

ON \mathbb{D}^* -EXTENSION PROPERTY OF THE HARTOGS DOMAINS

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Abstract

A complex analytic space is said to have the \mathbb{D}^* -extension property if and only if any holomorphic map from the punctured disk to the given space extends to a holomorphic map from the whole disk to the same space. A Hartogs domain H over the base X (a complex space) is a subset of $X \times \mathbb{C}$ where all the fibers over X are disks centered at the origin, possibly of infinite radius. Denote by ϕ the function giving the logarithm of the reciprocal of the radius of the fibers, so that, when X is pseudoconvex, H is pseudoconvex if and only if ϕ is plurisubharmonic.

We prove that H has the \mathbb{D}^* -extension property if and only if (i) X itself has the \mathbb{D}^* -extension property, (ii) ϕ takes only finite values and (iii) ϕ is plurisubharmonic. This implies the existence of domains which have the \mathbb{D}^* -extension property without being (Kobayashi) hyperbolic, and simplifies and generalizes the authors' previous such example.

1. Introduction

The “big Picard” theorem states that any holomorphic map f from the punctured unit disc \mathbb{D}^* into the Riemann sphere $\mathbb{P}_1(\mathbb{C})$ which omits three points can be extended to a holomorphic map $f: \mathbb{D} \rightarrow \mathbb{P}_1(\mathbb{C})$. Kwack [Kw] extended this theorem to a higher dimensional context. If f is a holomorphic map from \mathbb{D}^* into a hyperbolic space X such that, for a suitable sequence of points $z_k \in \mathbb{D}^*$ converging to the origin, $f(z_k)$ converges to a point $p_0 \in X$, then f extends to a holomorphic map from \mathbb{D} into X .

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The above-mentioned theorem of Kwack has strongly motivated the study of the extension problem of holomorphic maps through isolated singularities. At the same time, this result has suggested the study of the class of complex spaces having the following property.

Definition. Let X be a complex space. We say that X has the \mathbb{D}^* -extension property (\mathbb{D}^* -EP) iff for any holomorphic map f from $\mathbb{D}^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$ to X , there exists a map $\tilde{f} \in \text{Hol}(\mathbb{D}, X)$ (where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$) such that $\tilde{f}|_{\mathbb{D}^*} = f$.

Much attention has been devoted to the \mathbb{D}^* -EP and various theorems have been obtained by Kwack [Kw], Thai [Th], Thai and Thomas [Th-Tho], and others, see the monograph [Ko].

The first aim of this note is to prove the following

Theorem 1. Let $X_\varphi := \{(z, w) \in X \times \mathbb{C} : |w| < e^{-\varphi(z)}\}$ where $\varphi: X \rightarrow [-\infty, +\infty)$ is upper semi continuous.

Then X_φ has the \mathbb{D}^* -EP iff:

X has the \mathbb{D}^* -EP, $\varphi \in \text{PSH}(X)$ and $\varphi(z) > -\infty, \forall z \in X$.

Notice that this result admits as a corollary, when $X = \mathbb{D}$ and φ is not locally bounded, the existence of a domain in \mathbb{C}^2 which has the \mathbb{D}^* -EP without being Kobayashi hyperbolic. We thus simplify the proof of that result given in [Th-Tho], and generalize somewhat the class of counter-examples available.

The proof of this result can be carried out by elementary means. However, use of more powerful theorems makes for shorter proofs and more general results. We denote by Λ_d the Hausdorff measure in (real) dimension d .

Definition. We say that X has the n -PEP (resp. n -PPEP, (n, d) -EP) iff for any closed set $A \subset \mathbb{D}^n$ which is polar (resp. pluripolar, resp. of locally finite Λ_d measure), for any holomorphic map f from $\mathbb{D}^n \setminus A$ to X , there exists a map $\tilde{f} \in \text{Hol}(\mathbb{D}^n, X)$ (where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$) such that $\tilde{f}|_{\mathbb{D}^n \setminus A} = f$.

