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THERE IS NO ANALOG OF THE TRANSPOSE MAP FOR INFINITE MATRICES

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Abstract

In this note we show that there are no ring anti-isomorphism between row finite matrix rings. As a consequence we show that row finite and column finite matrix rings cannot be either isomorphic or Morita equivalent rings. We also show that anti-isomorphisms between endomorphism rings of infinitely generated projective modules may exist.

1. Introduction and notation

The motivation for this note was to find out the extent to which the transpose for infinite matrices behaves like that for finite matrices.

It is well-known that for any commutative ring, R, the transpose map $t : \mathbb{M}_n(R) \to \mathbb{M}_n(R)$ is an anti-automorphism. For infinite matrix rings the analogous transpose map yields an anti-isomorphism $t : \mathrm{RFM}_A(R) \to \mathrm{CFM}_A(R)$, where $\mathrm{RFM}_A(R)$ (resp. $\mathrm{CFM}_A(R)$) denotes the ring of row-finite (resp. column-finite) matrices, indexed by the set A, having entries in R. Unlike the finite case, this is obviously not an anti-AUTOmorphism of $\mathrm{RFM}_A(R)$.

Thus we are led to ask: for a commutative ring, R, does there exist an anti-automorphism of $\mathrm{RFM}_A(R)$? We answer this question in the negative, with a vengeance. In fact, we show in Theorem 2.2 that there are no anti-isomorphisms between rings of the form $\mathrm{End}({}_R R^{(A)})$ and $\mathrm{End}({}_S S^{(B)})$ for any rings R and S (commutative or not), and any infinite sets A and B. As a corollary we show that for any two rings R and S the rings $\mathrm{RFM}_A(R)$ and $\mathrm{CFM}_B(S)$ cannot be either isomorphic or Morita equivalent rings. This corollary is well known in the case of commutative domains and A, B countable infinite sets, or division rings. A particularly clear proof for commutative domains is given in [**2**] although

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we wish to call to the reader's attention that Theorem 1.8 (III) does not hold in the generality stated. We establish in Example 2.4 that (normalized) Morita context is the closest relation that may exist between such rings.

Finally we show in Example 2.5 that anti isomorphisms (in fact, involutions) between endomorphism rings of countably generated and locally free left projective modules may exist, so that there are modules very close to free that admit anti-isomorphisms.

Throughout this paper ring means associative ring with identity and for any ring R, R^{op} denotes the opposite ring of R [**3**, Definition 0.1.11]. We denote by R-mod (respectively mod-R) the category of all left (resp. right) unital R-modules. Homomorphisms of modules act opposite scalars. Thus, when f and g are endomorphisms of a left R-module ${}_RM$, their composition $g \circ f$ will be denoted by fg; in particular $\mathrm{End}({}_RM)$ will be the opposite ring of $\mathrm{Hom}_R(M, M)$ and, for a right R-module N_R, $\mathrm{End}(N_R) = \mathrm{Hom}_R(N, N)$.

For a given family of modules $\{M_\alpha\}_{\alpha \in A}$, where A is a set, $\oplus_A M_\alpha$ will mean the direct sum, while $\prod_A M_\alpha$ will mean the direct product. Direct sum and product of copies of a single module M, indexed by A will be written $M^{(A)}$ or $\oplus_A M$ and M^A or $\prod_A M$, respectively, as usual. For any set A, cardinality will be denoted by $|A|$, and following most of literature, when A is finite, say $|A| = n \in \mathbf{N}$ we will write M^n instead of $\oplus_A M_\alpha = \prod_A M_\alpha$.

For any infinite set A, we denote by $\mathrm{RFM}_A(R)$ (respectively $\mathrm{CFM}_A(R)$) the ring of $A \times A$ row-finite matrices (respectively column-finite matrices) over R. It is well-known that there exist ring isomorphisms $\mathrm{End}({}_RR^{(A)}) \cong \mathrm{RFM}_A(R)$ and $\mathrm{End}(R_R^{(A)}) \cong \mathrm{CFM}_A(R)$. Recall that a ring R has (left) SBN (Single Basis Number) when for every $n \in \mathbf{N}$, $R^n \cong R$ (as left R-modules). Note that, for every ring R, since $R^{(\mathbf{N})} \cong (R^{(\mathbf{N})})^n$ as left and right R-modules then both $\mathrm{RFM}(R)$ and $\mathrm{CFM}(R)$ have SBN; thus $\mathrm{RFM}(R) \cong \mathbb{M}_n(\mathrm{RFM}(R))$ and also $\mathrm{CFM}(R) \cong \mathbb{M}_n(\mathrm{CFM}(R))$.

