Publicacions Matemàtiques, Vol **41** (1997), 209–222.

THE WELL-BEHAVED CATALAN AND BROWNIAN AVERAGES AND THEIR APPLICATIONS TO REAL RESUMMATION

Frédéric Menous

Abstract

The aim of this expository paper is to introduce the well-behaved uniformizing averages, which are useful in resummation theory. These averages associate three essential, but often antithetic, properties: respecting convolution; preserving realness; reproducing lateral growth. These new objects are serviceable in real resummation and we sketch two typical applications: the unitary iteration of unitary diffeomorphisms and the real normalization of real, local, analytic, vector fields.

1. Introduction

This paper introduces the well-behaved uniformizing averages, which answer to the problem of *real* resummation: how to assign a real sum to a real divergent series of "natural origin". We first give some heuristics which point out the difficulties that relate to the real resummation. The well-behaved averages solve these problems and we will explain the action of these objects. We will also present some well-behaved averages. Once these objects are given, we sketch two simple examples of their use (among their large range of application): the unitary iteration of unitary diffeomorphisms and the real normalization of real, local, analytic, vector fields. Most of these results are due to J. Ecalle and a complete exposition can be found in [**1**].

2. Some heuristics. The need for well-behaved averages

2.1. The resummation scheme.

All the series and functions in z introduced here are considered at infinity.

Let $\tilde{\varphi}(z)$ be a real (with real coefficients) divergent series of "natural origin": for instance the formal solution of a local analytic equation or system:

$$E(\tilde{\varphi}) = 0. \tag{1}$$

The most simple resummation scheme for resumming $\tilde{\varphi}(z)$ goes like this:

$$\begin{array}{ccc} \widetilde{\varphi}(z) & \longrightarrow & \varphi(z) \\ & \searrow \quad \nearrow & \\ & \hat{\varphi}(\zeta) & \end{array} \tag{2}$$

We begin by subjecting $\widetilde{\varphi}(z)$ to the formal Borel transform (to obtain $\hat{\varphi}(\zeta)$) which, for instance, turns each monomial $z^{-\sigma}$ into $\zeta^{\sigma-1}/\Gamma(\sigma)$ $(\sigma > 0)$.

Then we carry out a Laplace transform:

$$\hat{\varphi}(\zeta) \longrightarrow \varphi(z) = \int_0^{+\infty} e^{-z\zeta} \hat{\varphi}(\zeta)\, d\zeta. \tag{3}$$

This procedure for turning the formal object $\widetilde{\varphi}(z)$ into a geometric one $\varphi(z)$ is the most simple one, but it is already representative of the difficulties arising from the need for a *real* resummation.

Although the move $\hat{\varphi}(\zeta) \mapsto \varphi(z)$ seems to be one single step, it actually involves three distinct substeps.

2.2. Three steps in one.

(i) First substep: calculating a germ. The function $\hat{\varphi}(\zeta)$ is obtained, by the Borel transform, as a germ near $\zeta = +0$ and, generally speaking, it converges only for small enough values of ζ.

(ii) Second substep: getting a global function. We must continue this germ from $+0$ to $+\infty$ so as to get a global function, which could be Laplace transformed. This will be generally possible by analytic continuation, owing to the "natural origin" of $\tilde{\varphi}(z)$.

(iii) Third substep: uniformizing the global function. Although there are no obstacles to analytic continuation, there may be analytic singularities. Indeed, the existence of singularities in the ζ-plane is precisely what causes the divergence of $\tilde{\varphi}$. There are often compelling reasons for them to be located on $\mathbf{R}^+$. Then $\hat{\varphi}$ is *multivalued* (many-branched) over $\mathbf{R}^+$ (its determination depends on the choice of a path of analytic continuation and on the way this one dodges the singularities.). If so, we must turn $\hat{\varphi}(\zeta)$, "in some suitable way", into a *univalued* (uniform) function $(\mathbf{m}\hat{\varphi})(\zeta)$, so as to be able to carry out the Laplace transform.

