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CLASSIFICATION OF DEGREE 2 POLYNOMIAL AUTOMORPHISMS OF $\mathbb{C}^3$

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Abstract

For the family of degree at most 2 polynomial self-maps of $\mathbb{C}^3$ with nowhere vanishing Jacobian determinant, we give the following classification: for any such map f, it is affinely conjugate to one of the following maps:

(i) An affine automorphism;

(ii) An elementary polynomial autormorphism

$$E(x,y,z) = (P(y,z) + ax, Q(z) + by, cz + d),$$

where P and Q are polynomials with $\max\{\deg(P), \deg(Q)\} = 2$ and $abc \neq 0$.

(iii)

$$\begin{cases} H_1(x,y,z) = (P(x,z) + ay, Q(z) + x, cz + d) \\ H_2(x,y,z) = (P(y,z) + ax, Q(y) + bz, y) \\ H_3(x,y,z) = (P(x,z) + ay, Q(x) + z, x) \\ H_4(x,y,z) = (P(x,y) + az, Q(y) + x, y) \\ H_5(x,y,z) = (P(x,y) + az, Q(x) + by, x) \end{cases}$$

where P and Q are polynomials with $\max\{\deg(P), \deg(Q)\} = 2$ and $abc \neq 0$.

1. Introduction

In this note, we will give a classification theorem for the family of degree at most 2 polynomial self-maps of $\mathbb{C}^3$ with nowhere vanishing Jacobian determinant. Note that any polynomial automorphism has a nowhere vanishing Jacobian determinant. Our Theorem 2.1 implies that any degree at most 2 polynomial self-map of $\mathbb{C}^3$ with nowhere vanishing Jacobian determinant is a polynomial automorphism.

Let $\mathcal{G}$ be the group of polynomial automorphisms of $\mathbb{C}^2$. Let $\mathcal{A}$ be the group of affine automorphisms of $\mathbb{C}^2$ and let $\mathcal{E}$ be the group of elementary polynomial automorphisms of $\mathbb{C}^2$ such that each $e \in \mathcal{E}$ is of the form:

$$e(x, y) = (ax + P(y), by + c)$$

where P is a polynomial and a, $b \neq 0$. Note that $\mathcal{E}$ is the group of all polynomial automorphisms that carry each line of the form $y =$ *constant* to a line of the form $y =$ *constant'*. Then Jung's Theorem [**J**] asserts that $\mathcal{G}$ is generated by $\mathcal{A}$ and $\mathcal{E}$. Applying Jung's Theorem, Friedland and Milnor [**FM**] classified the polynomial automorphisms of $\mathbb{C}^2$: *Any polynomial automorphism of $\mathbb{C}^2$ is affinely conjugate to one of the following types of maps: (i) an affine automorphism; (ii) an elementary polynomial automorphism; (iii) A finite composition of generalized Hénon mappings. Each generalized Hénon mapping is of the form*

$$h(x, y) = (P(x) - ay, x)$$

where p is a polynomial of x of degree at least 2 and $a \neq 0$.

It seems to be difficult to extend Jung's Theorem to $\mathbb{C}^n$ for $n \geq 3$. So we cannot follow Friedland and Milnor's proof to classify polynomial automorphisms in higher dimensions. But if we restrict to polynomials of degree at most 2 in $\mathbb{C}^2$, it is not necessary to apply Jung's Theorem for the classification, see [**HO**] for a proof. In this paper, we give the classification of degree at most 2 polynomial self-maps of $\mathbb{C}^3$ with nowhere vanishing Jacobian determinant up to affine conjugation. The organization of this paper is as follows: in Section 2, we give the statement of our classification Theorem 2.1 and we also include some remarks on the dynamical differences between the various classes in our theorem. In Section 3, the proof of Theorem 2.1 is given and in Section 4, we briefly give some discussions of some basic dynamical properties of these maps.

2. The statement of main theorem and some remarks

Theorem 2.1. *If $f : \mathbb{C}^3 \to \mathbb{C}^3$ is a degree at most 2 polynomial self-map with nowhere vanishing Jacobian determinant, then f is affinely conjugate to one of the following maps:*

(1) An affine automorphism;

(2) An elementary polynomial automorphism

$$E(x, y, z) = (P(y, z) + ax, Q(z) + by, cz + d),$$

where P is a polynomial of y, z of degree at most 2, Q is a polynomial of z of degree at most 2 and $abc \neq 0$. Note that it maps every hyperplane $z = k$ to a hyperplane $z = k'$ and maps every line $y = k_1$, $z = k_2$ to a line $y = k_1'$, $z = k_2'$;

(3)

$$\begin{cases} H_1(x,y,z) = (P(x,z) + ay, Q(z) + x, cz + d) \\ H_2(x,y,z) = (P(y,z) + ax, Q(y) + bz, y) \\ H_3(x,y,z) = (P(x,z) + ay, Q(x) + z, x) \\ H_4(x,y,z) = (P(x,y) + az, Q(y) + x, y) \\ H_5(x,y,z) = (P(x,y) + az, Q(x) + by, x) \end{cases}$$

where P and Q are polynomials with $\max\{\deg(P), \deg(Q)\} = 2$ *and* $abc \neq 0$.

