Publicacions Matemàtiques, Vol. **44** (2000), 295–307

EXISTENCE AND UNIQUENESS OF PERIODIC SOLUTIONS FOR A NONLINEAR REACTION-DIFFUSION PROBLEM

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Abstract

We consider a class of degenerate reaction-diffusion equations on a bounded domain with nonlinear flux on the boundary. These problems arise in the mathematical modelling of flow through porous media. We prove, under appropriate hypothesis, the existence and uniqueness of the nonnegative weak periodic solution. To establish our result, we use the Schauder fixed point theorem and some regularizing arguments.

1. Introduction

This paper deals with the existence and uniqueness of periodic solutions for the following nonlinear reaction-diffusion problem

$$u_t = \operatorname{div}(\nabla\varphi(u)) + c(x,t,u), \quad \text{in } Q := \Omega\times\mathbb{R} \tag{1}$$

$$-\partial\varphi(u)/\partial\nu = g(\varphi(u)), \quad \text{on } \partial\Omega\times\mathbb{R} \tag{2}$$

$$u(x,t+\omega) = u(x,t) \text{ and } u \geq 0 \quad \text{in } Q \tag{3}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and ν denotes the outward unit normal to $\partial\Omega$. Equation (1) models the filtration of a fluid in a homogenous, isotropic, rigid and unsaturated porous medium, with the lower order term $c(x,t,u)\geq 0$ in $\overline{\Omega}\times\mathbb{R}\times\mathbb{R}_+$, accounts for a reaction taking place in the medium. We consider the

Partially supported by G.N.A.F.A. and M.U.R.S.T. 40% Equazioni Differenziali.

following assumptions on the data

$$(H_\varphi)\begin{cases}\varphi\in C([0,\infty))\cap C^{2+\alpha}_{\rm loc}((0,\infty)),\ \varphi(0)=0 \text{ and } \varphi'(s)>0 \text{ for } s>0\\ \text{and there exist } r_0>0,\ \alpha_0>0,\ \alpha_1>0,\ 0\le m_0\le m_1<1\\ \text{such that } \alpha_0\varphi(r)^{m_0}\le\varphi'(r)\le\alpha_1\varphi(r)^{m_1}, \text{ for any } r\ge r_0.\end{cases}$$

$$(H_c)\begin{cases}\text{i) } c\in C(\overline{\Omega}\times\mathbb{R}\times\mathbb{R}_+),\ c(x,t+\omega,s)=c(t,s),\\ \quad c_u\in C(\overline{\Omega}\times\mathbb{R}\times\mathbb{R}_+\backslash\{0\})\\ \text{ii) } c_u\in L^\infty(Q\times[-M,M]) \text{ for every } M>0\\ \text{iii) } c(x,t,r)\le C_0\varphi(r)^\beta,\ C_0>0,\ \beta\in[0,1) \text{ and } r\ge r_0.\end{cases}$$

$$(H_g)\begin{cases}\text{a) } g\in C(\mathbb{R}_+),\ g\ge 0,\ g\in C^1(\mathbb{R}_+),\ |g'(r)|<\infty \text{ for all } r\ge 0\\ \text{b) there exists } d_0>0 \text{ such that } g(\varphi(r))\ge d_0\varphi(r) \text{ for all } r\ge r_0.\end{cases}$$

Remark. Hypotesis (H_c) iii) is enough to have the global existence result for the associated initial-boundary value problem (see [**2**]).

In problem (1)-(3), u denotes the moisture content in the soil represented by the domain Ω, therefore we require the condition $u\ge 0$. Assumption (H_φ) includes the case of degenerate equation i.e. $\varphi'(0)=0$, thus classical solution doesn't exist and a concept of weak solution has to be introduced; the weak solution is continuous but not smooth.

In this paper we prove the existence and uniqueness of the periodic weak solution to (1)-(3) under the above assumptions.

To prove the existence of periodic weak solutions we use as a preliminear step the Schauder fixed point theorem for the Poincaré map of a nondegenerate initial-boundary value problem associated to (1)-(3).

To this purpose, we will consider a sequence of approximated nondegenerate problems which can be solved in a classical sense.