2.3. What does "suitable way" mean?.

If we assume that $\hat{\varphi}(\zeta)$ is multivalued, we will turn it into a uniform function by making an average of the different analytic continuations of $\hat{\varphi}$ and then, the difficulty relates to the choice of a "suitable" (or well-behaved) uniformizing average $\mathbf{m}$. Here, suitable means three things:

P1: $\mathbf{m}$ must respect convolution: it is indispensable in all non-linear situations that $\mathbf{m}$ turns convolution into convolution. The Borel and Laplace transforms are algebra homomorphisms. Thus our average $\mathbf{m}$ must be an algebra homomorphism (for the convolution of "ramified" functions and the convolution of "uniform" functions) so as to assign to $\tilde{\varphi}(z)$ a sum $\varphi(z)$ which is also a solution of the original equation.

P2: $\mathbf{m}$ must respect realness: this is rather necessary if $\tilde{\varphi}(z)$ has real coefficients and if we want to assign a real sum for some compelling reason: for example, if it represents a physical or real-geometric object.

P3: $\mathbf{m}$ must respect the lateral growth: for a series $\tilde{\varphi}$ of natural origin, $\hat{\varphi}$ displays, generally speaking, the good growth rate (that is to say exponential growth) which allows to carry out the Laplace transform. But that statement must be restricted. In fact, this exponential growth is obtained only:

- on singularity-free axes Γ_θ (from 0 to infinity in the direction θ).
- on both sides (right and left) of a singularity-carrying axis.

If $\hat{\varphi}(\zeta)$ has singularities on $\mathbf{R}^+$, this exponential growth is, generally speaking, also ensured on paths Γ which are close to the positive axis and cross it only a finite number of times. But, if $\mathbf{R}^+$ carries infinitely many singularities, on paths Γ following $\mathbf{R}^+$, but with infinitely many crossings, the function $\hat{\varphi}(\zeta)$ often has *faster-than-exponential* growth:

$$| \hat{\varphi}(\zeta) | \leq c_0 \exp(c_1(| \zeta | + | \zeta | \log | \zeta |)). \tag{4}$$

Unfortunately, the uniformizing averages which are P1 and P2 will involve the analytic continuations of $\hat{\varphi}(\zeta)$ on such "often-crossing" paths.

Thus we must carefully choose the average $\mathbf{m}$ such that $(\mathbf{m}\hat{\varphi})(\zeta)$ has a no-faster-than-exponential growth ($|\ (\mathbf{m}\hat{\varphi})(\zeta)\ | \leq c_0 \exp(c_1\ |\ \zeta\ |)$).

Now an average $\mathbf{m}$ will be called a "well-behaved" uniformizing average if the three properties P1, P2, P3 hold.

In order to define and present such well-behaved averages, we need to introduce the convolution algebras of resurgent functions. But, for the sake of simplicity, we will restrict ourselves to the definition of the convolution algebra $RESUR(\mathbf{R}^+//\mathbf{N}, int.)$ of resurgent functions, with singularities over $\mathbf{N}^*$ and which are locally integrable. Nonetheless, the following statements can be extended to more general convolution algebras (with some different set of singularities and without the condition of local integrability).

2.4. The algebra $RESUR(\mathbf{R}^+//\mathbf{N}, int.)$.

Definition. The algebra $RESUR(\mathbf{R}^+//\mathbf{N}, int.)$ is defined as follows. Let $\hat{\varphi}(\zeta)$ be an element of this algebra, then:

- $\hat{\varphi}(\zeta)$ is defined and holomorphic at the root of $\mathbf{R}^+$ (on $]0, \epsilon[$).
- $\hat{\varphi}(\zeta)$ is analytically continuable along any path that follows $\mathbf{R}^+$ and dodges each point of $\mathbf{N}^*$ to the left or to the right, but without ever going back.
- All the determinations of $\hat{\varphi}(\zeta)$ are locally integrable on $\mathbf{R}^+$.

