorem 4.1. *Let B and H be Hopf algebras in $\mathcal{C}$. Assume that H is quasitriangular. Let $\tau \in \mathrm{Reg}(B \otimes H, K)$ be a skew pairing on (B, H). Then B is a Hopf algebra in the category ${}_H^H\mathcal{YD}$ and there exists a Hopf algebra isomorphism $w\colon B \infty H \to B \bowtie_\tau H$.*

Proof: By 2.5 of [**6**], $g\colon B \bowtie_\tau H \to H$, defined by

$$g = (\tau \otimes (\mu_H \circ c_{H,H})) \circ (B \otimes c_{H,H} \otimes H) \circ (B \otimes H \otimes \mathcal{R})$$

and $f = \eta_B \otimes H\colon H \to B \bowtie_\tau H$ are Hopf algebra morphisms such that $g \circ f = \mathrm{id}_H$. Then $(B \bowtie_\tau H, \rho_{B\bowtie_\tau H})$ where

$$\rho_{B\bowtie_\tau H} = (B \otimes H \otimes \tau \otimes (\mu_H \circ c_{H,H})) \circ (B \otimes H \otimes B \otimes c_{H,H} \otimes H) \circ (\delta_{B\otimes H} \otimes \mathcal{R})$$

is a right H-Cleft comodule algebra with total integral f and $(B \bowtie_\tau H, \psi_{B\bowtie_\tau H} = B \otimes \mu_H)$ is a right H-Cleft module coalgebra with total cointegral g. Thus, applying 3.2 of [**1**], the object $(B \bowtie_\tau H)_0$ defined by the equalizer diagram

$$(B \bowtie_\tau H)_0 \xrightarrow{i_{B\bowtie_\tau H}} B \bowtie_\tau H \underset{B\bowtie_\tau H \otimes \eta_H}{\overset{\rho_{B\bowtie_\tau H}}{\rightrightarrows}} B \bowtie_\tau H \otimes H$$

is a Hopf algebra in ${}^H_H\mathcal{YD}$ such that $(B \bowtie_\tau H)_0 \infty H$ and $B \bowtie_\tau H$ are isomorphic Hopf algebras. Moreover, $(B \bowtie_\tau H)^0$ defined by the coequalizer of $\psi_{B\bowtie_\tau H}$ and $B \otimes H \otimes \varepsilon_H$ is such that $(B \bowtie_\tau H)^0 = (B \bowtie_\tau H)_0$ and therefore $(B \bowtie_\tau H)^0 = (B \bowtie_\tau H)_0 = B$ because the coequalizer morphism of $\psi_{B\bowtie_\tau H}$ and $B \otimes H \otimes \varepsilon_H$ is $p = B \otimes \varepsilon_H$. As a consequence, $(B, \eta_B, m_B, \varepsilon_B, \delta_B, s_B)$ where $m_B = \mu_B \circ (B \otimes \varphi_B) \circ (i_{B\bowtie_\tau H} \otimes B)$ and $s_B = (\tau \otimes \varphi_B) \circ (B \otimes \mathcal{R} \otimes \lambda_B) \circ \delta_B$ is a Hopf algebra in ${}^H_H\mathcal{YD}$.

The isomorphism w between $B \infty H$ and $B \bowtie_\tau H$ is $w = \mu_{B\bowtie_\tau H} \circ (i_{B\bowtie_\tau H} \otimes f)$. Note that $w = (B \otimes \tau \otimes \mu_H) \circ (\delta_B \otimes H \otimes \lambda_H \otimes H) \circ (B \otimes \mathcal{R} \otimes H)$ because

$$\begin{aligned} i_{B\bowtie_\tau H} &= i_{B\bowtie_\tau H} \circ p \circ (B \otimes \eta_H) \\ &= \mu_{B\bowtie_\tau H} \circ (B \bowtie_\tau H \otimes f^{-1}) \circ \rho_{B\bowtie_\tau H} \circ (B \otimes \eta_H) \\ &= \mu_{B\bowtie_\tau H} \circ (B \otimes H \otimes \eta_B \otimes \lambda_H) \circ \rho_{B\bowtie_\tau H} \circ (B \otimes \eta_H) \\ &= (B \otimes \tau \otimes \lambda_H) \circ (\delta_B \otimes \mathcal{R}). \end{aligned}$$

Finally, we compute $\varphi_{(B\bowtie_\tau H)_0}$ and $r_{(B\bowtie_\tau H)_0}$,

$$\begin{aligned} &i_{B\bowtie_\tau H} \circ \varphi_{(B\bowtie_\tau H)_0} \\ &= \mu_{B\bowtie_\tau H} \circ (B \otimes H \otimes (\mu_{B\bowtie_\tau H} \circ c_{B\otimes H, B\otimes H})) \\ &\qquad \circ (f \otimes f^{-1} \otimes i_{B\bowtie_\tau H}) \circ (\delta_H \otimes B) \\ &= (B \otimes \mu_H) \circ (\varphi_B \otimes \psi_H \otimes \tau \otimes \mu_H) \circ (\delta_{H\otimes B} \otimes B \otimes H \otimes \lambda_H \otimes H) \\ &\qquad \circ (H \otimes \delta_B \otimes \mathcal{R} \otimes H) \circ (H \otimes c_{H,B}) \otimes (H \otimes \lambda_H \otimes B) \circ (\delta_H \otimes B) \\ &= (B \otimes \mu_H) \circ (\varphi_B \otimes \psi_H \otimes \tau \otimes H) \circ (\delta_{H\otimes B} \otimes B \otimes H \otimes (\lambda_H \circ \mu_H \circ c_{H,H})) \\ &\qquad \circ (H \otimes \delta_B \otimes \mathcal{R} \otimes H) \circ (H \otimes c_{H,B}) \circ (\delta_H \otimes B) \\ &= i_{B\bowtie_\tau H} \circ \varphi_B. \end{aligned}$$

Thus, $\varphi_{(B\bowtie_\tau H)_0} = p \circ i_{B\bowtie_\tau H} \circ \varphi_{(B\bowtie_\tau H)_0} = p \circ i_{B\bowtie_\tau H} \circ \varphi_B = \varphi_B$ and, as a consequence, the left H-module coalgebra (B, φ_B) is a left H-module algebra too.

On the other hand,

$$
\begin{aligned}
&r_{(B\bowtie_\tau H)_0}\\
&= (g\otimes p)\circ\delta_{B\otimes H}\circ i_{B\bowtie_\tau H}\\
&= (\tau\otimes c_{B,H})\circ(B\otimes H\otimes B\otimes\tau\otimes\mu_H)\circ(B\otimes H\otimes\delta_B\otimes c_{H,H}\otimes\lambda_H)\\
&\circ(B\otimes c_{B,H}\otimes H\otimes H\otimes H)\circ(\delta_B\otimes\mathcal{R}\otimes\mathcal{R}).
\end{aligned}
$$

□

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J. N. Alonso Alvarez:
Departamento de Matemáticas
Universidad de Vigo
Campus Universitario Lagoas-Marcosende
E-36280, Vigo
Spain
E-mail address: `jnalonso@uvigo.es`

J. M. Fernández Vilaboa:
Departamento de Alxebra
Universidad de Santiago de Compostela
E-15771, Santiago de Compostela
Spain
E-mail address: `vilaboa@zmat.usc.es`

R. González Rodríguez:
Departamento de Matemática Aplicada
Universidad de Vigo
Campus Universitario Lagoas-Marcosende
E-36280, Vigo
Spain
E-mail address: `rgon@dma.uvigo.es`

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