Remark 2.2.

$$\begin{cases} H_1^{-1}(x,y,z) = \left(y - Q\left(\frac{1}{c}(z-d)\right),\right. \\ \qquad\qquad \left.\frac{1}{a}\left[x - P\left(y - Q\left(\frac{1}{c}(z-d)\right), \frac{1}{c}(z-d)\right)\right], \frac{1}{c}(z-d)\right) \\ H_2^{-1}(x,y,z) = \left(\frac{1}{a}\left[x - P\left(z, \frac{1}{b}(y - Q(z))\right)\right], z, \frac{1}{b}[y - Q(z)]\right) \\ H_3^{-1}(x,y,z) = \left(z, \frac{1}{a}[x - P(z, y - Q(z))], y - Q(z)\right) \\ H_4^{-1}(x,y,z) = \left(y - Q(z), z, \frac{1}{a}[x - P(y - Q(z), z)], z\right) \\ H_5^{-1}(x,y,z) = \left(z, \frac{1}{b}[y - Q(z)], \frac{1}{a}\left[x - P\left(z, \frac{1}{b}(y - Q(z))\right)\right]\right). \end{cases}$$

Remark 2.3. If $\tau(x,y,z) = (y,z,x)$, then $H_4 = H_3 \circ \tau$, $H_2 = H_5 \circ \tau$, $H_1 = H_5 \circ \tau^2$.

Remark 2.4. Some Generic Dynamical differences between the various classes: Assume for simplicity that the constants $|a|$, $|b|$, $|c| < 1$.

First of all the elementary maps and the class H_1 distinguish themselves from the other classes by the fact that the maps fix a hypersurface and the orbits of all points outside this hypersurface converge to it. Hence the dynamics reduces to two dimensions. In the case of H_1, the maps reduces to a Hénon map on the fixed hypersurface $z = \alpha$. In the case of the elementary maps, the orbits in the fixed hypersurface $z = \alpha$ converge to the fixed curve $y = \beta$ on which the maps are automorphisms. In fact both the elementary maps and the maps of class H_1 are semidirect products over a mapping $A(z) = cz + d$, i.e. there is a function $z \to f_z(x, y) \in \mathrm{Aut}(\mathbb{C}^2)$ such that $F(x, y, z) = (f_z(x, y), A(z))$.

Hence we need only to find dynamical differences between the classes H_2, H_3, H_4, H_5.

First we can observe that it is natural to consider the maps H_2, H_3 together as opposed to the maps H_4, H_5. There is a dynamical difference in the asymptotic dynamics. For the maps H_2, H_3, the orbits generically converge to one point at infinity. For example, for the map H_3, if $P(x, y) = Ax^2 + \cdots$, $Q(x) = Bx^2 + \cdots$, then this is the point $[A\colon B\colon 0\colon 0]$ at infinity in projective coordinates. On the other hand, for the maps H_4, H_5 the generic orbit converges to a complex line at infinity.

It remains to distinguish dynamically the maps H_2 and H_3 as well as to distinguish the maps H_4 and H_5.

Comparing the maps H_2 and H_3, we observe that the map H_2 is a Hénon map in the last two coordinates, $(y, x) \to (Q(y) + bz, y)$. In other words such a map F is a semi-direct product over a mapping $h(y, z) \in \mathrm{Aut}(\mathbb{C}^2)$, i.e. there is an analytic function $(y, z) \to A_{y,z} \in \mathrm{Aut}(\mathbb{C})$ such that $F(x, y, z) = (A_{y,z}(x), h(y, z))$. This sets H_2 apart from H_3.

Comparing the maps H_4 and H_5 we consider again their behaviour at infinity. We see that there is a P^1 at infinity which is mapped to itself. For H_4 this map is a second degree polynomial, while for H_5 this map is rational of degree 2, i.e. has a more complicated dynamics.

3. The Proof of Theorem 2.1

Let G be the family of degree at most 2 polynomial self-map of $\mathbb{C}^3$ with nowhere vanishing Jacobian determinant. For any $f \in G$, we can write f in the following form:

$$f(x, y, z) = (f_1(x, y, z), f_2(x, y, z), f_3(x, y, z)).$$

Because the degree of f is at most 2, the Jacobian matrix of f is as follows:

$$f'(x, y, z) = \begin{pmatrix} w_1(x, y, z) & w_2(x, y, z) & w_3(x, y, z) \\ w_4(x, y, z) & w_5(x, y, z) & w_6(x, y, z) \\ w_7(x, y, z) & w_8(x, y, z) & w_9(x, y, z) \end{pmatrix}$$

where $w_j(x, y, z) = a_j x + b_j y + c_j z + d_j$ for $1 \le j \le 9$.

