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WEAKLY SUFFICIENT SETS FOR $A^{-\infty}(\mathbb{D})$

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Abstract

In the space $A^{-\infty}(\mathbb{D})$ of functions of polynomial growth, weakly sufficient sets are those such that the topology induced by restriction to the set coincides with the topology of the original space. Horowitz, Korenblum and Pinchuk defined sampling sets for $A^{-\infty}(\mathbb{D})$ as those such that the restriction of a function to the set determines the type of growth of the function. We show that sampling sets are always weakly sufficient, that weakly sufficient sets are always of uniqueness, and provide examples of discrete sets that show that the converse implications do not hold.

0. Introduction and statement of results.

Let $\mathbb{D}$ be the unit disk in the complex plane. The space $A^{-\infty}(\mathbb{D})$ of functions of polynomial growth can be introduced as the smallest algebra closed under differentiation and containing the bounded holomorphic functions, or as the dual of $A^{\infty}(\mathbb{D})$, the space of holomorphic functions smooth up to the boundary. More precisely, for any $p > 0$, define

$$A^{-p}(\mathbb{D}) := \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|_{A^{-p}} := \sup_{|z|<1} (1-|z|)^p |f(z)| < \infty \right\}.$$

The space $A^{-p}(\mathbb{D})$ is a Banach space, and we define $A^{-\infty}(\mathbb{D}) := \cup_{p>0} A^{-p}(\mathbb{D})$. Basic references about $A^{-\infty}(\mathbb{D})$ are [**H1**], [**K1**], [**K2**].

Definition. A subset $S \subset \mathbb{D}$ is a *set of uniqueness* for a function space X iff $f \in X$ and $f(z) = 0$ for all $z \in S$ imply that $f = 0$.

Sets of uniqueness for $A^{-\infty}(\mathbb{D})$ (resp. $A^{-p}(\mathbb{D})$) are precisely those sets which are not zero-sets for $A^{-\infty}(\mathbb{D})$ (resp. $A^{-p}(\mathbb{D})$) and have been characterized in [**K1**] (resp. studied extensively in [**H1**], [**Se1**], [**Se2**]).

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The space $A^{-\infty}(\mathbb{D})$ is endowed with the inductive limit topology induced by the spaces $A^{-p}(\mathbb{D})$. A sequence $\{f_n\} \subset A^{-\infty}(\mathbb{D})$ converges to 0 if and only if there exists $p > 0$ such that $f_n \in A^{-p}(\mathbb{D})$ for all n, and $\lim_{n\to\infty} \|f_n\|_{A^{-p}} = 0$.

Let $S \subset \mathbb{D}$. We define

$$\|f\|_{A^{-p}(S)} := \sup_{z\in S}(1-|z|)^p|f(z)|;$$

obviously $\|f\|_{A^{-p}(S)} \le \|f\|_{A^{-p}}$. Let

$$A^{-p}(S) := \left\{f \in A^{-\infty}(\mathbb{D}) : \|f\|_{A^{-p}(S)} < \infty\right\} \supset A^{-p}(\mathbb{D}).$$

Definition. A set $S \subset \mathbb{D}$ is a *weakly sufficient set* for $A^{-\infty}(\mathbb{D})$ iff the inductive limit topology induced by the spaces $A^{-p}(S)$ is the same as the topology of $A^{-\infty}(\mathbb{D})$.

More explicitly, a sequence $\{f_n\} \subset A^{-\infty}(\mathbb{D})$ is said to converge to 0 in $A^{-\infty}(S)$ iff there exists $p > 0$ such that $\{f_n\} \subset A^{-p}(S)$ and $\lim_{n\to\infty} \|f_n\|_{A^{-p}(S)} = 0$. The set S is weakly sufficient for $A^{-\infty}(\mathbb{D})$ if and only if any sequence which converges to 0 in $A^{-\infty}(S)$ converges to zero in $A^{-\infty}(\mathbb{D})$.

This definition (in a more general context) originates with [**Sc**]. The notion of *sufficient sets* is actually more complicated to define, but coincides in the present situation with that of weakly sufficient sets [**Ab**], [**Na**].

Applying the above definition to the constant sequence $f_n = f$, where we pick an f vanishing on the set S, we see that any weakly sufficient set for $A^{-\infty}(\mathbb{D})$ must be a set of uniqueness for $A^{-\infty}(\mathbb{D})$. Of course the converse implication does not hold, since for instance any non-discrete set relatively compact in the disk will be of uniqueness without being weakly sufficient. In fact, since the intersection of any zero-set for $A^{-\infty}(\mathbb{D})$ with a single radius verifies the Blaschke condition [**K1**], any non-Blaschke sequence along a single radius is a discrete set of uniqueness for $A^{-\infty}(\mathbb{D})$ which is not weakly sufficient. More specifically, taking for instance $S := \{1 - 1/n, n \in \mathbb{Z}_+^*\}$, and

$$f_n(z) := B_n(z)\frac{1}{(1+z)^n},$$

where B_n is a Blaschke product with simple zeroes at $\{1 - 1/k, 1 \le k \le n\}$, then $f_n \to 0$ in $A^{-\infty}(S)$, but not in $A^{-\infty}(\mathbb{D})$.