The uniqueness of the periodic weak solution will be established using an adaptation of the method of [**4**] and the assumption that $g(\varphi(u))=d_0\varphi(u)$.

Initial and Dirichlet's boundary value problems have been studied to equation (1) by many authors; we quote for example [**6**], [**7**], [**12**] and [**13**].

Recently, [**2**] and [**1**] have studied the asymptotic behavior and the blow-up in finite time of solutions for equations of type (1) with various boundary conditions.

To the knowledge of the author, it seems that the topic considered in the present paper has not been discussed previously. We only are aware of the paper [**8**] which treats the periodic case for (1)-(3) but for $g = 0$, $\varphi(u) = u^m$, $m > 1$ and a zero order term which is linear w.r.t.u.

Our assumptions, allows to consider more general φ and zero order terms.

2. Existence of periodic solutions

Since our equation (1) may degenerates, we make the following definition of periodic solution

Definition 1. A function $u \in C(I; L^1(\Omega)) \cap L^\infty(Q_I)$ $(Q_I := \Omega \times (t_0, t_1))$ is said to be a periodic weak solution of (1)-(3) if for any compact interval $I = [t_0, t_1] \in \mathbb{R}$, satisfies $-\partial\varphi(u(x,t))/\partial\nu = g(\varphi(u))$ on $\partial\Omega \times I$, $u(x, t+\omega) = u(x,t)$, $\varphi(u) \in L^2((t_0,t_1); H^1(\Omega))$ and

$$\int_{t_0}^{t_1} \int_\Omega (u\zeta_t + \varphi(u)\Delta\zeta + c(x,t,u)\zeta(x,t))\, dx\, dt$$

$$= \int_\Omega (u(x,t_1)\zeta(x,t_1) - u(x,t_0)\zeta(x,t_0)\, dx$$

$$+ \int_{t_0}^{t_1} \int_{\partial\Omega} g(\varphi(u))\zeta(x,t)\, dx\, dt$$

$$+ \int_{t_0}^{t_1} \int_{\partial\Omega} \varphi(u)\partial\zeta(x,t)/\partial\nu\, d\sigma\, dt,$$

for any ζ, ζ_t, $\Delta\zeta \in L^2(\Omega \times (t_0, t_1))$, and $\partial\zeta/\partial\nu \in L^2(\partial\Omega \times (t_0, t_1))$.

To show the existence of periodic weak solutions to (1)-(3), we begin to show the existence of a periodic solution for the approximated problem

$$(4) \qquad u_{\varepsilon t} = \operatorname{div}(\nabla\varphi_\varepsilon(u_\varepsilon)) + c_\varepsilon(x,t,u_\varepsilon), \quad \text{in } Q$$

$$(5) \qquad -\partial\varphi_\varepsilon(u_\varepsilon)/\partial\nu = g_\varepsilon(\varphi_\varepsilon(u_\varepsilon)), \quad \text{on } \partial\Omega \times (0,\infty)$$

$$(6) \qquad u_\varepsilon(x,t+\omega) = u_\varepsilon(x,t) \text{ and } u_\varepsilon \geq 0, \quad \text{in } Q$$

where φ_ε, c_ε and g_ε are smooth approximations of φ, c and g, constructed by convolutions with mollifiers functions.

Construct $\varphi_\varepsilon(s)$, $c_\varepsilon(x,t,s)$ and $g_\varepsilon(s)$ such that

$$(H_\varepsilon)\quad \begin{cases} \varphi_\varepsilon \in C^{2+\alpha}_{\text{loc}}([0,\infty)),\ \varphi_\varepsilon(0)=0,\ \varphi_\varepsilon(s)=\varphi(s),\ \text{for } s\ge \varepsilon/2 \\ \varphi'_\varepsilon(s)\ge\varepsilon,\ \text{for all } s\ge 0, \\ \varphi_\varepsilon\to\varphi \text{ uniformly on compact subsets of } \mathbb{R}_+. \end{cases}$$

$$(H_{c_\varepsilon})\quad \begin{cases} c_\varepsilon\in C^1(\overline{Q}_T\times\mathbb{R}_+),\ c_\varepsilon(x,t,0)=0,\ c_\varepsilon(x,t,s)=c(x,t,s), \\ \text{for all } (x,t)\in\overline{Q}_T \text{ and } s\ge\varepsilon/2, \\ c_\varepsilon\to c \text{ uniformly on compact subsets of } \overline{Q}_T\times\mathbb{R}_+. \end{cases}$$