Since the determinant of the Jacobian matrix $f'(x,y,z)$ is a nonzero constant, all coefficients of the polynomial $\det(f')$ must be *zero* except the constant. In particular, the coefficients of x^3, y^3 and z^3 must be zero, i.e.,

$$\det(A) = \det(B) = \det(C) = 0 \tag{1}$$

where

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}, \ B = \begin{pmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{pmatrix}, \ C = \begin{pmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{pmatrix}. \tag{2}$$

It is easy to see that both b_1 and a_2 are the coefficient of xy in $f_1(x,y,z)$, this implies that $b_1 = a_2$. By looking at the coefficients of xy, yz, and xz in $f = (f_1, f_2, f_3)$, we obtain the following table:

$b_1 = a_2$	$c_1 = a_3$	$c_2 = b_3$
$b_4 = a_5$	$c_4 = a_6$	$c_5 = b_6$
$b_7 = a_8$	$c_7 = a_9$	$c_8 = b_9$

Table (ABC)

By using the above table we can write our function f in the following form:

$$f(x,y,z) = (\phi_1(x,y,z) + L_1(x,y,z), \phi_2(x,y,z) + L_2(x,y,z), \phi_3(x,y,z) + L_3(x,y,z)) \tag{3}$$

where

$$\begin{cases} \phi_1(x,y,z) = \frac{1}{2}a_1x^2 + \frac{1}{2}b_2y^2 + \frac{1}{2}c_3z^2 + b_1xy + c_2yz + c_1xz \\ \phi_2(x,y,z) = \frac{1}{2}a_4x^2 + \frac{1}{2}b_5y^2 + \frac{1}{2}c_6z^2 + b_4xy + c_5yz + c_4xz \\ \phi_3(x,y,z) = \frac{1}{2}a_7x^2 + \frac{1}{2}b_8y^2 + \frac{1}{2}c_9z^2 + b_7xy + c_8yz + c_7xz \\ L_1(x,y,z) = d_1x + d_2y + d_3z + e_1 \\ L_2(x,y,z) = d_4x + d_5y + d_6z + e_2 \\ L_3(x,y,z) = d_7x + d_8y + d_9z + e_3. \end{cases}$$

Let's introduce the following trivial lemma which is useful in our proof of Theorem 2.1.

Lemma 3.1. *Let $F = (f_1, \dots, f_n)$ be a polynomial self-map of $\mathbb{C}^n$ of degree at most 2, with nowhere vanishing Jacobian. Let g and h be any affine automorphisms of $\mathbb{C}^n$. We denote the degree 2 homogeneous part of F by $(\phi_1, \dots, \phi_n)$ and the degree 2 homogeneous part of $g \circ F \circ h$ by $(\psi_1, \dots, \psi_n)$. Then the following statements are equivalent:*

(i) *There exist constants $\alpha_1, \dots, \alpha_n$ with $\sum_{j=1}^n |\alpha_j| \neq 0$ such that*

$$\sum_{j=1}^n \alpha_j \phi_j \equiv 0,$$

(ii) *There exist constants β_j with $\sum_{j=1}^n |\beta_j| \neq 0$ such that*

$$\sum_{j=1}^n \beta_j \psi_j = 0.$$

Proof: Clear. ■

Remark 3.2. For the map F in above lemma, if we want to prove that $\sum_{j=1}^n \alpha_j \phi_j \equiv 0$, we can simplify J_F, the Jacobian matrix of F, by composing constant invertible matrices in both sides of J_F. Note that $J_{(g \circ F \circ h)} = J_g(F(h)) J_F(h) J_h = J_g J_F(h) J_h$, i.e., we have to use the new variables for the Jacobian matrix of F, but this doesn't matter because J_h is a constant matrix and therefore we may keep the original notation as the new variables.

Let's recall the following result from [**HO**]:

Lemma 3.3. *Let $f(x, y) = (f_1(x, y), f_2(x, y)) = (P_1(x, y) + A_1(x, y), P_2(x, y) + A_2(x, y))$ be a polynomial self-map of $\mathbb{C}^2$ of degree at most 2, with nowhere vanishing Jacobian, where $P_j(x, y)$ is the corresponding degree 2 homogeneous polynomial of f_j and $A_j = f_j - P_j$. Then the homogeneous polynomials P_1 and P_2 are proportional.*

Lemma 3.4. *If for all constants α, β and γ with $|\alpha| + |\beta| + |\gamma| \neq 0$, we have $\alpha\phi_1 + \beta\phi_2 + \gamma\phi_3 \not\equiv 0$, then there exist affine automorphisms g and h such that*

$$\psi_1 = x^2, \quad \psi_2 = xy, \quad \psi_3 = y^2, \tag{4}$$

where (ψ_1, ψ_2, ψ_3) is the degree 2 homogeneous part of the map $g \circ f \circ h$.

Proof: Let $\phi = [\phi_1 : \phi_2 : \phi_3] : P^2 \to P^2$. Since the Jacobian determinant of ϕ is 0, the rank of ϕ is at most 1. If $\phi_1 \equiv 0$, then

$1\phi_1 + 0\phi_2 + 0\phi_3 \equiv 0$. So WLOG, we may assume that $\phi_j \not\equiv 0$ for $j = 1, 2, 3$.

(i) If the rank of ϕ is 0, i.e., ϕ is constant, then we may assume that $[\phi_1 \colon \phi_2 \colon \phi_3] = [1 \colon 0 \colon 0]$. In this case, we have

$$0\phi_1 + 1\phi_2 + 0\phi_3 \equiv 0.$$

(ii) If rank of ϕ is 1, we may assume that $\phi([1 \colon 0 \colon 0]) = [1 \colon 0 \colon 0]$ and that $\phi([x \colon 0 \colon 1])$ is non-constant.