2. Anti isomorphisms

We shall see that for any two rings R and S, the existence of an anti isomorphism between $\mathrm{RFM}_A(R)$ and $\mathrm{RFM}_B(S)$ for some sets A and B implies that either A or B must be finite. To do this, we begin with the following proposition:

Proposition 2.1. *Let R be a ring and A an infinite set. Then $(R^{(A)})^A$ cannot be isomorphic to $R^{(A)}$ as left R-modules.*

Proof: It is well-known that there is the first ordinal number α such that $|\alpha| = |A|$. Then for every ordinal β, such that $\beta < \alpha$, we must have that $|\beta| < |\alpha|$.

Since $|\alpha| = |A|$ then we can endow A with a linear ordering $(A, \leq)$, given by α. By this and properties of α above, we have that for every $a \in A$, the set $S_a = \{x \in A \mid x \leq a\}$ satisfies $|S_a| < |A|$.

Now suppose there exists an R-isomorphism $\varphi : {R^{(A)}}^A \to R^{(A)}$. Let $\{e_a\}_{a \in (A,\leq)}$ be the canonical (well-ordered) basis for $R^{(A)}$ and let $\{x_a\}_{a \in (A,\leq)}$ such that $x_a = \varphi^{-1}(e_a)$. Then, $\{x_a\}_{a \in (A,\leq)}$ is a well-ordered basis for ${R^{(A)}}^A$. Let $\rho_a : {R^{(A)}}^A \to R_a^{(A)}$ and $\eta_a : R^{(A)} \to R_a$ be the usual projections. Define, for each $a \in A$, the set $N_a = \cup_{b \leq a} \operatorname{Supp}(x_b \rho_a)$, where $\operatorname{Supp}(x) = \{a \in A \mid x\eta_a \neq 0\}$ for $x \in R^{(A)}$.

Since $\operatorname{Supp}(x_b \rho_a)$ is a finite set and $|S_a| < |A|$ then $|N_a| < |A|$ and hence $N_a \subset A$ (strictly). So, for each $a \in A$ we may choose $\xi(a) \in A \backslash N_a$; that is, for each $a \in A$, there exists $\xi(a) \in A$ such that $x_b \rho_a \eta_{\xi(a)} = 0$ for all $b \leq a$.

Now define $x \in {R^{(A)}}^A$ such that, for each $a \in A$, $x\rho_a\eta_{\xi(a)} = 1$ and $x\rho_a\eta_b = 0$ if $b \neq \xi(a)$. We claim that x cannot be a linear combination of $\{x_a\}_{a \in (A,\leq)}$.

To see this, suppose $x = \sum_{i=1}^n r_i x_{a_i}$ where $a_1 < \cdots < a_n$ are elements of A and $r_1, \ldots, r_n$ are elements of R.

By definition of x we must have

$$1 = x\rho_{a_n}\eta_{\xi(a_n)} = \sum_{i=1}^n r_i x_{a_i} \rho_{a_n} \eta_{\xi(a_n)} = 0 \text{ (as } a_i \leq a_n\text{). A contradiction.}$$

Therefore, ${R^{(A)}}^A$ cannot be isomorphic to $R^{(A)}$. ■

Now we can prove our main result.

Theorem 2.2. *Let R and S be rings, A and B sets and δ : $\operatorname{End}({}_R R^{(A)}) \to \operatorname{End}({}_S S^{(B)})$ a ring anti isomorphism. Then A and B cannot be infinite sets simultaneously.*

Proof: Suppose both A and B are infinite sets and suppose WLOG $|A| \geq |B|$. Let $E = \operatorname{End}(R^{\mathrm{op}(A)}_{R^{\mathrm{op}}})$ and $F = \operatorname{End}({}_S S^{(B)})$. Since both F and E are anti isomorphic to $\operatorname{End}({}_R R^{(A)})$, we have that there is a ring isomorphism $\sigma : E \to F$. Since $|B| \leq |A|$ then $(R^{\mathrm{op}(A)}_{R^{\mathrm{op}}})^{(B)} \cong R^{\mathrm{op}(A)}_{R^{\mathrm{op}}}$ and so, there is a natural isomorphism between the functors

$$\operatorname{Hom}_{R^{\mathrm{op}}}((R^{\mathrm{op}(A)})^{(B)}, -) \cong \operatorname{Hom}_{R^{\mathrm{op}}}(R^{\mathrm{op}(A)}, -)$$

which on applying these isomorphisms to $R^{\mathrm{op}(A)}$ yields that ${}_E E \cong {}_E E^B$.

Using σ^{-1} to endow left E-modules with left F-module structure we have that (by properties of change of rings functors), as left F-modules, ${}_F E \cong {}_F F$ and ${}_F E^B \cong {}_F E$; so that ${}_F F^B \cong {}_F F$.