(H_{g_ε}) α) $g_\varepsilon\in C^\infty(\mathbb{R}_+)$, $g_\varepsilon\to g$ uniformly on compact subsets of $\mathbb{R}_+$,
β) $g_\varepsilon(\varphi_\varepsilon(r))\ge d_0\varphi_\varepsilon(r)$ for all $r\ge r_0$,
γ) g_ε is uniformly Lipschitz continuous on $\mathbb{R}_+$.

We shall show for (4)-(6) the existence of periodic weak solutions as fixed points for the Poincaré map of a suitable initial-boundary value problem. Hence, we need construct a closed, convex and nonempty set where to find these fixed points for the following sequence of nondegenerate initial-boundary value problems.

$$u_{\varepsilon t}=\operatorname{div}(\nabla\varphi_\varepsilon(u_\varepsilon))+c_\varepsilon(x,t,u_\varepsilon),\quad \text{in } Q_T,\ T\ge\omega \tag{7}$$
$$-\partial\varphi_\varepsilon(u_\varepsilon)/\partial\nu=g_\varepsilon\varphi_\varepsilon(u_\varepsilon),\quad \text{on } \partial\Omega\times(0,T) \tag{8}$$
$$u_\varepsilon(x,0)=u_{0\varepsilon}(x),\quad \text{on } \Omega \tag{9}$$

where $(H_{0\varepsilon})$ $u_{0\varepsilon}\in C^2(\overline{\Omega})$, such that $0\le u_{0\varepsilon}(x)$ for all x in $\overline{\Omega}$ and satisfying the compatibility condition

$$-\partial\varphi_\varepsilon(u_{0\varepsilon})/\partial\nu=g_\varepsilon(\varphi_\varepsilon(u_{0\varepsilon}))\quad\text{on}\quad\partial\Omega.$$

Problem (7)-(9) is an approximation of the following problem

$$u_t=\operatorname{div}(\nabla\varphi(u))+c(x,t,u),\quad \text{in } Q_T,\ T\ge\omega \tag{10}$$
$$-\partial\varphi(u)/\partial\nu=g(\varphi(u)),\quad \text{on } \partial\Omega\times(0,T) \tag{11}$$
$$u(x,0)=u_0(x),\quad \text{on } \Omega \tag{12}$$

where

$$\begin{cases} u_0\in L^\infty(\Omega),\ 0\le u_0(x),\ \text{a.e.} & \text{in } \Omega,\ \|u_{0\varepsilon}\|_\infty\le\|u_0\|_\infty \quad \text{and} \\ u_{0\varepsilon}\to u_0 & \text{in } L^2(\Omega) \text{ as } \varepsilon\to 0^+. \end{cases} \tag{13}$$

It is well known that there exists a unique classical solution u_ε of (7)-(9) (see [**9**]). Moreover, by a result of [**2**], the following estimates hold

$$\|u_\varepsilon\|_\infty \le C_1(\|u_0\|_\infty) \tag{14}$$

$$\int_0^T \int_\Omega |\nabla \varphi_\varepsilon(u_\varepsilon)|^2 \, dx \, dt \le C_2(\|u_0\|_\infty), \quad \text{for all } \varepsilon > 0. \tag{15}$$

In [**2**] is showed that the set of stationary solutions of (10)-(12) is bounded in $L^\infty(\Omega)$ i.e. the solutions $\overline{u}(x)$ of

$$\text{(SP)} \qquad \begin{cases} \operatorname{div}(\nabla\varphi(\overline{u})) + C_0\varphi(\overline{u})^\beta = 0 & \text{in } \Omega \\ -\partial\varphi(\overline{u})/\partial\nu = g(\varphi(\overline{u})) & \text{on } \partial\Omega \end{cases}$$

satisfy $\|\overline{u}\|_\infty = M$. It is easy to verify that ε is a subsolution of (SP), hence $\overline{u}(x) \ge \varepsilon$ a.e. in Ω.