Moreover, the convolution is defined by:

$$\hat{\varphi}_3(\zeta) = (\hat{\varphi}_1 * \hat{\varphi}_2)(\zeta) = \int_0^\zeta \hat{\varphi}_1(\zeta_1)\, \hat{\varphi}_2(\zeta - \zeta_1)\, d\zeta_1 \quad (0 < \zeta \ll 1) \tag{5}$$
$$(\hat{\varphi}_1, \hat{\varphi}_2 \in RESUR(\mathbf{R}^+//\mathbf{N}, int.)).$$

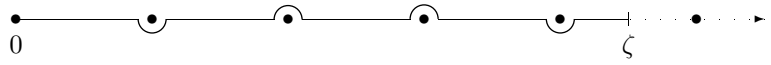
This expression is purely local (at $\zeta = 0$) and the germ $\hat{\varphi}_3(\zeta)$ must then be extended, by analytic continuation, to a global function.

For details see [**1**].

Now, for a function of $RESUR(\mathbf{R}^+//\mathbf{N}, int.)$, we can give the following notation:

Let $\hat{\varphi}(\zeta)$ be a function of $RESUR(\mathbf{R}^+//\mathbf{N}, int.)$ and $(\varepsilon_1, \ldots, \varepsilon_n)$ be a sequence of n plus or minus signs, then, for ζ in $]n, n+1[$, we will note $\hat{\varphi}^{\varepsilon_1,\cdots,\varepsilon_n}(\zeta)$ the analytic continuation of $\hat{\varphi}$ from 0 to ζ on the path that follows $\mathbf{R}^+$ and dodges each singularity k $(1 \leq k \leq n)$ to the *right* (resp. to the *left*) if $\varepsilon_k = +$ (resp. $\varepsilon_k = -$).

Example. If $\zeta \in]4,5[$, then $\hat{\varphi}^{+,-,-,+}(\zeta)$ is the analytic continuation of $\hat{\varphi}$ along the following path:



Of course, $\hat{\varphi}^{\emptyset}(\zeta)$ $(O < \zeta < 1)$ is the unique determination of $\hat{\varphi}$ on $]0,1[$.

Once this notation is given, for any fixed integer n, a function $\hat{\varphi}$ of $RESUR(\mathbf{R}^+//\mathbf{N}, int.)$ has 2^n possibly different determinations $\hat{\varphi}^{\varepsilon_1,\cdots,\varepsilon_n}(\zeta)$ *over* the interval $]n, n+1[$ and a uniformizing average $\mathbf{m}$ will return an actual average of these 2^n determinations.

2.5. The uniformizing averages.

A uniformizing average $\mathbf{m}$ is a uniformizing projection of the space $RESUR(\mathbf{R}^+//\mathbf{N}, int.)$ into the space $UNIF(\mathbf{R}^+, int.)$ of uniform, locally integrable functions on $\mathbf{R}^+$.

It can be define as a collection of "weights":

$$\mathbf{m} = \{\mathbf{m}^{\varepsilon_1,\cdots,\varepsilon_n};\ n \in \mathbf{N};\ \varepsilon_i = \pm;\ \mathbf{m}^{\varepsilon_1,\cdots,\varepsilon_n} \in \mathbf{C}\} \tag{6}$$

subject to the self-consistency relations:

$$\sum_{\varepsilon_n=\pm} \mathbf{m}^{\varepsilon_1,\cdots,\varepsilon_n} = \mathbf{m}^{\varepsilon_1,\cdots,\varepsilon_{n-1}} \text{ (resp. } \mathbf{m}^{\emptyset} = 1) \text{ if } n > 1 \text{ (resp. } n = 1) \tag{7}$$

and the action of the average $\mathbf{m}$ on a function $\hat{\varphi}$ of $RESUR(\mathbf{R}^+//\mathbf{N}, int.)$ is defined as follows:

$$\forall n \in \mathbf{N};\ \forall \zeta \in]n, n+1[\quad (\mathbf{m}\hat{\varphi})(\zeta) = \sum_{\varepsilon_1=\pm\cdots\pm\varepsilon_n=\pm} \mathbf{m}^{\varepsilon_1,\cdots,\varepsilon_n} \hat{\varphi}^{\varepsilon_1,\cdots,\varepsilon_n}(\zeta). \tag{8}$$