Re-using the notations of Theorem 1, we have, for any $n \geq 1, 2n-2 < d < 2n-1$:

Theorem 2. X_φ has the n -PEP (resp. n -PPEP, (n, d) -EP) iff:

X has the n -PEP (resp. n -PPEP, (n, d) -EP), $\varphi \in \text{PSH}(X)$ and $\varphi(z) > -\infty, \forall z \in X$.

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2. Proof of Theorem 1

1) *Sufficiency:*

Let $H \in \text{Hol}(\mathbb{D}^*, X_\varphi), H = (h_1, h_2)$. Since X has the \mathbb{D}^* -EP, h_1 extends to a map $\tilde{h}_1, \tilde{h}_1: \mathbb{D} \rightarrow X$. Let $z_0 = \tilde{h}_1(0)$. We need the following result on the local growth of (pluri-)subharmonic functions.

Theorem ([Ho1, Corollary 4.4.6, p. 98] or [Ho2, Corollary 4.2.10, p. 261]). *If $\varphi \in PSH(\Omega) \setminus \{-\infty\}$, where Ω is a connected pseudoconvex open set, then $e^{-\varphi}$ is locally integrable in a dense open subset G containing all points z where $\varphi(z) > -\infty$.*

In particular, since $\varphi(z_0)$ is finite, $e^{-2\varphi} \in L^1_{\text{loc}}$ in a neighborhood of z_0 . But then $h_2 \in L^2_{\text{loc}}$, which implies that h_2 extends holomorphically across 0, as can easily be deduced from the Laurent series expansion.

Finally, to see that we actually have $\log |\tilde{h}_2(0)| < -\varphi(\tilde{h}_1(0))$, we apply the maximum principle to the subharmonic function $\log |\tilde{h}_2| + \varphi(\tilde{h}_1)$, as was done in [Th-Tho].

A direct proof of the extendability of h_2 may be given without recourse to Hörmander’s result.

Since φ is u.s.c, there exists $r_0 > 0, M \in \mathbb{R}$ such that $\varphi(z) \leq M, \forall z \in \overline{D}(z_0, r_0)$. Without loss of generality, suppose $M \leq 0$. Therefore we have

$$(1) \quad \log |h_2(\xi)| \leq -\varphi(\tilde{h}_1(\xi))$$

and since $\varphi \leq 0, \log_+ |h_2(\xi)| \leq -\varphi_1(\xi)$, where $\varphi_1(\xi) := \varphi(\tilde{h}_1(\xi))$, therefore $\varphi_1 \in \text{SH}(\mathbb{D}, \mathbb{R}_-)$.

By the mean value inequality for subharmonic functions, (1) implies: $\forall r \in (0, r_0)$,

$$\frac{1}{\pi r^2} \int_{D(0,r)} \log_+ |h_2(\xi)| d\lambda_2(\xi) \leq -\varphi_1(0) < +\infty.$$

We want to show that this implies that h_2 has a removable singularity at the origin.

Expand h_2 as a Laurent series

$$h_2(\xi) = \sum_{n \in \mathbb{Z}} a_n \xi^n.$$

Then for r small enough,

$$\left| \sum_{n>0} a_n \xi^n \right| \leq e,$$

so

$$\log_+ |h_0(\xi)| \leq 1 + \log_+ |h_2(\xi)|,$$

where we set

$$h_0(\xi) := \sum_{n \leq 0} a_n \xi^n,$$

and we are reduced to $\forall r > 0$

$$\frac{1}{\pi r^2} \int_{D(0,r)} \log_+ |h_0(\xi)| d\lambda_2(\xi) \leq C < +\infty.$$

Set $f(\xi) := h_0(1/\xi)$. This is now an entire function. Under the change of variable $\psi = \frac{1}{\xi}$, we get

$$\frac{1}{\pi r^2} \int_{D(0,r)} \log_+ |h_0(\xi)| d\lambda_2(\xi) = \frac{1}{\pi r^2} \int_{\mathbb{C} \setminus \bar{D}(0,1/r)} \log_+ |f(\psi)| \frac{1}{|\psi|^4} d\lambda_2(\psi).$$

Now $\log_+ |f| \in \text{SH}(\mathbb{C})$, so

$$m(\rho) := \int_0^{2\pi} \log_+ |f(\rho e^{i\theta})| \frac{d\theta}{2\pi}$$

is an increasing function of ρ .