If $u_{0\varepsilon}$ is chosen in such way that besides satisfies

$$\varepsilon \le u_{0\varepsilon}(x) \le \overline{u}(x) \le M, \quad \text{a.e. in} \quad \Omega,$$

then $\overline{u}(x)$ is a supersolution and ε is a subsolution to (7)-(9). Thus we have

$$\varepsilon \le u_\varepsilon(x,t) \le \overline{u}(x) \le M, \quad \text{for a.e.} \quad (x,t) \in Q_T. \tag{16}$$

Applying the continuity result of [**5**], the following regularity property for the solutions of (7)-(9) holds.

Proposition 1 ([**5**]). *Since $u_{0\varepsilon}$ is continuous on $\overline{\Omega}$, the sequence $\{u_\varepsilon\}$ of the solutions to (7)-(9) is equicontinuous in $\overline{Q}_T$ i.e. there exists $\omega_0 : \mathbb{R}_+ \to \mathbb{R}_+$, $\omega_0(0) = 0$, continuous and nondecreasing such that*

$$|u_\varepsilon(x_1,t_1) - u_\varepsilon(x_2,t_2)| \le \omega_0(|x_1 - x_2| + |t_1 - t_2|^{1/2})$$

for any (x_1,t_1), $(x_2,t_2) \in \overline{\Omega} \times [0,T]$. The function $s \to \omega_0(s)$, depends on the essential bound of u_ε in $\overline{Q}_T$.

If consider the Poincaré map associated to problem (7)-(9) and defined by

$$F(u_{0\varepsilon}(\cdot)) = u_\varepsilon(\cdot,\omega)$$

where u_ε is the unique solution of (7)-(9) and introduce the closed, nonempty, bounded and convex set

$$K_\varepsilon := \{w \in C(\overline{\Omega}) : \varepsilon \le w(x) \le M, \quad \text{for any} \quad x \in \overline{\Omega}\}$$

then, by (16) and Proposition 1, we get

1. i) $F(K_\varepsilon) \subset K_\varepsilon$.
2. ii) $F(K_\varepsilon)$ is relatively compact in $C(\overline{\Omega})$.

Remains to prove that

1. iii) $F \mid K_\varepsilon$ is continuous.

To this purpose, we show

Proposition 2. *If $u_{0\varepsilon}^n$, $u_{0\varepsilon} \in K_\varepsilon$ and $u_{0\varepsilon}^n \to u_{0\varepsilon}$ uniformly in $\overline{\Omega}$ as $n \to \infty$, then if u_ε^n and u_ε are solutions to (7)-(9) of initial data $u_{0\varepsilon}^n$ and $u_{0\varepsilon}$ respectively, we have that $u_\varepsilon^n(\cdot, t)$ converges to $u_\varepsilon(\cdot, t)$ uniformly as $n \to \infty$, for any $t \in [0, T]$.*

Proof: Multiplying (7) by $\operatorname{sgn}(u_\varepsilon^n - u_\varepsilon)$ and integrating on Q_t one has

$$\int_\Omega |u_\varepsilon^n(x,t) - u_\varepsilon(x,t)|\, dx \le \int_\Omega |u_{0\varepsilon}^n(x) - u_{0\varepsilon}(x)|\, dx + L \int_0^t \int_\Omega |u_\varepsilon^n(x,t) - u_\varepsilon(x,t)|\, dx\, dt,$$

because of the local Lipschitz continuity of $c_\varepsilon(x, t, \cdot)$ with Lipschitz constant L. Applying Gronwall's lemma, it is easy to see that $u_\varepsilon^n(x,t)$ converges to $u_\varepsilon(x,t)$ strongly in $L^1(\Omega)$ as n goes to infinity. Consequently, for a subsequence, we have that $u_\varepsilon^n(x,t)$ converges to $u_\varepsilon(x,t)$ for a.e. $x \in \Omega$. Since $u_\varepsilon^n(x,t) \le M$, by the Lebesgue theorem, we conclude that $u_\varepsilon^n(x,t) \to u_\varepsilon(x,t)$ in $L^p(\Omega)$ for any $1 \le p \le \infty$. Since $u_\varepsilon^n(\cdot,t)$, $u_\varepsilon(\cdot,t) \in C(\overline{\Omega})$ the uniform convergence holds.