Thus an average $\mathbf{m}$ turns a multivalued function into a uniform one by averaging its different determinations and it is important to point out that the self-consistency relations are a necessity: for instance, whenever a function $\hat{\varphi}$ of $RESUR(\mathbf{R}^+//\mathbf{N}, int.)$ has only *fictive* singularities, that is to say $\hat{\varphi}$ is uniform, then we would like to obtain $\mathbf{m}\hat{\varphi} = \hat{\varphi}$, which is ensured by the self-consistency relations.

Once these definitions are given, there exists more precise statements for the properties P1, P2, P3:

P1: An average $\mathbf{m}$ respects convolution if and only if, for any two functions $\hat{\varphi}$ and $\hat{\psi}$ in $RESUR(\mathbf{R}^+//\mathbf{N}, int.)$:

$$\mathbf{m}(\hat{\varphi} * \hat{\psi}) = (\mathbf{m}\hat{\varphi}) * (\mathbf{m}\hat{\psi}) \tag{9}$$

where the first star $*$ (resp. the second) denotes the convolution on $RESUR(\mathbf{R}^+//\mathbf{N}, int.)$ (resp. on $UNIF(\mathbf{R}^+, int.)$).

This condition is ensured if and only if the weights of $\mathbf{m}$ verify a universal multiplication table which reads, for example:

$$(10)\qquad \begin{cases} \mathbf{m}^{+}\mathbf{m}^{+} = \mathbf{m}^{+,+} - \mathbf{m}^{-,+} \\ \mathbf{m}^{+}\mathbf{m}^{-} = \mathbf{m}^{+,-} + \mathbf{m}^{-,+} \\ \mathbf{m}^{-}\mathbf{m}^{-} = \mathbf{m}^{-,-} - \mathbf{m}^{+,-} \\ \mathbf{m}^{+}\mathbf{m}^{+,+} = \mathbf{m}^{+,+,+} - \mathbf{m}^{+,-,+} - \mathbf{m}^{-,+,+} \\ \qquad\qquad \vdots \end{cases}$$

For proofs and complements, see [**1**], [**4**].

P2: The fact that an average $\mathbf{m}$ respects realness can easily be read on its weights. Let $\hat{\varphi}$ be a function of $RESUR(\mathbf{R}^+//\mathbf{N}, int.)$ and let us assume that $\hat{\varphi}$ is the formal Borel transform of a real divergent series. Then $\hat{\varphi}(\zeta)$ is *real* for small enough real values of ζ and assumes complex conjugate values on complex conjugate paths of analytic continuation:

$$(11)\quad \forall n \in \mathbf{N};\ \forall \zeta \in]n, n+1[;\ \forall \varepsilon_i \in \{+,-\}\ \hat{\varphi}^{\varepsilon_1,\cdots,\varepsilon_n}(\zeta) = \overline{\hat{\varphi}^{\bar{\varepsilon}_1,\cdots,\bar{\varepsilon}_n}(\zeta)}$$

where $\bar{\varepsilon}_i$ is the opposite sign to ε.

Therefore, a uniformizing average $\mathbf{m}$ respects realness ($(\mathbf{m}\hat{\varphi})$ is real on $\mathbf{R}^+$) if and only if:

$$(12)\qquad \mathbf{m}^{\varepsilon_1,\cdots,\varepsilon_n} = \overline{\mathbf{m}}^{\bar{\varepsilon}_1,\cdots,\bar{\varepsilon}_n} \quad (\forall n \geq 0;\ \forall \varepsilon_i \in \{+,-\}).$$

P3: Although this appears to be the main demand, we won't go into details in this introductory paper (see [**1**]). This condition does not reduce to growth conditions on the weights. It actually involves some compensation phenomena. This faster-than-exponential growth on "often-crossing" paths is a precise mechanism, which has to do with the nature of the "acting alien algebra". But, whenever a function $\hat{\varphi}$ is of "natural origin", the analysis of this nuisance (with the "Bridge equation") shows how to construct, independently of $\hat{\varphi}$ itself, "well-behaved" averages $\mathbf{m}$ which produce mean values $\mathbf{m}\hat{\varphi}$ with the requisite exponential growth.