Let

$$(\phi_1, \phi_2, \phi_3)|_{[x:0:1]} = (a_1x^2 + b_1x + c_1, a_2x^2 + b_2x + c_2, a_3x^3 + b_3x + c_3).$$

Then since $\phi([1 \colon 0 \colon 0]) = [1 \colon 0 \colon 0]$, $a_1 \neq 0$, $a_2 = a_3 = 0$.

Hence

$$(\phi_1, \phi_2, \phi_3)|_{[x:0:1]} = (a_1x^2 + b_1x + c_1, b_2x + c_2, b_3x + c_3).$$

Since $\phi([x \colon 0 \colon 1])$ is non-constant, it follows that b_2 or $b_3 \neq 0$. By Lemma 3.1 we may assume that $b_2 \neq 0$ and $b_1 = b_3 = 0$.

Hence

$$(\phi_1, \phi_2, \phi_3)|_{x:0:1]} = (a_1x^2 + c_1, b_2x + c_2, c_3).$$

If $c_3 = 0$, then we have $\phi_3 = 0$, so we have $0\phi_1 + 0\phi_2 + 1\phi_3 \neq 0$, which is impossible. Hence $c_3 \neq 0$, so we may assume that

$$(\phi_1, \phi_2, \phi_3)|_{[x:0:1]} = (x^2, x, 1).$$

Hence $\phi(y = 0) = (XZ = Y^2)$. Sice ϕ has rank 1, $\phi(P^2) \subset (XZ = Y^2)$, so

$$\phi_1\phi_3 \equiv \phi_2^2. \tag{5}$$

(1) If $\phi_1 = c\phi_2$ for a nonzero constant c, then $\phi_1 - c^{-1}\phi_2 + 0\phi_3 \equiv 0$.

(2) If $\phi_3 = c\phi_2$ for a nonzero constant c, then we have $0\phi_1 + c^{-1}\phi_2 - \phi_3 \equiv 0$.

(3) Hence $\phi_1 \neq c_1\phi_2$ and $\phi_3 \neq c_2\phi_2$ for any constants c_1 and c_2. Then ϕ_2 must be a product of two nonproportional linear factors, $\phi = L_1L_2$. This implies that $\phi_1 = c_1L_1^2$ and $\phi_3 = c_3L_2^2$ or vice versa. Setting $L_1 = x$ and $L_2 = y$ and scaling we finish the proof of the lemma. ■

Lemma 3.5. *There exist constants α, β and γ such that $|\alpha| + |\beta| + |\gamma| \neq 0$ and*

$$\alpha\phi_1 + \beta\phi_2 + \gamma\phi_3 \equiv 0.$$

Proof: If for all constants α, β and γ with $|\alpha| + |\beta| + |\gamma| \neq 0$, $\alpha\phi_1 + \beta\phi_2 + \gamma\phi_3 \not\equiv 0$, then by Lemma 3.1 and Lemma 3.4 we may assume that $\psi_1 = x^2$, $\psi_2 = xy$, $\psi_3 = y^2$.

In this case, the Jacobian matrix of f is as follows:

$$f'(x, y, z) = \begin{pmatrix} 2x + d_1 & d_2 & d_3 \\ y + d_4 & x + d_5 & d_6 \\ d_7 & 2y + d_8 & d_9 \end{pmatrix}.$$

Since the Jacobian determinant is a nonzero constant, then

$$\det \begin{pmatrix} 2x + d_1 & d_2 & d_3 \\ y + d_4 & x + d_5 & d_6 \\ d_7 & 2y + d_8 & d_9 \end{pmatrix} = \det \begin{pmatrix} d_1 & d_2 & d_3 \\ d_4 & d_5 & d_6 \\ d_7 & d_8 & d_9 \end{pmatrix} = \text{ const. } \neq 0.$$

But

$$\det \begin{pmatrix} 2x + d_1 & d_2 & d_3 \\ y + d_4 & x + d_5 & d_6 \\ d_7 & 2y + d_8 & d_9 \end{pmatrix} = 2d_9x^2 - 4d_6xy + 2d_3y^2 + \cdots .$$

This implies that $d_3 = d_6 = d_9 = 0$, i.e.,

$$\det \begin{pmatrix} d_1 & d_2 & d_3 \\ d_4 & d_5 & d_6 \\ d_7 & d_8 & d_9 \end{pmatrix} = 0.$$

This is a contradiction. ■

Proof of the Theorem 2.1: If $\deg(f) = 1$, then it is easy to see that f is an affine automorphism.