Thus, by the Schauder fixed point theorem it follows that there exists a fixed point for the Poincaré map, which is a periodic solution to (4)-(6). From (16) we get, for a subsequence if necessary, that

$$u_\varepsilon \rightharpoonup u, \quad \text{in} \quad L^2(\Omega). \tag{17}$$

Since the set of $\varphi_\varepsilon(u_\varepsilon)$ is relatively compact in $L^2(Q_T)$, we have that u_ε strongly converges to u in $L^2(Q_T)$. In fact set $v_\varepsilon = \varphi(u_\varepsilon)$, v_ε is bounded in $L^2(0,T;H^1(\Omega)) \subset L^2(0,T;W^{s,2}(\Omega))$, $0 < s < 1$. If we suppose that

$$\varphi^{-1} \quad \text{is Hölder continuous of order} \quad \theta \in (0,1) \tag{18}$$

for a classical result (see [**3**]) one has

$$\|u_\varepsilon(t)\|^{1/\theta}_{w^{\theta s,2/\theta}(\Omega)} \le \|v_\varepsilon(t)\|_{w^{s,2}(\Omega)} \|\varphi^{-1}\|^{1/\theta}_{\text{Hölder}}$$

that integrate with respect to t, gives

$$\|u_\varepsilon\|^{2/\theta}_{L^{2/\theta}(0,T;w^{\theta s,2/\theta}(\Omega))} \le \|v_\varepsilon\|_{L^2(0,T;w^{s,2}(\Omega))} \|\varphi^{-1}\|^{2/\theta}_{\text{Hölder}}.$$

Moreover, (see [**3**])

$$W^{\theta s,2/\theta}(\Omega) \subset L^2(\Omega)$$

with compact injection. Then,

$$u_\varepsilon \to u, \quad \text{in} \quad L^2(Q_T) \quad \text{and a.e. .} \tag{19}$$

From (18) and the Lebesgue theorem one has

$$\varphi_\varepsilon(u_\varepsilon) \to \varphi(u), \quad \text{in} \quad L^2(\Omega_T) \tag{20}$$

thus, we get by (15) and (20) that $\nabla\varphi_\varepsilon(u_\varepsilon) \rightharpoonup \nabla\varphi(u)$, in $L^2(Q_T)$ and

$$\varphi_\varepsilon(u_\varepsilon) \rightharpoonup \varphi(u), \quad \text{in} \quad H^1(\Omega). \tag{21}$$

Theorem 3.4.5 of [**11**] states that if (21) holds, then $\varphi_\varepsilon(u_\varepsilon)$ converges to $\varphi(u)$ in $L^2(0,T;L^1(\partial\Omega))$. Now, by the uniform convergence of g_ε on compact set of $\mathbb{R}_+$ and its uniformly Lipschitz continuity, it is easy to see that

$$g_\varepsilon(\varphi_\varepsilon(u_\varepsilon)) \to g(\varphi(u)) \text{ in } L^2(0,T;L^2(\partial\Omega)).$$

In fact

$$\begin{aligned}\int_0^T \int_{\partial\Omega} &|g_\varepsilon(\varphi_\varepsilon(u_\varepsilon)) - g(\varphi(u))|^2 \, d\sigma \, dt \\ &\le L\Big(\int_0^T \int_{\partial\Omega} |\varphi_\varepsilon(u_\varepsilon) - \varphi(u)|^2 \, d\sigma \, dt \\ &\quad + \int_0^T \int_{\partial\Omega} |g_\varepsilon(\varphi(u)) - g(\varphi(u))|^2 \, d\sigma \, dt\Big) \to 0 \quad \text{as} \quad \varepsilon \to 0.\end{aligned}$$

Finally, in [**2**] is proven that $u \in C([0,T];\, L^1(\Omega))$, $\varphi(u) \in L^2(0,T;H^1(\Omega))$ thus, u is a periodic weak solution to (1)-(3). □