These three properties tend to be mutually exclusive but the main fact is that such well-behaved averages exist. We shall give now some examples of uniformizing averages (well-behaved or not).

3. Examples of uniformizing averages

3.1. The right-lateral average mur and the left-lateral average mul.

It can be proved that they are the only convolution-preserving averages that involve only one determination over each interval. In term of weights:

(13) $\mathbf{mur}^{\varepsilon_1,\dots,\varepsilon_n} = 1$ (resp. 0) if $\varepsilon_1 = \varepsilon_2 = \dots = +$ (resp. otherwise)

(14) $\mathbf{mul}^{\varepsilon_1,\dots,\varepsilon_n} = 1$ (resp. 0) if $\varepsilon_1 = \varepsilon_2 = \dots = -$ (resp. otherwise)

mur and **mul** are P1 (convolution) and P3 (lateral growth) but clearly fail to preserve realness.

3.2. The median average mun.

The weights of **mun** depend only on the number p (resp. q) of $+$ signs (resp. $-$ signs) in the address $(\varepsilon_1, \dots, \varepsilon_n)$:

$$(15) \qquad \mathbf{mun}^{\varepsilon_1,\dots,\varepsilon_n} \equiv \frac{\Gamma(p+1/2)\,\Gamma(q+1/2)}{\Gamma(1/2)\Gamma(p+q+1)\Gamma(1/2)} \equiv \frac{(2p)!\,(2q)!}{4^{p+q}\,(p+q)!\,p!\,q!}$$

and **mul**, **mur** and **mun** can be embedded in an interval of averages (depending also on the number of $+$ and $-$ signs) $\mathbf{mu}_{\alpha,\beta}$ with weights:

$$(16) \quad \mathbf{mu}_{\alpha,\beta}^{\varepsilon_1,\dots,\varepsilon_n} \equiv \frac{\Gamma(p+\alpha)\,\Gamma(q+\beta)}{\Gamma(\alpha)\Gamma(p+q+1)\Gamma(\beta)} \quad (\alpha,\beta \in \mathbf{R}^+;\ \alpha+\beta=1)$$

$$(17) \qquad \mathbf{mu}_{1,0} = \mathbf{mur};\ \mathbf{mu}_{1/2,1/2} = \mathbf{mun};\ \mathbf{mu}_{0,1} = \mathbf{mul}.$$

The averages $\mathbf{mu}_{\alpha,\beta}$ respect convolution (P1). Only **mun** respects realness but only **mur** and **mul** respect lateral growth: no one of these averages is "well-behaved".

3.3. The Catalan average man.

The weights of **man** assume rational values, and can be obtained by the following formula:

$$(18) \qquad \mathbf{man}^{\varepsilon_1,\dots,\varepsilon_n} \equiv 4^{-n} ca_{n_1} ca_{n_2} \dots ca_{n_s}\,(1+n_s)$$

with the classical Catalan numbers:

$$(19) \qquad ca_n \stackrel{\text{def}}{=} \frac{(2n)!}{n!\,(n+1)!} \quad (ca_n \in \mathbf{N})$$

which in this case are indexed by the integers $n_1, n_2, \dots, n_s$ which denote the numbers of identical consecutive signs within the address $(\varepsilon_1, \dots, \varepsilon_n)$:

(20) $(\varepsilon_1, \dots, \varepsilon_n) = (\pm)^{n_1}(\mp)^{n_2} \dots (\varepsilon_n)^{n_s}$ (of course $n_1 + \dots + n_s = n$).