In an analogous way, we could say that S is weakly sufficient for $A^{-p}(\mathbb{D})$ if for any sequence $\{f_n\} \subset A^{-p}(\mathbb{D})$ such that $\|f_n\|_{A^{-p}(S)} \to 0$, then $\|f_n\|_{A^{-p}} \to 0$. Again, S must be a set of uniqueness for $A^{-p}(\mathbb{D})$, so $\|\cdot\|_{A^{-p}(S)}$ is a norm on $A^{-p}(\mathbb{D})$. Weak sufficiency here means that the identity map is continuous from $(A^{-p}(\mathbb{D}), \|\cdot\|_{A^{-p}(S)})$ to $A^{-p}(\mathbb{D})$, thus there exists $L > 0$ such that $\|f\|_{A^{-p}} \leq L\|f\|_{A^{-p}(S)}$ for all $f \in A^{-p}(\mathbb{D})$. This is exactly the definition of S being a *sampling* set for $A^{-p}(\mathbb{D})$; these sets have been characterized in [**Se1**].

Definition. A set $S \subset \mathbb{D}$ is a *sampling* set for $A^{-\infty}(\mathbb{D})$ iff for any $p < p'$, $A^{-p}(S) \subset A^{-p'}(\mathbb{D})$.

This notion is due to Horowitz, Korenblum and Pinchuk [**HKP**], who introduced it in the following way: for any set $S \subset \mathbb{D}$, $f \in A^{-\infty}(\mathbb{D})$, define the *type* of f as

$$T_S(f) := \inf\left\{p > 0 : f \in A^{-p}(S)\right\} = \overline{\lim}_{z\in S,|z|\to 1} \frac{\log_+ |f(z)|}{|\log(1-|z|)|}.$$

Then S is sampling for $A^{-\infty}(\mathbb{D})$ if and only if $T_S(f) = T_{\mathbb{D}}(f)$ for any $f \in A^{-\infty}(\mathbb{D})$. This corresponds to what Abanin calls "effective sets" in an analogous context [**Ab**].

It is proved in [**HKP**] that if S is sampling for $A^{-p}(\mathbb{D})$ for all $p > 0$, then S is sampling for $A^{-\infty}(\mathbb{D})$, but that there exist sampling sets for $A^{-\infty}(\mathbb{D})$ which are not sampling for *any* $A^{-p}(\mathbb{D})$, $p > 0$.

An overall picture of the situation is given by the following.

Theorem. *For a set $S \subset \mathbb{D}$, consider the following assertions:*

(i) *S is sampling for $A^{-\infty}(\mathbb{D})$;*

(ii) *S is weakly sufficient for $A^{-\infty}(\mathbb{D})$;*

(iii) *S is of uniqueness for $A^{-\infty}(\mathbb{D})$.*

Then each assertion implies the next, and both converse implications fail.

Furthermore, the counterexamples can be taken to be (discrete) symmetric sequences in the sense of [**HKP**], *and there exist such sets which are weakly sufficient for $A^{-\infty}(\mathbb{D})$, and are not sampling for $A^{-\infty}(\mathbb{D})$ nor for any $A^{-p}(\mathbb{D})$, $p > 0$.*

The plan of this paper is as follows: that (ii) implies (iii) was remarked above; in Section 1 we give some equivalent characterizations of weakly sufficient sets, and prove that (i) implies (iii), and subsequently (ii); in Section 2 we study the counterexamples.

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1. Characterizations of weak sufficiency.

The first implication of the Theorem is an immediate consequence of the following characterization.