If $\deg(f) = 2$, then by Lemma 3.5 we can assume that there exist constants k_1 and k_2 such that

$$\phi_3 = k_1\phi_1 + k_2\phi_2.$$

Then

$$f = (\phi_1 + L_1, \phi_2 + L_2, k_1\phi_1 + k_2\phi_2 + L_3). \tag{6}$$

Let $g_1(x,y,z) = (x,y,z-k_1x-k_2y)$, then $g_1^{-1}(x,y,z) = (x,y,z+k_1x+k_2y)$. Then

$$
\begin{aligned}
F_1(x,y,z) :&= g_1 \circ f \circ g_1^{-1}(x,y,z) \\
&= g_1 \circ f(x,y,z+k_1x+k_2y) \\
&= (\phi_4(x,y,z) + L_4(x,y,z), \phi_5(x,y,z) \\
&\quad + L_5(x,y,z), L_6(x,y,z))
\end{aligned} \tag{7}
$$

where ϕ_j are degree 2 homogeneous polynomial and $L_j(x,y,z)$ are degree 1 polynomial and written as $L_j(x,y,z) := \alpha_j x + \beta_j y + \gamma_j z + \rho_j$. In particular, $|\alpha_6| + |\beta_6| + |\gamma_6| \neq 0$.

We will classify L_6 into the following 3 cases:

Case (i): $\alpha_6 \neq 0$.

Let $g_2(x,y,z) = (\alpha_6 x + \beta_6 y + \gamma_6 z + \rho_6, y, z)$, then $g_2^{-1}(x,y,z) = \left(\frac{1}{\alpha_6}(x - \beta_6 y - \gamma_6 z - \rho_6), y, z\right)$. Then

$$
\begin{aligned}
F_2(x,y,z) :&= g_2 \circ F_1 \circ g_2^{-1}(x,y,z) \\
&= (\phi_7(x,y,z) + L_7(x,y,z), \phi_8(x,y,z) + L_8(x,y,z), x)
\end{aligned} \tag{8}
$$

where ϕ_j are degree 2 homogeneous polynomials and $L_j(x,y,z)$ are degree 1 polynomials and written as $L_j(x,y,z) := \alpha_j x + \beta_j y + \gamma_j z + \rho_j$.

Then the Jacobian matrix of F_2 has the following form:

$$
F_2'(x,y,z) = \begin{pmatrix} (\phi_7 + L_7)'_x & (\phi_7 + L_7)'_y & (\phi_7 + L_7)'_z \\ (\phi_8 + L_8)'_x & (\phi_8 + L_8)'_y & (\phi_8 + L_8)'_z \\ 1 & 0 & 0 \end{pmatrix}. \tag{9}
$$

Then since $\det(F_2'(x,y,z))$ is a nonzero constant,

$$
\det \begin{pmatrix} (\phi_7 + L_7)'_y & (\phi_7 + L_7)'_z \\ (\phi_8 + L_8)'_y & (\phi_8 + L_8)'_z \end{pmatrix} = a \text{ nonzero constant.}
$$

Then for any fixed x, we may consider

$$
F_x(y,z) = (\phi_7 + L_7, \phi_8 + L_8)
$$

as a degree 2 polynomial self-map of $\mathbb{C}^2$ with nowhere vanishing Jacobian determinant.

We can write

$$
\begin{aligned}
\phi_7 + L_7 &= a_7x^2 + b_7y^2 + c_7z^2 + d_7yz + e_7xy + f_7xz + \cdots \\
\phi_8 + L_8 &= a_8x^2 + b_8y^2 + c_8z^2 + d_8yz + e_8xy + f_8xz + \cdots
\end{aligned}
$$

By Lemma 3.3 it follows that the homogeneous polynomials $b_7y^2 + c_7z^2 + d_7yz$ and $b_8y^2 + c_8z^2 + d_8yz$ are proportional. Moreover we make the following claim:

Claim. *The homogeneous polynomials* $b_7y^2+c_7z^2+d_7yz+e_7xy+f_7xz$ *and* $b_8y^2 + c_8z^2 + d_8yz + e_8xy + f_8xz$ *are also proportional.*

$$J_{F_x}(y,z) = \begin{pmatrix} 2b_7y + d_7z + e_7x + k_1 & 2c_7z + d_7y + f_7x + k_2 \\ 2b_8y + d_8z + e_8x + k_3 & 2c_8z + d_8y + f_8x + k_4 \end{pmatrix}$$

and $\det J_{F_x}(y,z) = a$ nonzero constant independent on x.

(1) Since $b_7y^2 + c_7z^2 + d_7yz$ and $b_8y^2 + c_8z^2 + d_8yz$ are proportional, we may compose an invertible constant matrix, say M_1, to the left of the matrix of $J_{F_x}(y,z)$ to kill the y, z terms in the first row or second row. WLOG, we may assume that we killed the y, z terms in the first row. Now our Jacobian matrix becomes

$$M_1J_{F_x}(y,z) = \begin{pmatrix} e_7x + k_1 & f_7x + k_2 \\ 2b_8y + d_8z + e_8x + k_3 & 2c_8z + d_8y + f_8x + k_4 \end{pmatrix}.$$

Note that e_7, k_1, f_7, k_2 are different from the original values, for simplicity we still use the same notation. The following discussion will follow the same rule.

(2) If $e_7 = f_7 = 0$, then we are done.