Like **mun**, the Catalan average **man** may be embedded in an interval of averages $\mathbf{ma}_{\alpha,\beta}$ (as usual $\alpha + \beta = 1$) with weights:

$$\mathbf{ma}_{\alpha,\beta}^{\varepsilon_1,\dots,\varepsilon_n} \stackrel{\text{def}}{=} (\alpha\beta)^n (ca_{n_1} ca_{n_2} \dots ca_{n_{s-1}}) Ca_{n_s}((\alpha/\beta)^{\varepsilon_n}). \tag{21}$$

Once again n_i denotes the cardinality of the i^{th} cluster of identical signs. The new formula, however, alongside with the Catalan numbers ca_n, also involves the Catalan polynomials Ca_n, which are distinguished by a capital C and inductively definable by:

$$Ca_0(x) = 1 \tag{22}$$

$$Ca_{1+n}(x) = -(1 + x^{-1})ca_n + (x + 2 + x^{-1})Ca_n(x). \tag{23}$$

All negative powers of x cancel out, and it may be noted that:

$$Ca_n(0) = ca_n;\ Ca_n(1) = (1 + n)ca_n \tag{24}$$

$$\lim_{x \to -1} (x + 1)^{-1} Ca_n(x) = ca_{n-1}. \tag{25}$$

The Catalan average **man** is interesting because it is a *well-behaved uniformizing average* (P1, P2, P3). Moreover, under a "rescaling" and a "passage to the limit", the Catalan average gives rise to the so-called Brownian average **mown**. For details, see [**5**], [**1**], [**4**].

3.4. The Brownian average mown.

If we define, for any positive integers m_i:

$$\mathbf{man}^{m_1,\varepsilon_1,\dots,m_n,\varepsilon_n} \stackrel{\text{def}}{=} \sum_{\eta_k^i = \pm} \mathbf{man}^{\eta_1^1 \dots \eta_{m_1-1}^1 \varepsilon_1 \eta_1^2 \dots \eta_{m_2-1}^2 \varepsilon_2 \dots \varepsilon_{n-1} \eta_1^n \dots \eta_{m_n-1}^n \varepsilon_n} \tag{26}$$

then the following limits exist for any address $(\varepsilon_1, \dots, \varepsilon_n)$ and define the weights of an average:

$$\mathbf{mown}^{\varepsilon_1,\dots,\varepsilon_n} = \lim_{m \to +\infty} \mathbf{man}^{m,\varepsilon_1,\dots,m,\varepsilon_n}. \tag{27}$$

These weights determine the Brownian average **mown**, which is also a well-behaved uniformizing average. For **man** and **mown** this point follows from their direct study (by using the family $\mathbf{ma}_{\alpha,\beta}$ [**5**]). But this also follows from the fact that these averages belong to a larger set of well-behaved convolution averages, introduced and studied by J. Ecalle: "The averages induced by a diffusion".

3.5. Averages induced by a diffusion.

We fix an integrable function f on $\mathbf{R}$ such that:

$$\int_{-\infty}^{+\infty} f(x)\,dx = 1. \tag{28}$$

The function f may be viewed as representing the probability distribution at the time $t = 1$, on the vertical axis $1+i\mathbf{R}$, of a particle starting from the origin at $t = 0$, moving along $\mathbf{R}^+$ with unit speed, and diffusing randomly in the vertical direction. To any such "diffusion", we may associate a uniformizing average $\mathbf{m}$ with weights defined as follows:

Definition (J. Ecalle). $\mathbf{m}^{\varepsilon_1,\cdots,\varepsilon_n}$ is the probability for the particle to hit the half-axis $n + i\varepsilon_n\mathbf{R}^+$ at the time n after successively crossing each half-axis $j + i\varepsilon_j\mathbf{R}^+$ $(1 \leq j < n)$ at the time j.