Proposition 1. *The following properties are equivalent:*

(i) *S is a weakly sufficient set for $A^{-\infty}(\mathbb{D})$;*

(ii) *For any $p > 0$, there exists $q > 0$, $L > 0$ such that*

$$\forall\, f \in A^{-\infty}(\mathbb{D}),\ \|f\|_{A^{-q}} \leq L\|f\|_{A^{-p}(S)};$$

(iii) *For any $p > 0$, $A^{-p}(S)$ is a Banach space;*

(iv) *For any $p > 0$, there exists $q > 0$ such that $A^{-p}(S) \subset A^{-q}(\mathbb{D})$.*

Remark. The equivalence of (i) and (ii) is relatively easy, and can be found in several sources, e.g. [**Sc**], [**Ab**]; in the latter, Abanin proved that (i) is equivalent to (iv) together with the additional property that S be of uniqueness, in a more general context. Here, (iv) itself implies that S is of uniqueness, and one could then apply Abanin's result. However, we would like to present our direct proof that (iv) implies (ii) for the reader's convenience, and because it might shed some light on the behavior of functions in $A^{-\infty}(\mathbb{D})$. The equivalence of (i) with (iii) can be found in [**Sc**] in the framework of entire functions, but the relevant proofs are actually quite general, and apply here.

Lemma 2. *Suppose that there exist $q \geq p > 0$ such that $A^{-p}(S) \subset A^{-q}(\mathbb{D})$. Then S is a set of uniqueness for $A^{-\infty}(\mathbb{D})$.*

Proof: We thank Pr. Charles Horowitz for communicating us the following elegant argument.

Suppose that $f \in A^{-\infty}(\mathbb{D}) \setminus \{0\}$ and that $f|_S \equiv 0$. Take a function $g \in A^{-2q}(\mathbb{D}) \setminus \{0\}$, such that $g^{-1}\{0\}$ is a set of uniqueness for $A^{-q}(\mathbb{D})$; such a function exists by [**H**, Theorem 1].

Then $fg \in A^{-\infty}(\mathbb{D})$, $\|fg\|_{A^{-p}(S)} = 0$, so $fg \in A^{-p}(S)$. On the other hand, since fg vanishes on a set of uniqueness for $A^{-q}(\mathbb{D})$, and $fg \neq 0$, then $fg \notin A^{-q}(\mathbb{D})$, contradicting the assumption. ∎

The proof that (iv) implies (ii) now reduces to the following proposition.

Proposition 3. *Suppose $S \subset \mathbb{D}$ is a set of uniqueness for $A^{-q}(\mathbb{D})$. Suppose that $A^{-p}(S) \subset A^{-q}(\mathbb{D})$. Then there exists $M > 0$ such that for all $f \in A^{-p}(S)$, $\|f\|_{A^{-q}} \leq M\|f\|_{A^{-p}(S)}$.*

Proof: We first need a lemma. For any $g \in A^{-q}(\mathbb{D})$, define

$$r_q(g) := r(g) := \inf\{|\zeta| : \|g\|_{A^{-q}} = (1-|\zeta|)^q|g(\zeta)|\},$$

if the above set is not empty, and $r_q(g) = 1$ if it is.

Lemma 4. *Suppose S is a set of uniqueness for $A^{-q}(\mathbb{D})$ and $\{g_n\} \subset A^{-p}(S)$ a sequence such that*

$$\lim_{n\to\infty} \frac{\|g_n\|_{A^{-q}}}{\|g_n\|_{A^{-p}(S)}} = \infty;$$

then $\lim_{n\to\infty} r(g_n) = 1$.

Proof: Suppose instead that there exists a subsequence, denoted again by $\{g_n\}$, such that $\lim_{n\to\infty} r(g_n) = r_0 < 1$. Let $h_n := \|g_n\|_{A^{-q}}^{-1} g_n$.

$$1 = \|h_n\|_{A^{-q}} = (1-|\zeta_n|)^q|h_n(\zeta_n)|,$$

and passing to a subsequence, we may assume that $\zeta_n \to \zeta_0 \in \overline{D}(0, r_0)$, and furthermore that $h_n(\zeta_n) \to \eta$ with $|\eta| = (1-|\zeta_0|)^{-q} \neq 0$.

Since $\{h_n\}$ is a normal family, there is a subsequence, denoted again by $\{h_n\}$, which converges uniformly on compact subsets of $\mathbb{D}$ to $h \in A^{-q}(\mathbb{D})$, $\|h\|_{A^{-q}} \leq 1$, and $h(\zeta_0) = \eta$, so that $h \neq 0$.

On the other hand, for any $\xi \in S$,

$$|h_n(\xi)| \leq (1-|\xi|)^{-p}\frac{\|g_n\|_{A^{-p}(S)}}{\|g_n\|_{A^{-q}}} \to 0 \text{ as } n \to \infty,$$

so $h|_S \equiv 0$, contradicting the assumption that S is a set of uniqueness for $A^{-q}(\mathbb{D})$.