(3) If $e_7 \neq 0$, $f_7 \neq 0$, then we may compose an invertible constant matrix to the right of the matrix $M_1J_{F_x}$ to kill f_7. Now the matrix becomes

$$\begin{pmatrix} e_7x + k_1 & k_2 \\ 2b_8y + d_8z + e_8x + k_3 & 2c_8z + d_8y + f_8x + k_4 \end{pmatrix}.$$

Since the determinant of this matrix is a nonzero constant, we have $c_8 = d_8 = f_8 = 0$. If $e_8 \neq 0$, we will use the fact $e_7 \neq 0$ to kill e_8. So now the matrix ix

$$\begin{pmatrix} e_7x + k_1 & k_2 \\ 2b_8y + k_3 & k_4 \end{pmatrix}.$$

The determinant of this matrix is $e_7k_4x - 2b_8yk_2 + k_1k_4 - k_2k_3$ which is a nonzero constant. This implies that $e_7k_4 = 0$, $b_8k_2 = 0$. $e_7 \neq 0$ implies that $k_4 = 0$. k_2 cannot be 0 otherwise the determinant is 0, so $b_8 = 0$.

Now our matrix is pretty simple:

$$\begin{pmatrix} e_7x + k_1 & k_2 \\ k_3 & 0 \end{pmatrix}.$$

It is easy to see that our claim is true for the map F_x with above matrix as its Jacobian matrix. By Lemma 3.1 and Remark 3.2, we proved our claim.

(4) If one of $\{e_7, f_7\}$ is 0 and the other is different from 0, then go to the case (3). ■

By the above claim we can write

$$\begin{aligned} \phi_7 + L_7 &= a_7x^2 + \psi_7(x,y,z) + L_7(x,y,z) \\ \phi_8 + L_8 &= a_8x^2 + \psi_8(x,y,z) + L_8(x,y,z) \end{aligned}$$

where $\psi_7(x,y,z) = b_7y^2 + c_7z^2 + d_7yz + e_7xy + f_7xz$ and $\psi_8(x,y,z) = b_8y^2 + c_8z^2 + d_8yz + e_8xy + f_8xz$ are proportional. This implies that either (i-a) $\psi_8 \equiv k\psi_7$ for some constant k; or (i-b) $\psi_7 \equiv 0$.

Case (i-a): $\psi_8 = k\psi_7$ for some constant k.

Let $g_3(x,y,z) = (x, y - kx, z)$, then $g_3^{-1}(x,y,z) = (x, y + kx, z)$,

$$\begin{aligned} F_3(x,y,z) : &= g_3 \circ F_2 \circ g_3^{-1}(x,y,z) \\ &= (a_7x^2 + \psi_7(x, y+kx, z) + L_7(x, y+kx, z), a_8'x^2 \\ &\quad + L_8'(x,y,z), x) \\ &= (p_1(x) + \psi(x,y,z) + l_7(y,z), p_2(x) + l_8(y,z), x) \end{aligned}$$

where a_8' is the new coefficient of x^2 after the compositions, L_8' is the new linear function, p_1, p_2 are the polynomial of x of the corresponding coordinates, l_7, l_8 are linear functions of y, z of the corresponding coordinates, $\psi(x,y,z)$ is the degree 2 homogeneous polynomial except the x^2 term in $a_7x^2 + \psi_7(x, y+kx, z) + L_7(x, y+kx, z)$.

Now we assume that the linear function $l_8(y,z) = \alpha_8 y + \beta_8 z$ with $|\alpha_8| + |\beta_8| \neq 0$.

If $\alpha_8 \neq 0$, let $g_4(x,y,z) = (x, \alpha_8 y + \beta_8 z, z)$, then

$$\begin{aligned} H_3(x,y,z) : &= g_4 \circ F_3 \circ g_4^{-1}(x,y,z) \\ &= \left(p_1(x) + \psi\left(x, \frac{1}{\alpha_8}(y - \beta_8 z), z\right)\right. \\ &\quad \left. + l_7\left(\frac{1}{\alpha_8}(y - \beta_8 z), z\right), \alpha_8(p_2(x) + y) + \beta_8 x, x \right). \end{aligned}$$

Since the Jacobian determinant is a nonzero constant, $\frac{\partial}{\partial z}(p_1(x)+\psi(x, \frac{1}{\alpha_8}(y-\beta_8 z), z) + l_7(\frac{1}{\alpha_8}(y-\beta_8 z), z))$ = nonzero constant. So we may rewrite H_3 in the following form:

$$H_3(x,y,z) = (P(x,y) + az, Q(x) + by, x) \tag{10}$$

where P and Q are polynomials with $\max\{\deg(P), \deg(Q)\} = 2$ and $a \neq 0$.

If $\alpha_8 = 0$, then $\beta_8 \neq 0$. In this case, following a similar discussion as above, we may rewrite our map F_3 as the following form:

$$F_3(x,y,z) = (p(x,z) + cy, q(x) + \beta_8 z, x) \tag{11}$$

where $p(x,z)$ is a degree at most 2 polynomial of x and z.

Let $g(x,y,z) = (\beta_8 x, y, \beta_8 z)$ and let $H_1 = g \circ F_3 \circ g^{-1}$, then H_1 has the following form:

$$H_3(x,y,z) = (P(x,z) + ay, Q(x) + z, x) \tag{12}$$

where P and Q are polynomials with $\max\{\deg(P), \deg(Q)\} = 2$ and $a \neq 0$.

Case (i-b): $\psi_7 \equiv 0$.