3. Uniqueness

To get the uniqueness result, suppose that u_ε and v are periodic solutions of (4)-(6) with boundary data g_ε, respectively, g such that

$$\varepsilon \leq \max\{u_\varepsilon(x,t), v(x,t)\} \leq M$$

then

$$\begin{aligned}
\int_0^T \int_\Omega [(u_\varepsilon - v)\zeta_t + &(\varphi(u_\varepsilon) - \varphi(v))\Delta\zeta + (c(x,t,u_\varepsilon) - c(x,t,u))\zeta]\, dx\, dt \\
&= \int_\Omega (u_\varepsilon(x,T) - v(x,T))\zeta(x,T)\, dx \\
&\quad - \int_\Omega (u_\varepsilon(x,0) - v(x,0))\zeta(x,0)\, dx \\
&\quad + \int_0^T \int_{\partial\Omega} (g_\varepsilon(\varphi_\varepsilon(u_\varepsilon)) - g(\varphi(v)))\zeta(x,t)\, d\sigma\, dt \\
&\quad + \int_0^T \int_{\partial\Omega} (\varphi(u_\varepsilon) - \varphi(v))\partial\zeta(x,t)/\partial\nu\, d\sigma\, dt
\end{aligned} \tag{22}$$

for any $\zeta \in L^2(Q_T)$ such that ζ_t, $\Delta\zeta \in L^2(Q_T)$ and $\partial\zeta/\partial\nu \in L^2(\partial\Omega \times (0,T))$.

Proceeding as in [**4**], define

$$\Phi_\varepsilon(x,t) := \int_0^1 \varphi_u(\theta u_\varepsilon(x,t) + (1-\theta)v(x,t))\, d\theta$$

and

$$C_\varepsilon(x,t) := \int_0^1 c_u(x,t,\theta u_\varepsilon(x,t) + (1-\theta)v(x,t))\, d\theta$$

then

$$(u_\varepsilon - v)\Phi_\varepsilon(x,t) = \varphi(u_\varepsilon) - \varphi(v)$$

and

$$(u_\varepsilon - v)C_\varepsilon(x,t) = c(x,t,u_\varepsilon) - c(x,t,v).$$

Hence,

$$\begin{aligned}\int_0^T \int_\Omega (u_\varepsilon - v)(\zeta_t + \Phi_\varepsilon \Delta\zeta + C_\varepsilon \zeta)\, dx\, dt \\ = \int_\Omega (u_\varepsilon(x,T) - v(x,T))\zeta(x,T)\, dx \\ - \int_\Omega (u_\varepsilon(x,0) - v(x,0))\zeta(x,0)\, dx \\ + \int_0^T \int_{\partial\Omega} (g_\varepsilon(\varphi_\varepsilon(u_\varepsilon)) - g(\varphi(v)))\zeta(x,t)\, d\sigma\, dt \\ + \int_0^T \int_{\partial\Omega} (\varphi(u_\varepsilon) - \varphi(v))\partial\zeta(x,t)/\partial\nu\, d\sigma\, dt.\end{aligned} \tag{23}$$

From [**7**], there exist some positive constants α_1, α_2 depending only on ε and M, such that

$$\alpha_1(\varepsilon) \le \Phi_\varepsilon(x,t) \le \alpha_2(M), \quad \text{for any } (x,t) \in \overline{Q}_T$$
$$|C_\varepsilon(x,t)| \le L_1(M), \quad \text{for any } (x,t) \in \overline{Q}_T.$$