End of Proof of Proposition 3: Suppose there is no $M > 0$ as in the conclusion. Then we can find $\{g_{m,\varepsilon}, m \in \mathbb{Z}_+, \varepsilon > 0\} \subset A^{-p}(S)$ such that

(1) $$\|g_{m,\varepsilon}\|_{A^{-q}} = 3^m,$$

(2) $$\|g_{m,\varepsilon}\|_{A^{-p}(S)} \leq m^{-2},$$

and, because of Lemma 6,

(3) $$r_q(g_{m,\varepsilon}) \geq 1-\varepsilon.$$

Set $p_1 = 0$, $\varepsilon_1 = 1$, $h_1(z) = g_{1,1}(z)$, $r_1 = r_q(g_{1,1})$, and define for any $m \geq 1$,

$$h_m(z) := z^{p_m} g_{m,\varepsilon_m}(z),$$

where $p_m \in \mathbb{Z}_+$, $\varepsilon_m > 0$, and $r_m \in (0,1)$ will be defined inductively as follows.

Choose $p_m \geq 3^m$ large enough so that, for any $k \leq m-1$, $r_k^{p_m} \leq 3^{2(k-m)}/2$.

Now choose $\varepsilon_m < \min_{0 \leq k \leq m-1}(1-r_k)$, so that $(1-\varepsilon_m)^{p_m} > 9/10$, and g_{m,ε_m} according to the above requirements. Then we can choose $r_m \in [1-\varepsilon_m, 1)$ such that

$$\sup_{|z|=r_m} (1-|z|)^q |g_{m,\varepsilon_m}(z)| \geq \frac{9}{10} \|g_{m,\varepsilon_m}\|_{A^{-q}},$$

thus

$$\sup_{|z|=r_m} (1-|z|)^q |h_m(z)| \geq (1-\varepsilon_m)^{p_m} \cdot \frac{9}{10} \cdot 3^m \geq \left(\frac{9}{10}\right)^2 \cdot 3^m.$$

Now set $h(z) := \sum_m h_m(z)$. This is analogous to a gap series. The proof concludes with the following.

Claim. *The series $\sum_m h_m$ converges in $A^{-q-2}(\mathbb{D})$ (and thus in $A^{-\infty}(\mathbb{D})$) and in $A^{-p}(S)$, but $h \notin A^{-q}(\mathbb{D})$.*

Proof: First note that

$$\begin{aligned}\sup_{z\in\mathbb{D}}(1-|z|)^{q+2}|h_m(z)| &\leq \sup_{z\in\mathbb{D}}(1-|z|)^2|z|^{p_m}\cdot(1-|z|)^q|g_{m,\varepsilon_m}(z)| \\ &\leq \frac{p_m^{p_m}2^2}{(p_m+2)^{p_m+2}}3^m \leq 4\cdot 3^m p_m^{-2} \leq 4\cdot 3^{-m},\end{aligned}$$

so that $\sum h_m$ converges in $A^{-q-2}(\mathbb{D})$, and in particular uniformly on compact subsets of $\mathbb{D}$, and $h \in A^{-\infty}(\mathbb{D})$.

Furthermore, $\|h_m\|_{A^{-p}(S)} \leq \|g_{m,\varepsilon_m}\|_{A^{-p}(S)} \leq m^{-2}$, so in fact $|h(z)| \leq C(1-|z|)^{-p}$ for $z \in S$, and $h \in A^{-p}(S)$.

On the other hand,

$$\begin{aligned}\sup_{|z|=r_m}|h(z)| &\geq \sup_{|z|=r_m}|h_m(z)| - \sum_{k\neq m}\sup_{|z|=r_m}|h_k(z)| \\ &\geq (1-r_m)^{-q}\left(\left(\frac{9}{10}\right)^2\cdot 3^m - \sum_{k\neq m} r_m^{p_k}\|g_{k,\varepsilon_k}\|_{A^{-q}}\right) \\ &\geq (1-r_m)^{-q}\left(\left(\frac{9}{10}\right)^2\cdot 3^m - \sum_{k=1}^{m-1}3^k - \sum_{k=m+1}^{\infty}3^k\cdot 3^{2(m-k)}/2\right) \\ &\geq (1-r_m)^{-q}\frac{3^m}{20}.\end{aligned}$$

From this we deduce $(1-r_m)^q \sup_{|z|=r_m}|h(z)| \geq 3^m/20 \to \infty$ as $m\to\infty$, so $h \notin A^{-q}(\mathbb{D})$, which is the desired contradiction. ■

Open Questions.