Passing to polar coordinates, we get that

$$\begin{aligned} C &\geq \frac{1}{\pi r^2} \int_{\frac{1}{r}}^{\infty} \left(\int_0^{2\pi} \log_+ |f(\rho e^{i\theta})| d\theta \right) \frac{1}{\rho^3} d\rho \\ &= \frac{2}{r^2} \int_{\frac{1}{r}}^{\infty} \frac{m(\rho)}{\rho^3} d\rho \\ &\geq \frac{2}{r^2} m(1/r) \int_{\frac{1}{r}}^{\infty} \frac{d\rho}{\rho^3} = m\left(\frac{1}{r}\right). \end{aligned}$$

Therefore $m(\rho)$ is bounded as $\rho \rightarrow \infty$. But then, since $\log_+ |f| \in \text{SH}(\mathbb{C})$, it must be bounded above on \mathbb{C} , since by the Poisson formula

$$\log_+ |f(z_0)| \leq \int_0^{2\pi} \frac{1 - \left| \frac{z_0}{\rho} \right|^2}{\left| \frac{z_0}{\rho} - e^{i\theta} \right|^2} \log_+ |f(\rho e^{i\theta})| \frac{d\theta}{2\pi} \leq \frac{1 + \left| \frac{z_0}{\rho} \right|}{1 - \left| \frac{z_0}{\rho} \right|} m(\rho) \leq 3C,$$

for $\rho \geq 2|z_0|$, so f is constant by Liouville's theorem.

Therefore h has a removable singularity at 0.

2) *Necessity*:

To prove that X has the \mathbb{D}^* -EP, if $h \in \text{Hol}(\mathbb{D}^*, X)$, then the map H given by $H(\xi) := (h(\xi), 0)$ is holomorphic from \mathbb{D}^* to X_φ and must therefore admit an extension \tilde{H} such that (by continuity) $\tilde{H}(\mathbb{D}) \subset X \times \{0\} \subset X_\varphi$. Writing $\tilde{H} = (\tilde{h}, 0)$, we obtain the required extension of h .

If there exists $z_0 \in X$ such that $\varphi(z_0) = -\infty$ then the complex line $\{(z_0, w) : w \in \mathbb{C}\} \subset X_\varphi$, so X_φ does not have the \mathbb{D}^* -EP (take $h(\xi) = (z_0, 1/\xi)$, see [Th]).

There remains to show that $\varphi \in \text{PSH}(X)$. We first do this for the case where X is an open set in \mathbb{C}^n .

Lemma. *Let $\Omega \subset \mathbb{C}^n$ be a domain with the \mathbb{D}^* -EP. Then Ω is pseudoconvex.*

This lemma (which we alluded to in [Th-Tho]) is a consequence of a theorem of Shiffman [Si] (see also [So-Th]): if for any sequence $\{f_n\} \subset \text{Hol}(\mathbb{D}, X)$, convergence of $\{f_n|_{\mathbb{D}^*}\}$ in $\text{Hol}(\mathbb{D}^*, X)$ implies convergence of $\{f_n\}$ (“weak disk condition”), then X has the Hartogs extension condition (which implies pseudoconvexity for open sets in \mathbb{C}^n).

But a domain in \mathbb{C}^n with the \mathbb{D}^* -EP verifies the weak disk condition (simply extend the limit mapping and then apply the maximum principle on all coordinates).

However, there is a direct and elementary proof which avoids the use of Shiffman’s theorem. For the reader’s convenience, and since some colleagues of ours seemed to find it nice, we include it here.

Direct Proof of the Lemma: Let Φ be a holomorphic embedding of the closed unit bidisk $\overline{\mathbb{D}^2}$ into \mathbb{C}^n . Call *Hartogs figure* the image under Φ of the set $H_0 := \{|z_1| \leq 1, z_2 = 0\} \cup \{|z_1| = 1, |z_2| \leq 1\}$. Recall that Ω is pseudoconvex if and only if for every Hartogs figure contained in Ω , $\Phi(\mathbb{D}^2)$ is also contained in Ω [Ra].