Let $\zeta_{\varepsilon,m}$ denotes the solution of the backward linear parabolic problem with smooth coefficients

$$\zeta_{\varepsilon,mt} + \Phi_{\varepsilon,m}\Delta\zeta_{\varepsilon,m} + C_{\varepsilon,m}\zeta_{\varepsilon,m} = f, \quad \text{in } Q_T \tag{24}$$
$$\zeta_{\varepsilon,m}(x,T) = \theta(x), \quad \text{in } \overline{\Omega} \tag{25}$$
$$\partial\zeta_{\varepsilon,m}(x,t)/\partial\nu = -d_0\zeta_{\varepsilon,m}, \quad \text{on } \partial\Omega \times (0,T) \tag{26}$$

with $\Phi_{\varepsilon,m}$, $C_{\varepsilon,m}$, $f \in C^\infty(\overline{Q}_T)$, $\Phi_{\varepsilon,m} \to \Phi_\varepsilon$, $C_{\varepsilon,m} \to C_\varepsilon$ uniformly in $\overline{Q}_T$ as m goes to infinity and $\theta \in C_0^\infty(\overline{\Omega})$, $0 \le \theta(x) \le 1$. Also for $\Phi_{\varepsilon,m}$ and $C_{\varepsilon,m}$ holds

$$\alpha_1(\varepsilon) \le \Phi_{\varepsilon,m}(x,t) \le \alpha_2(M), \quad \text{for any } (x,t) \in \overline{Q}_T$$
$$|C_{\varepsilon,m}(x,t)| \le L_1(M), \quad \text{for any } (x,t) \in \overline{Q}_T.$$

The existence, uniqueness and regularity of $\zeta_{\varepsilon,m}(x,t)$ is proven in [**8**]. The following esimates will be need

Lemma 3 ([**7**])**.** *Let $\zeta(x,t) := \zeta_{\varepsilon,m}(x,t)$ be the solution to (24)-(26). Then,*

$$\int_0^T \int_\Omega |\nabla\zeta(x,t)|^2\, dx\, dt \le k_1; \quad \int_0^T \int_\Omega |\Delta\zeta(x,t)|^2\, dx\, dt \le k_1. \tag{27}$$

If $f \leq 0$ *we have*

$$0 \leq \zeta(x,t) \leq k_2, \quad \text{for any} \quad (x,t) \in \overline{Q}_T, \tag{28}$$

where $k_1 := k_1(\varepsilon, L_1, \|f\|_2)$, $k_2 := k_2(\|f\|_\infty)$.

The main result of this section is the following

Theorem 4. *For any* $f \in C^\infty(\overline{Q}_T)$ *and any* $\theta \in C_0^\infty(\overline{\Omega})$, $0 \leq \theta(x) \leq 1$ *we get*

$$\begin{aligned} \int_0^T \int_\Omega &(u_\varepsilon(x,T) - v(x,T))\theta(x)\,dx \\ &- \int_0^T \int_\Omega (u_\varepsilon(x,t) - v(x,t)) f(x,t)\,dx\,dt \\ &\leq k_2 \int_\Omega |u_\varepsilon(x,0) - v(x,0)|\,dx \\ &+ \int_0^T \int_{\partial\Omega} (g_\varepsilon(\varphi_\varepsilon(u_\varepsilon)) - g(\varphi(v)))\zeta(x,t)\,d\sigma\,dx \\ &- d_0 \int_0^T \int_{\partial\Omega} (\varphi(u_\varepsilon) - \varphi(v))\zeta(x,t)\,d\sigma\,dt. \end{aligned} \tag{29}$$

Proof: Substituting $\zeta(x,t)$ in (23) this yields

$$\begin{aligned} \int_0^T \int_\Omega &(u_\varepsilon(x,t) - v(x,t))(\Phi_\varepsilon(x,t) - \Phi_{\varepsilon,m}(x,t))\Delta\zeta\,dx\,dt \\ &+ \int_0^T \int_\Omega (u_\varepsilon(x,t) - v(x,t))(C_\varepsilon(x,t) - C_{\varepsilon,m}(x,t))\zeta(x,t)\,dx\,dt \\ &+ \int_0^T \int_\Omega (u_\varepsilon(x,t) - v(x,t)) f(x,t)\,dx\,dt \\ = &\int_\Omega (u_\varepsilon(x,T) - v(x,T))\theta(x)\,dx \\ &- \int_\Omega (u_\varepsilon(x,0) - v(x,0))\zeta(x,0)\,dx \\ &+ \int_0^T \int_{\partial\Omega} (g_\varepsilon(\varphi_\varepsilon(u_\varepsilon)) - g(\varphi(v)))\zeta(x,t)\,d\sigma\,dt \\ &- d_0 \int_0^T \int_{\partial\Omega} (\varphi(u_\varepsilon) - \varphi(v))\zeta(x,t)\,d\sigma\,dt. \end{aligned} \tag{30}$$