Now, note that since $S_F^{(B)}$ (as right F-module) is finitely presented then by [**3**, Exercise 2.11.9', p. 337] we have $S^{(B)} \bigotimes_F \prod_B F \cong \prod_B S^{(B)} \bigotimes_F F$. Now

$$S^{(B)} \cong S^{(B)} \bigotimes_F F \cong S^{(B)} \bigotimes_F \prod_B F \cong \prod_B S^{(B)} \bigotimes_F F \cong S^{(B)^B}.$$

But this contradicts Proposition 2.1 and hence A and B cannot be infinite sets simultaneously. ■

Corollary 2.3. *For any two rings R and S, and any infinite sets A and B, the rings* $\mathrm{RFM}_A(R)$ *and* $\mathrm{CFM}_B(S)$ *cannot be isomorphic rings. In fact,* $\mathrm{RFM}_A(R)$ *and* $\mathrm{CFM}_B(S)$ *cannot be Morita equivalent rings.*

Proof: The first statement is a direct consequence of Theorem 2.2. For the second statement, take $E = \mathrm{RFM}_A(R)$ and $F = \mathrm{CFM}_B(S)$. By [**4**, Theorem 1] we know that, if E and F are Morita equivalent rings then there exists $n \in \mathbf{N}$ such that $E \cong \mathbb{M}_n(F)$ as rings; but also $\mathbb{M}_n(F) \cong F$ as rings, because F has SBN. So that $E \cong F$ which is impossible by Theorem 2.2. ■

At present, we have seen that for any two rings R and S the rings $\mathrm{RFM}(R)$ and $\mathrm{CFM}(S)$ cannot be either isomorphic or Morita equivalent. Another possibly relationship between $\mathrm{RFM}(R)$ and $\mathrm{CFM}(S)$ would be the existence of a Morita context. We recall from [**2**, Definition III.1.4, (see p. 51)] that a (left) R-module M is (left) slender if for every homomorphism $\theta : R^{\mathbb{N}} \to M$, we have that, $\theta(e_n) = 0$ for all but finitely many $n \in \mathbb{N}$; where e_n is the element of $R^{\mathbb{N}}$ such that $\mathrm{Supp}(e_n) = \{n\}$ and $\rho_n(e_n) = 1$; where $\rho_n : R^{\mathbb{N}} \to R_n$ are the usual projections.

Example 2.4. The rings $R = \mathrm{RFM}(\mathbb{Z})$, $S = \mathrm{CFM}(\mathbb{Z})$ and bimodules ${}_R M_S = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{(\mathbb{N})}, \mathbb{Z}^{\mathbb{N}})$, ${}_S N_R = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\mathbb{N}}, \mathbb{Z}^{(\mathbb{N})})$ make $\langle M, N\rangle$ a Morita context.

We shall construct a Morita context as in [**1**, Exercise 22.11]. To do this, note that by [**2**, Corollary III.2.3] $\mathbb{Z}$ is slender so that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\mathbb{N}}, \mathbb{Z}) \cong \oplus_{\mathbb{N}} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$ as abelian groups, which implies that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\mathbb{N}}, \mathbb{Z}^{\mathbb{N}}) \cong (\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})^{(\mathbb{N})})^{\mathbb{N}}$. Thus $\mathrm{End}({}_{\mathbb{Z}}\mathbb{Z}^{\mathbb{N}}) \cong \mathrm{CFM}(\mathbb{Z})$ as rings. Now by [**1**, Exercise 22.11(1)] we have that $\langle M, N\rangle$ is a Morita context. ■

Remark. From Example 2.4 it is easy to construct new examples with noncommutative rings, by using [**2**, Corollary III.2.11] which, in particular, states that if R is slender so is $\mathrm{RFM}(R)$.

Note also that, for any slender ring R, even if R is commutative, the ring $\mathrm{RFM}(R)$ is left slender but not right slender because of the isomorphism of right $\mathrm{RFM}(R)$-modules $\mathrm{RFM}(R) \cong \mathrm{RFM}(R)^{\mathbf{N}}$.

In view of Theorem 2.2 we ask for modules which are close to infinitely generated free, but admit endomorphism rings with involution. We will give examples of such modules.

Example 2.5. There exist a ring R and a left infinitely generated projective module ${}_RP$ (which satisfies $P^n \cong P$ for all $n \in \mathbb{N}$) such that the ring $\mathrm{End}({}_RP)$ has an involution.

Let S be any ring and ${}_SQ = {}_SS^{(\mathbb{N})}$. Set $T = \mathrm{End}({}_SQ)$ and $U = T^{\mathrm{op}}$.

Setting $R = S \times U$, $P = Q \times U$ we have that $\mathrm{End}({}_RP) \cong T \times U$ as rings and note that $P \cong P^n$ for every $n \in \mathbb{N}$.

Now let $\delta : T \times U \to T \times U$ such that, for every $(a, b) \in T \times U$, $\delta(a, b) = (b, a)$. It is clear that δ is an involution. ∎

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