1. *Is it possible to have* $A^{-p}(S) \subset A^{-q}(\mathbb{D})$ *for some values of* p *(and* q *depending on* p*), but not for others?*
2. *When* S *is weakly sufficient, one could define*

$$q(p) := \inf\{q : A^{-p}(S) \subset A^{-q}(\mathbb{D})\}.$$

By taking integer powers of holomorphic functions, it is easy to see that $q(p/m) \leq q(p)/m$*, for any* $p > 0$*,* $m \in \mathbb{Z}_+$*. Must we have a constant* $M \geq 1$ *such that* $q(p) = Mp$*? It is the case in all examples studied below.*

2. A family of examples.

As in [**HKP**], we begin by studying the case of unions of concentric circles. Let $\{r_n\}_{n\geq 1}$ be an increasing sequence contained in the interval $(0,1)$ and tending to 1. We set

$$E(\{r_n\}) := \bigcup_{n\geq 1}\{z : |z| = r_n\}.$$

Horowitz, Korenblum and Pinchuk proved that $E(\{r_n\})$ is sampling for $A^{-\infty}(\mathbb{D})$ if and only if

$$\overline{\lim}_{n\to\infty}\frac{|\log(1-r_{n+1})|}{|\log(1-r_n)|} = 1.$$

Proposition 5. *$S := E(\{r_n\})$ is weakly sufficient for $A^{-\infty}(\mathbb{D})$ if and only if*

$$M := \overline{\lim}_{n\to\infty} \frac{|\log(1-r_{n+1})|}{|\log(1-r_n)|} < \infty.$$

Furthermore, when $M < \infty$, $A^{-p}(S) \subset A^{-q}(\mathbb{D})$ for any $q > Mp$.

Taking $1 < M < \infty$, we have thus exhibited sets which are weakly sufficient without being sampling for $A^{-\infty}(\mathbb{D})$. Furthermore, they are not sampling for any $A^{-p}(\mathbb{D})$, $p > 0$. Indeed, S is sampling for $A^{-p}(\mathbb{D})$ if and only if it contains a separated sequence of lower density greater than p [**Se1**], where the lower density of a set $S \subset \mathbb{D}$ is defined by

$$D_-(S) := \liminf_{r\to 1} \inf_{z\in\mathbb{D}} \left(\frac{\sum_{\zeta\in S, \frac{1}{2}<|\varphi_z(\zeta)|<r} |\log|\varphi_z(\zeta)||}{|\log(1-r)|} \right),$$

where $\varphi_z(\zeta) := \frac{z-\zeta}{1-\bar{z}\zeta}$ is the usual Möbius automorphism of the disk.

It turns out that here $E(\{r_n\})$ itself is of lower density zero, because for well chosen $z \in \mathbb{D}$, the summation in the numerator runs over the empty set. Indeed, pick a subsequence $\{r_{n_k}\}$ increasing to 1 such that for some $\varepsilon > 0$, $(1 - r_{1+n_k}) \le (1 - r_{n_k})^{1+\varepsilon}$, and let

$$z_k := 1 - (1 - r_{n_k})^{1+\varepsilon/2} \in (r_{n_k}, r_{1+n_k}).$$

Thus for $\zeta \in E(\{r_n\})$, $|\zeta| < z_k$, we have

$$\left|\frac{z_k - \zeta}{1 - z_k\zeta}\right| \ge \left|\frac{z_k - r_{n_k}e^{i\theta}}{1 - z_k r_{n_k} e^{i\theta}}\right| \ge \frac{(1-r_{n_k}) - (1-z_k)}{(1-r_{n_k}) + (1-z_k)} = \frac{1-(1-r_{n_k})^{\varepsilon/2}}{1+(1-r_{n_k})^{\varepsilon/2}} > r$$

for $k \ge k(r)$, for any $r < 1$. Similarly, for $\zeta \in E(\{r_n\})$, $|\zeta| > z_k$,

$$\left|\frac{z_k - \zeta}{1 - z_k\zeta}\right| \ge \frac{(1-z_k) - (1-r_{1+n_k})}{(1-r_{1+n_k}) + (1-z_k)} = \frac{1 - \frac{(1-r_{1+n_k})}{(1-z_k)}}{1 + \frac{(1-r_{1+n_k})}{(1-z_k)}} \ge \frac{1-(1-r_{n_k})^{\varepsilon/2}}{1+(1-r_{n_k})^{\varepsilon/2}} > r$$

for $k \ge k(r)$. So the infimum in the definition of $D_-(E(\{r_n\}))$ is equal to 0 for any $r < 1$.

Of course the sets $E(\{r_n\})$ are not discrete; we shall show below (Proposition 8) how to pass to discrete sets.