In this case, $F_2(x,y,z) = (p_1(x) + l_7(y,z), a_8 x^2 + \psi_8 + L_8, x)$ where $p_1(x)$ is polynomial of x and $l_7(x,y) = \alpha_7 y + \beta_7 z$.

If $\alpha_7 \neq 0$, let $g(x,y,z) = (x, \alpha_7 y + \beta_7 z, z)$, since the Jacobian determinant of $g \circ F_2 \circ g^{-1}$ is a nonzero constant, it is easy to check that $g \circ F_2 \circ g^{-1}$ has the following form:

$$G_2(x,y,z) = (q(x) + y, p(x,y) + az, x) \tag{13}$$

where P and Q are polynomials with $\max\{\deg(P), \deg(Q)\} = 2$ and $a \neq 0$.

If $\alpha_7 = 0$, then $\beta_7 \neq 0$. In this case, $F_2 = (p_1(x) + \beta_7 z, a_8 x^2 + \psi_8 + L_8, x)$. Using the fact that the Jacobian determinant of F_2 is a nonzero constant, we have that $\frac{\partial}{\partial y}(a_8 x^2 + \psi_8 + L_8)$ = nonzero constant. So we can rewrite F_2 in the following form:

$$G_4(x,y,z) = (q(x) + bz, p(x,z) + ay, x) \tag{14}$$

where P and Q are polynomials with $\max\{\deg(P), \deg(Q)\} = 2$ and $ab \neq 0$.

Let $g(x,y,z)=(y,x,z)$, then

$$H_4 := g\circ G_2\circ g^{-1} = (P(x,y)+az, Q(y)+x, y) \tag{15}$$
$$H_2 := g\circ G_4\circ g^{-1} = (P(y,z)+ax, Q(y)+bz, y) \tag{16}$$

where P and Q are polynomials with $\max\{\deg(P),\deg(Q)\} = 2$ and $ab\neq 0$.

Therefore, in the case (i) the map F_2 is affinely conjugate to one of the maps H_2, H_3, H_4, H_5.

Case (ii): $\alpha_6 = 0$ and $\beta_6 \neq 0$.

In this case, we construct the following affine map g_5:

$$g_5(x,y,z) = (x, \beta_6 y + \gamma_6 z + \rho_6, z).$$

Then $g_5^{-1}(x,y,z) = (x, \frac{1}{\beta_6}(y-\gamma_6 z-\rho_6), z)$.

$$\begin{aligned} F_5(x,y,z) :&= g_5\circ F_1\circ g_5^{-1}(x,y,z)\\ &= (\phi_9+L_9, \phi_{10}+L_{10}, y), \end{aligned} \tag{17}$$

where ϕ_j are degree 2 homogeneous polynomials and $L_j(x,y,z)$ are degree 1 polynomials and written as $L_j(x,y,z) := \alpha_j x+\beta_j y+\gamma_j z+\rho_j$.

Then the Jacobian matrix of F_5 has the following form:

$$F_5'(x,y,z) = \begin{pmatrix} (\phi_9+L_9)_x' & (\phi_9+L_9)_y' & (\phi_9+L_9)_z' \\ (\phi_{10}+L_{10})_x' & (\phi_{10}+L_{10})_y' & (\phi_{10}+L_{10})_z' \\ 0 & 1 & 0 \end{pmatrix}. \tag{18}$$

The property that $\det(F_5'(x,y,z)) = a$ nonzero constant implies that

$$\det\begin{pmatrix} (\phi_9+L_9)_x' & (\phi_9+L_9)_z' \\ (\phi_{10}+L_{10})_x' & (\phi_{10}+L_{10})_z' \end{pmatrix} = a \text{ nonzero constant.}$$

Then following the same arguments in the proof of case (i), we can prove that F_5 is conjugate to one of the following maps:

$$\begin{aligned} H_2(x,y,z) &= (P(y,z)+ax, Q(y)+bz, y)\\ H_4(x,y,z) &= (P(x,y)+az, Q(y)+x, y)\\ G_3(x,y,z) &= (q(y)+z, p(y,z)+ax, y)\\ G_5(x,y,z) &= (q(y)+bx, p(x,y)+az, y) \end{aligned}$$

where P and Q are polynomials with $\max\{\deg(P), \deg(Q)\} = 2$ and $ab \neq 0$.

But G_j is conjugate to H_j by the affine map $g(x, y, z) = (y, x, z)$ for $j = 3, 5$. So F_5 is affinely conjugate to one of the maps H_2, H_3, H_4, H_5.

Case (iii): If $\alpha_6 = \beta_6 = 0$.