By Lemma 3, one concludes

$$\begin{aligned}
&\int_\Omega (u_\varepsilon(x,T) - v(x,T))\theta(x)\,dx \\
&\qquad - \int_0^T \int_\Omega (u_\varepsilon(x,t) - v(x,t)) f(x,t)\,dx\,dt \\
&\quad \le k_2 \int_\Omega |u_\varepsilon(x,0) - v(x,0)|\,dx \\
&\qquad + \int_0^T \int_{\partial\Omega} (g_\varepsilon(\varphi_\varepsilon(u_\varepsilon)) - g(\varphi(v)))\zeta(x,t)\,d\sigma\,dt \qquad (31)\\
&\qquad + \max_{\overline{Q}_T} |u_\varepsilon(x,t) - v(x,t)| [\max_{\overline{Q}_T} |\Phi_\varepsilon(x,t) - \Phi_{\varepsilon,m}(x,t)| (T|\Omega|k_1)^{1/2} \\
&\qquad + \max_{\overline{Q}_T} |C_\varepsilon(x,t) - C_{\varepsilon,m}(x,t)| k_2 T|\Omega|] \\
&\qquad - d_0 \int_0^T \int_{\partial\Omega} (\varphi(u_\varepsilon) - \varphi(v))\zeta(x,t)\,d\sigma\,dt.
\end{aligned}$$

Going to the limit as $m \to \infty$ in (31), we obtain the desired result. □

Corollary 5. *Let v and u be any periodic weak solutions of (1)-(3). Then, we have*

$$\begin{aligned}
&\int_0^T \int_\Omega |u(x,t) - v(x,t)|^2\,dx\,dt \\
&\qquad \le k_2 \int_\Omega |u(x,0) - v(x,0)|\,dx \\
&\qquad\quad + \int_0^T \int_{\partial\Omega} (g(\varphi(u)) - g(\varphi(v)))\zeta(x,t)\,d\sigma\,dt \qquad (32)\\
&\qquad\quad - d_0 \int_0^T \int_{\partial\Omega} (\varphi(u) - \varphi(v))\zeta(x,t)\,d\sigma\,dt.
\end{aligned}$$

Proof: If in (29) we choose $\theta(x) = 0$ and $f = f_k \in C^\infty(\overline{Q}_T)$, $f_k \to -(u_\varepsilon - v)$ in $L^2(Q_T)$ when k goes to infinity, one has

$$
\begin{aligned}
&\int_0^T \int_\Omega |u_\varepsilon(x,t) - v(x,t)|^2 \, dx \, dt \\
&\qquad \le k_2 \int_\Omega |u_\varepsilon(x,0) - v(x,0)| \, dx \\
&\qquad + \int_0^T \int_{\partial\Omega} (g_\varepsilon(\varphi_\varepsilon(u_\varepsilon)) - g(\varphi(v)))\zeta(x,t) \, d\sigma \, dx \\
&\qquad - d_0 \int_0^T \int_{\partial\Omega} (\varphi(u_\varepsilon) - \varphi(v))\zeta(x,t) \, d\sigma \, dt.
\end{aligned} \tag{33}
$$

Letting $\varepsilon \to 0^+$, since $g_\varepsilon(\varphi_\varepsilon(u_\varepsilon))$ converges to $g(\varphi(u))$ in $L^2(0,T;L^2(\partial\Omega))$ one obtains (32). □

Corollary 6. *Problem (1)-(3) with $g(\varphi(u)) = d_0\varphi(u)$, has a unique periodic weak solution.*

Proof: Choosing $T = n\omega$ in (32) because of the periodicity of u and v, we get

$$
n \int_0^\omega \int_\Omega |u(x,t) - v(x,t)|^2 \, dx \, dt \le k_2 \int_\Omega |u(x,0) - v(x,0)| \, dx \le k_3,
$$

for any $n \in \mathbb{N}$, hence $u = v$. □

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Rebut el 4 de maig de 1999.