Proof of Proposition 5: Let $M' > M$; there is n_0 such that for any $n \geq n_0$, $(1 - r_{n+1}) \geq (1 - r_n)^{M'}$. Let $q := M'p$. For any $f \in A^{-\infty}(\mathbb{D})$ and z such that $|z| \leq r_{n_0}$, $(1-|z|)^q |f(z)| \leq \sup_{|z| \leq r_{n_0}} |f(z)|$, so to prove that $f \in A^{-q}$ it is enough to show that $(1-|z|)^q |f(z)| \leq C$ for $|z| \geq r_{n_0}$. Let n be the unique integer such that $r_n < |z| \leq r_{n+1}$. Then

$$\begin{aligned}(1-|z|)^q |f(z)| &\leq (1-r_n)^q \sup_{|z|=r_{n+1}} |f(z)| \\ &\leq (1-r_{n+1})^p \sup_{|z|=r_{n+1}} |f(z)| \leq \|f\|_{A^{-p}(S)},\end{aligned}$$

with $S := E(\{r_n\})$.

The converse can be proved by calling upon a consequence of a theorem of Horowitz [**H2**], given in [**HKP**, Theorem 3.1]: for any increasing function k of $r \in [0,1]$ such that $\sup_{0 \leq r < 1} \left(k(r) - k(r^2)\right) < \infty$, there exists f analytic in $\mathbb{D}$ such that for $0 < r < 1$

$$\max_{|z|=r} \log |f(z)| = k(r) + O(1).$$

To simplify calculations, we use the auxiliary variable $u := \log \frac{1}{1-r}$ and let $\tilde{k}(u) := k(r)$. Then the hypothesis on k is satisfied if $\sup_{0<u} \left(\tilde{k}(u) - \tilde{k}(u - \log 2)\right) < \infty$, for instance if $\tilde{k}$ is Lipschitz.

Also, letting $m_f(u) := \max_{|z|=1-e^{-u}} \log |f(z)|$, $f \in A^{-q}$ if and only if $m_f(u) = qu + O(1)$; and, letting $u_n := \log \frac{1}{1-r_n}$ and $S := E(\{r_n\})$, $f \in A^{-p}(S)$ if and only if $m_f(u_n) = pu_n + O(1)$.

Now assume that $\overline{\lim}_{n\to\infty} \frac{|\log(1-r_{n+1})|}{|\log(1-r_n)|} = \infty$, i.e. $\overline{\lim}_{n\to\infty} \frac{u_{n+1}}{u_n} = \infty$. Choose a subsequence $\{u_{n_j}\}_{j\geq 0}$ such that $u_{1+n_j}/u_{n_j} \to \infty$ as $j \to \infty$. Given any $p > 0$, $q > 0$, we want a sequence f_j such that $\|f_j\|_{A^{-p}(S)}$ remains bounded and $\|f_j\|_{A^{-q}} \to \infty$. It will be enough to pick the f_j given by Horowitz's theorem, with

$$\tilde{k}_j(u) := pu + (q+1) \left(\frac{u_{1+n_j} - u_{n_j}}{2} - \left| u - \frac{u_{1+n_j} + u_{n_j}}{2} \right| \right)_+ .$$

This function is Lipschitz with constant $p + q + 1$, and equal to pu outside the interval (u_{n_j}, u_{1+n_j}), so the corresponding $f_j \in A^{-p}(S)$. Furthermore, it is easily checked that

$$\tilde{k}_j \left(\frac{u_{1+n_j} + u_{n_j}}{2} \right) - q \frac{u_{1+n_j} + u_{n_j}}{2} \to \infty \text{ as } j \to \infty,$$

which proves that $\|f_j\|_{A^{-q}} \to \infty$. ■

However, we can also bypass Horowitz's powerful theorem and exhibit a sequence of very simple functions —monomials $f_k(z) := c_k z^{m_k}$, which converges in $A^{-\infty}(S)$ but not in $A^{-\infty}(\mathbb{D})$. We record the elementary calculations that will be needed.

For $\alpha > 0$, $0 \le x \le 1$, set $g_\alpha(x) := (1-x)x^\alpha$ and $G_\alpha(u) := g_\alpha(1-e^{-u})$, $u \ge 0$.

Lemma 6.

(i) $$\sup_{0\le x\le 1} g_\alpha(x) = g_\alpha\left(\frac{\alpha}{\alpha+1}\right) = G_\alpha(\log(1+\alpha)) = \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}};$$

$\log G_\alpha(\log(1+\alpha)+h) - \log G_\alpha(\log(1+\alpha)) \le -h + 1 - e^{-h}$, *thus for any* $h > 0$,

(ii) $$G_\alpha(\log(1+\alpha)+h) \le G_\alpha(\log(1+\alpha))e^{-(h-1)_+},$$

(iii) $$G_\alpha(\log(1+\alpha)-h) \le G_\alpha(\log(1+\alpha))e^{-h^2/2}.$$

Suppose that $\overline{\lim}_{n\to\infty}\frac{|\log(1-r_{n+1})|}{|\log(1-r_n)|} = \infty$. Pick a subsequence $\{r_{n_k}\}$ increasing to 1 such that for all $k \ge 1$,

$$\log\frac{1}{1-r_{1+n_k}} \ge k\log\frac{1}{1-r_{n_k}}.$$

We will show that for any $\beta > 0$, there is a sequence $\{f_k\}$ converging to 0 in $A^{-\beta}(E(\{r_n\}))$, but not convergent in $A^{-\infty}(\mathbb{D})$.