In this case, $\gamma_6 \neq 0$. Then we have

$$F_1(x, y, z) = (\phi_4 + L_4, \phi_5 + L_5, \gamma_6 z + \rho_6).$$

Then the Jacobian matrix of F_1 has the following form:

$$F_1'(x, y, z) = \begin{pmatrix} (\phi_4 + L_4)'_x & (\phi_4 + L_4)'_y & (\phi_4 + L_4)'_z \\ (\phi_5 + L_5)'_x & (\phi_5 + L_5)'_y & (\phi_5 + L_5)'_z \\ 0 & 0 & \gamma_6 \end{pmatrix}. \tag{19}$$

The fact $\det(F_1'(x, y, z)) = a$ nonzero contant implies that

$$\det \begin{pmatrix} (\phi_4 + L_4)'_x & (\phi_4 + L_4)'_y \\ (\phi_5 + L_5)'_x & (\phi_5 + L_5)'_y \end{pmatrix} = a \text{ nonzero constant.}$$

Then the same arguments as in the proof of case (i) implies that F_1 is conjugate to one of the following maps:

$$H_1(x, y, z) = (P(x, z) + ay, Q(z) + x, cz + d) \tag{20}$$

$$E(x, y, z) = (P(y, z) + ax, Q(z) + by, cz + d) \tag{21}$$

where P and Q are polynomials with $\max\{\deg(P), \deg(Q)\} = 2$ and $abc \neq 0$.

We finish the proof of Theorem 2.1. ■

4. Some dynamical properties of H_j

Remark 4.1. For an elementary map $E(x, y, z) = (P(y, z) + ax, by + Q(z), cz + d)$, E has at most 1 isolated fixed point, in fact, it is easy to check the following facts:

(i) $If c \neq 1$, $a \neq 1$, $b \neq 1$, then E has only one isolated fixed point;

(ii) If $c = 1$, $d \neq 0$, then E has no fixed point;

(iii) If $c = 1$, $d = 0$, then E has a fixed z-plane for any given z and the set of fixed points of map E, $\mathrm{Fix}(E)$, has only the following possibilities:

  (1) Empty (for example: $b = 1$, $Q(z) = 1$);

  (2) Entire-curves;

  (3) 2-dimensional complex surfaces.

Remark 4.2. For the map $H_1(x,y,z) = (P(x,z)+ay, x+Q(z), cz+d)$, if $P(x,z)$ doesn't depend on x, then H_1^2 is a special case of our elementary map E. So the discussion of its dynamics goes to the study of elementary maps. If $c = 1$, $d = 0$ and $P(x,z) = kx^2 + \cdots$ with $k \neq 0$, then this map is essentially an Hénon map of $\mathbb{C}^2$ for any fixed z. Therefore in general we believe that H_1 has interesting dynamics if $P(x,z) = kx^2 + \cdots$ with $k \neq 0$.

Remark 4.3. For the map $H_2(x,y,z) = (Py,z) + ax, z + Q(y), y)$, if $a = 1$ and $P(y,z) = 0$, then the map H_2 is essentially an Hénon map of $\mathbb{C}^2$ for any fixed x. So if Q is a degree 2 polynomial, we believe that the map H_2 has interesting dynamics. Otherwise we have some trivial examples like this: $H_2(x,y,z) = (y^2 - x^2 + x, y, z)$. It is easy to see that H_2^2 is an identity map.

Remark 4.4. For $H_3(x,y,z) = (P(x,z) + ay, z + Q(x), x)$, we have the following facts about its fixed points:

(i) If $P(x,x) + aQ(x) = 0$, then we have 2 possibilities:

(i.a) If $a = 1$, then $\mathrm{Fix}(H_3)$ is a entire curve;

(i.b) If $a \neq 1$, then H_3 has only one fixed point $(0, Q(0), 0)$.

(ii) If $P(x,x) + aQ(x) \neq 0$ and

(ii.a) The degree of the polynomial $P(x,x) + aQ(x) \leq 1$, then it is easy to see that the $\mathrm{Fix}(H_3)$ could be an empty set, one point or an entire curve;

(ii.b) The degree of the polynomial $P(x,x) + aQ(x) = 2$, then it is easy to see that the H_3 has exactly 2 isolated fixed points counted with multiplicity. We believe that the H_1 in this case has rich dynamics.

Example 4.5. There are some interesting examples of H_3 in the case of $P(x,x) + aQ(x) = 0$ and $a = 1$ with H_3^3 is an identity map. For example, $H_3(x,y,z) = (z^2 + y, z - x^2, x)$.

Remark 4.6. Follow the same discussion as in Remark 3.6, we know that $H_4(x,y,z) = (P(x,y) + az, x + Q(y), y)$ has 2 isolated fixed points counted multiplicity if $P(y - Q(y), y) + Q(y) + (a-1)y$ is a degree 2 polynomial of y. We also believe that H_4 has rich dynamics in this case.

Remark 4.7. For the map $H_5(x,y,z) = (P(x,y) + az, by + Q(x), x)$, if $b = 1$ and $Q(x) = 0$, then the map H_5 is essentially an Hénon map of

$\mathbb{C}^2$ for any fixed y. If P and Q are degree 2 polynomials, we believe that the map H_5 has interesting dynamics and different from the dynamics of Hénon map in $\mathbb{C}^2$.

There are also some uninteresting examples like this: $H_3(x, y, z) = (2xz, 2y - x^2, x)$. It is easy to check this map has only one periodic point which is the fixed point at the origin.

Remark 4.8. For every map of H_j and E, the degree of its inverse polynomial could be 3 or 4 if the $\deg(P) = \deg(Q) = 2$. But it must be 2 if either $\deg(P) \leq 1$ or $\deg(Q) \leq 1$.

Remark 4.9. The detailed discussion of the dynamical properties of our maps H_j and E will appear in our forthcoming papers.

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