Therefore, assuming Ω is not pseudoconvex, we obtain the following situation: there exists a holomorphic embedding Φ such that $\Omega_1 := \Phi^{-1}(\Omega) \cap \mathbb{D}^2$ is open, $\mathbb{D}^2 \setminus \Omega_1 \neq \emptyset$, and $\overline{\mathbb{D}^2} \setminus \Omega_1 \cap H_0 = \emptyset$.

Let

$$r_2 := \inf\{|z_2| : (z_1, z_2) \in \mathbb{D}^2 \setminus \Omega_1\};$$

our hypotheses mean that $0 < r_2 < 1$, and they also imply that the set

$$K := (\mathbb{D}^2 \setminus \Omega_1) \cap \left\{ |z_2| \leq \frac{1+r_2}{2} \right\}$$

is compact in \mathbb{D}^2 , and therefore $r_1 := \max_{(z_1, z_2) \in K} |z_1| < 1$. Now set

$$\delta_\varepsilon(z) := \varepsilon|z_1|^2 - |z_2|^2.$$

There exists a point $z^0 = (z_1^0, z_2^0)$ so that $\delta_\varepsilon(z^0) = \max_K \delta_\varepsilon$. For ε small enough ($|\varepsilon|r_1^2 - (\frac{1+r_2}{2})^2 < -r_2^2$), $z^0 \in K \setminus \{|z_2| = \frac{1+r_2}{2}\}$, so there exists a neighborhood V of z^0 such that $V \cap \Omega_1 = V \setminus K$. It is now enough to find an analytic disk f with center $f(0) \in K$ and $f(D(0, r) \setminus \{0\}) \subset V \setminus K$, for $r > 0$ small enough. We may in fact pick an affine disk, namely

$$f(\xi) := (z_1^0 + \bar{z}_2^0 \xi, z_2^0 + \varepsilon \bar{z}_1^0 \xi).$$

Observe that $f(\mathbb{C})$ is a line tangent to the level hypersurface of δ_ε corresponding to the value $\delta_\varepsilon(z^0)$, and in fact an elementary calculation shows that

$$\delta_\varepsilon(f(\xi)) = \delta_\varepsilon(z^0) + \varepsilon|\xi|^2(|z_2^0|^2 - \varepsilon|z_1^0|^2) > \delta_\varepsilon(z^0)$$

for $\varepsilon > 0$ and small enough (z_1^0 and z_2^0 do depend on ε , but we have the condition as soon as $r_2^2 - \varepsilon r_1^2 > 0$), which completes the proof of the lemma. \square

We may remark that setting $f_t(\xi) := f(\xi) + t\nabla\delta_\varepsilon(z^0)$, the map $\Phi \circ f$ gives a disk violating the \mathbb{D}^* -EP for the Ω which we had assumed non-pseudoconvex, and the maps $\Phi \circ f_t$ provide a refined failure of the Kontinuitätsatz for Ω (contact with the boundary occurs at one point exactly).

Together with the above lemma, the following will complete the proof of necessity.

Claim. *If X_φ has the \mathbb{D}^* -EP, and $\varphi \notin \text{PSH}(X)$ then there exists $\Omega \subset \mathbb{C}^2$, having the \mathbb{D}^* -EP and Ω not pseudoconvex.*

We need the following characterization. Denote by $\mathcal{H}a(\mathbb{D})$ the space of harmonic functions on the disk.

Fact. $\varphi \in \text{PSH}(X)$ iff $\forall f \in \text{Hol}(\overline{\mathbb{D}}, X), \forall u \in \mathcal{H}a(\mathbb{D}) \cap \mathcal{C}^0(\overline{\mathbb{D}})$, such that $\varphi \circ f(e^{i\theta}) \leq u(e^{i\theta}) \forall \theta \in \mathbb{R}$, then $\varphi \circ f(0) \leq u(0)$.

This follows immediately from the theorem of Fornaess and Narasimhan [Fo-Na] which characterizes plurisubharmonic functions on complex spaces as those whose pullback under any analytic disk is subharmonic, and the characterization of subharmonicity by the mean value inequality (see e.g. [Ho1, Theorem 1.6.3, p. 16]).