For each k, define m_k to be the smallest integer such that

$$\log\left(1+\frac{m_k}{\beta}\right) \ge 2\log\frac{1}{1-r_{n_k}},$$

and let $p_k := \min\left(k^{1/2}, (\log m_k)^{1/2}\right)$. Then m_k, p_k, and $m_k p_k^{-1}$ all tend to infinity as k tends to infinity.

Set $f_k(z) := c_k z^{m_k}$, where

$$c_k := \left(\sup_{0\le x\le 1}(1-x)^{p_k}x^{m_k}\right)^{-1} = \left(G_{m_k p_k^{-1}}(\log(1+m_k p_k^{-1}))\right)^{-p_k}.$$

By construction, for a given p and any k such that $p_k \ge p$, $\|f_k\|_{A^{-p}} \ge \|f_k\|_{A^{-p_k}} = 1$, and it is easy to see that $\|f_k\|_{A^{-p}}$ tends in fact to infinity.

The proof concludes with the following.

Claim. $\lim_{k\to\infty} \|f_k\|_{A^{-\beta}(E(\{r_n\}))} = 0$.

Proof of the Claim: By Lemma 8(i), $(1-|z|)^\beta |f_k(z)|$ is maximal for $|z| = \frac{m_k}{\beta}(1+\frac{m_k}{\beta})^{-1} \in (r_{n_k}, r_{1+n_k})$ for $k \geq 3$, by the choice of m_k. So

$$\|f_k\|_{A^{-\beta}(E(\{r_n\}))} = \max\left((1-r_{n_k})^\beta f_k(r_{n_k}), (1-r_{1+n_k})^\beta f_k(r_{1+n_k})\right).$$

To estimate this quantity, let

$$h_1 := \log\left(1+\frac{m_k}{\beta}\right) - \log\frac{1}{1-r_{n_k}},$$
$$h_2 := \log\frac{1}{1-r_{1+n_k}} - \log\left(1+\frac{m_k}{\beta}\right).$$

Then

$$(1-r_{n_k})^\beta f_k(r_{n_k}) = \frac{G_{\frac{m_k}{\beta}}(\log(1+\frac{m_k}{\beta})-h_1)^\beta}{G_{m_k p_k^{-1}}(\log(1+m_k p_k^{-1}))^{p_k}},$$

and by Lemma 6(iii),

$$\log\left((1-r_{n_k})^\beta f_k(r_{n_k})\right) \leq -\beta\frac{h_1^2}{2} + \beta\log\left(G_{\frac{m_k}{\beta}}\left(\log\left(1+\frac{m_k}{\beta}\right)\right)\right)$$
$$- p_k\log\left(G_{m_k p_k^{-1}}(\log(1+m_k p_k^{-1}))\right).$$

Lemma 7. *Suppose that p_k, m_k, and $m_k p_k^{-1}$ tend to infinity as k tends to infinity, and that $p_k \leq \log m_k$. Then, for k large enough,*

$$\beta\log\left(G_{\frac{m_k}{\beta}}\left(\log\left(1+\frac{m_k}{\beta}\right)\right)\right)$$
$$- p_k\log\left(G_{m_k p_k^{-1}}(\log(1+m_k p_k^{-1}))\right) \leq p_k\log m_k.$$

The above Lemma is easily checked and implies that

$$\log\left((1-r_{n_k})^\beta f_k(r_{n_k})\right) \leq -\beta\frac{h_1^2}{2} + p_k\log m_k$$
$$\leq -\frac{\beta}{8}\left(\log\left(1+\frac{m_k}{\beta}\right)\right)^2 + (\log m_k)^{3/2} \to -\infty$$

as $k\to\infty$.

On the other hand,

$$(1-r_{1+n_k})^\beta f_k(r_{1+n_k}) = \frac{G_{\frac{m_k}{\beta}}(\log(1+\frac{m_k}{\beta})+h_2)^\beta}{G_{m_k p_k^{-1}}(\log(1+m_k p_k^{-1}))^{p_k}},$$

so applying Lemma 6(ii) and Lemma 7 in succession,

$$\log\left((1-r_{1+n_k})^\beta f_k(r_{1+n_k})\right) \le -\beta\left((k-3)\log\frac{1}{1-r_{n_k}}-1\right)_+ + p_k\log m_k$$
$$\le \beta-\beta(k-3)\log\frac{1}{1-r_{n_k}}+3\sqrt{k}\log\frac{1}{1-r_{n_k}} \to -\infty$$

as $k\to\infty$. This implies finally that $\|f_k\|_{A^{-\beta}(E(\{r_n\}))}\to 0$ as $k\to\infty$. ∎

We are now going to describe *discrete* weakly sufficient sets which are included in the $E(\{r_n\})$. Let $g:\mathbb{R}_+\to\mathbb{R}_+$ be an increasing function such that $g(0)=0$. Set

$$a_{n,\ell} := r_n\exp\left(2\pi i\ell g(1-r_n)\right), \quad \ell\in\mathbb{Z}_+, \quad 0\le \ell < \frac{1}{g(1-r_n)}.$$

This is a slight generalization of the symmetric sequences of [**HKP**].

Proposition 8. *Suppose that* $\{r_n\}$ *is an increasing sequence of radii tending to* 1 *and that*

$$M := \overline{\lim}_{n\to\infty}\frac{|\log(1-r_{n+1})|}{|\log(1-r_n)|}<\infty.$$

Suppose also that, for any $m>0$, $\lim_{x\to 0}x^{-m}g(x)=0$. *Then*

$$S := \left\{a_{n,\ell} : 1\le k, 0\le \ell < g(1-r_n)^{-1}\right\}$$

is a weakly sufficient set for $A^{-\infty}(\mathbb{D})$.

Proof: By Proposition 1, it is enough to show that for any $p>0$, there exists $q>0$ such that $A^{-p}(S)\subset A^{-q}(\mathbb{D})$. By Proposition 5, it will be enough to show that $A^{-p}(S)\subset A^{-p}(E(\{r_n\}))$.

Let $f\in A^{-p}(S)$; $f\in A^{-\infty}(\mathbb{D})$, so there exists $q_f>0$ so that $|f(z)|\le \|f\|_{A^{-q_f}}(1-|z|)^{-q_f}$ for all $z\in\mathbb{D}$. Let $z=r_ne^{i\theta}$; there exists ℓ such that $|\theta-2\pi\ell g(1-r_n)|<2\pi g(1-r_n)$, so

$$|f(r_ne^{i\theta})| \le |f(a_{n,\ell})| + 2\pi g(1-r_n)\sup_{|\zeta|=r_n}|f'(\zeta)|$$
$$\le \|f\|_{A^{-p}(S)}(1-r_n)^{-p}+C_f g(1-r_n)(1-r_n)^{-q_f-1}$$
$$\le (1-r_n)^{-p}\left[\|f\|_{A^{-p}(S)}+C_f g(1-r_n)(1-r_n)^{p-q_f-1}\right]$$
$$\le C(1-r_n)^{-p}$$

for $n\ge n_f$. The values $z=r_ne^{i\theta}$, $n<n_f$, are dealt with by the maximum principle. ∎

End of the Proof of the Theorem: (ii) does not imply (i):

Take a set $S = \{a_{n,\ell}\} \subset E(\{r_n\})$ with $M > 1$. S is weakly sufficient for $A^{-\infty}(\mathbb{D})$ by Proposition 8, and cannot be sampling since $E(\{r_n\})$ is not.

(iii) does not imply (ii):

It follows from [**K1**, 3.1.4] that if a set S as above (with $0 \leq M \leq \infty$) is a zero set for $A^{-\infty}(\mathbb{D})$, then

$$\sum_{\ell}(1 - |a_{n,\ell}|) = \frac{(1 - r_n)}{g(1 - r_n)} = O\left(\log \frac{1}{1 - r_n}\right).$$

Taking $g(x) \leq Cx^{1+\varepsilon}$ for some $\varepsilon > 0$ and $M = \infty$, we find a discrete set S which is of uniqueness for $A^{-\infty}(\mathbb{D})$, without being weakly sufficient.

Remark. As a corollary of the proof of Proposition 5, this set of uniqueness possesses the additional property that there does not exists any choice of p, $q > 0$ such that $A^{-p}(S) \subset A^{-q}(\mathbb{D})$, so that the converse to Lemma 2 does not hold either.

Open Question. *When $M = 1$ and $g(x) \leq Cx^{1+\varepsilon}$, it follows from* [**HKP**] *that S must be sampling, and thus weakly sufficient. When $1 < M < \infty$ and $g(x) \leq Cx^{1+\varepsilon}$, must S be weakly sufficient ? One can see that it is of uniqueness.*

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