Now suppose $\varphi \notin \text{PSH}(X)$. Then $\exists f \in \text{Hol}(\overline{\mathbb{D}}, X), u \in \mathcal{H}a(\mathbb{D}) \cap \mathcal{C}^0(\overline{\mathbb{D}})$, such that $\varphi(f(0)) > u(0), \varphi(f(e^{i\theta})) \leq u(e^{i\theta}), \forall \theta \in \mathbb{R}$.

Let

$$\begin{aligned} \Omega &:= \{(z, w) \in \mathbb{D} \times \mathbb{C} : (f(z), w) \in X_\varphi\} \\ &= \{(z, w) \in \mathbb{D} \times \mathbb{C} : |w| < e^{\varphi \circ f(z)}\} =: \mathbb{D}_{\varphi \circ f}. \end{aligned}$$

Since $\varphi \circ f$ is not subharmonic, the classical result about Hartogs domains implies that Ω is not pseudoconvex. The following claim then completes our proof.

Claim. Ω has the \mathbb{D}^* -EP.

Proof of the Claim: Let $h \in \text{Hol}(\mathbb{D}^*, \Omega)$, $h(\xi) = (h_1(\xi), h_2(\xi))$.

Now $h_1(\xi) \in \mathbb{D}$ for all ξ , so $h_1(\xi)$ extends to $\tilde{h}_1 \in \text{Hol}(\mathbb{D}, \mathbb{D})$. The map $\xi \mapsto (f \circ h_1(\xi), h_2(\xi))$ is holomorphic from \mathbb{D}^* to X_φ by construction, so it extends to $F \in \text{Hol}(\mathbb{D}, X_\varphi)$.

Let $F(\xi) = (F_1(\xi), F_2(\xi))$.

F_2 provides an extension of h_2 . It remains to see that $\tilde{h} = (\tilde{h}_1, F_2) \in \text{Hol}(\mathbb{D}, \Omega)$, that is, that $|f(\tilde{h}_1(0))| < e^{-\varphi(F_2(0))}$.

Since $|f(\tilde{h}_1(0))| = \lim_{\xi \rightarrow 0} |f(h_1(\xi))| = \lim_{\xi \rightarrow 0} |F_1(\xi)| = |F_1(0)| < e^{-\varphi(F_2(0))}$ because $F \in \text{Hol}(\mathbb{D}, X_\varphi)$, we are done. \square

3. Proof of Theorem 2

The direct implication proceeds as in the previous section, noticing that each of the extension properties we have defined implies the \mathbb{D}^* -EP. (In the case where $n \geq 2$, given a map $f \in \text{Hol}(\mathbb{D}^*, X)$, simply consider the map $F \in \text{Hol}(\mathbb{D}^n \setminus \{z_1 = 0\}, X)$ given by $F(z_1, \dots, z_n) := f(z_1)$.)

To prove the converse implication, recall the following result of extension.

Theorem ([Ha-Po, Theorem 1, (d)]). *Suppose A is a closed subset of an open set $\Omega \subset \mathbb{C}^n$ and that $f \in \text{Hol}(\Omega \setminus A)$. Let $2 \leq p < \infty$ and p' be the conjugate exponent ($\frac{1}{p} + \frac{1}{p'} = 1$). If $f \in L^p_{\text{loc}}(\Omega)$ and $\Lambda_{2n-p'}(A)$ is locally finite, then $f \in \text{Hol}(\Omega)$.*

Now given d as in the theorem, set $p' := 2n - d$ and p its conjugate exponent. Given a map $h = (h_1, h_2) \in \text{Hol}(\mathbb{D}^n \setminus A, X_\varphi)$, h_1 extends to $\tilde{h}_1 \in \text{Hol}(\mathbb{D}^n, X)$ by the extension property for X which is included in the hypothesis, and $p\varphi \circ \tilde{h}_1$ is a finite-valued plurisubharmonic function, so locally integrable, therefore $|h_2| \leq e^{-\varphi \circ \tilde{h}_1}$ verifies all the hypotheses of the Harvey-Polking theorem.

If A is polar, it is a classical fact that $\Lambda_d(A) = 0$ for any $d > 2n - 2$ [La] and the same is true for pluri-polar sets [KI].

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