Analytically, this translates into the following formula:

$$\mathbf{m}^{\varepsilon_1,\cdots,\varepsilon_n} \tag{29}$$
$$= \int f(x_1)\ldots f(x_n)\sigma_{\varepsilon_1}(x_1)\sigma_{\varepsilon_2}(x_1+x_2)\ldots\sigma_{\varepsilon_n}(x_1+\cdots+x_n)\,dx_1\ldots dx_n$$

with integration over $\mathbf{R}^n$ and with the classical step functions σ_+ and σ_-:

$$\sigma_\pm(x) \equiv 1\,(\text{resp. } 0) \text{ if } \pm x > 0\,(\text{resp. } \pm x \leq 0). \tag{30}$$

J. Ecalle proved that any average induced by a diffusion respects both convolution and lateral growth. Moreover, as soon as the function f is even ($f(x) = f(-x)$), the average also respects realness and thus is a well-behaved uniformizing average (for details see [**1**]).

Now it can be checked that both **man** and **mown** are induced by a diffusion:

$$\textbf{man} \text{ induced by } f(x) = \frac{1}{2}\exp(-\mid x \mid) \tag{31}$$

$$\textbf{mown} \text{ induced by } f(x) = \frac{1}{2\sqrt{\pi}}\exp\left(-\frac{x^2}{4}\right). \tag{32}$$

The construction and the study of the averages induced by a diffusion are due to J. Ecalle. Moreover, this method for building well-behaved averages allows to *generalize* them, so that they can act on resurgent functions having their singularities in any discrete additive semi-group of $\mathbf{R}^+$. J. Ecalle also proved that the condition of local integrability, for the resurgent functions, is not a necessity. Thus the well-behaved averages have a great range of application and we will present, in the two following sections, some illustrations of the use of such averages (for details, see [**1**] ,[**4**]).

4. First illustration: unitary iteration of unitary diffeomorphisms

A local diffeomorphism U of $\mathbf{C}$ is said to be *unitary* if it is reciprocal to its own complex conjugate:

$$\{U \text{ unitary}\} \Longleftrightarrow \left\{U \circ \overline{U} = \mathrm{id}\right\} \quad (U, \overline{U} : \mathbf{C} \mapsto \mathbf{C};\ \overline{U}(z) = \overline{U(\bar{z})}). \tag{33}$$

We shall focus for simplicity on unitary and identity-tangent diffeomorphisms of $\mathbf{C}$. Moreover, it will be convenient to consider diffeomorphisms that are formally conjugate to the shift $T_{2i\pi}$ ($T_{2i\pi} : z \mapsto z + 2i\pi$). Thus we may consider:

$$U : z \mapsto U(z) = z + 2\pi i + \sum_{n\geq 2} a_n z^{-n} \quad (z \sim \infty;\ a_n \in \mathbf{C}) \tag{34}$$

$$U = ({}^*\widetilde{U}) \circ (T_{2\pi i}) \circ (\widetilde{U}^*) \text{ and } ({}^*\widetilde{U}) \circ (\widetilde{U}^*) = \mathrm{id} \tag{35}$$

with

$$U(z) \in z + \mathbf{C}\{z^{-1}\};\ \widetilde{U}^*(z) \text{ and } {}^*\widetilde{U}(z) \text{ both in } z + \mathbf{R}[[z^{-1}]]. \tag{36}$$

These unitary diffeomorphisms have a geometric meaning because they are the holonomy diffeomorphisms of some real resonant vector fields.

The mapping U is unitary if and only if the formal power series $\widetilde{U}^*$ and ${}^*\widetilde{U}$ are real and, in this case, the formal iterates of *real* order w:

$$\widetilde{U}^{\circ w} \stackrel{\text{def}}{=} ({}^*\widetilde{U}) \circ (T_{2\pi i w}) \circ (\widetilde{U}^*) \quad (\text{with } T_{2\pi i w} \stackrel{\text{def}}{=} z + 2\pi i w) \tag{37}$$

are themselves formally unitary.

If we want to resum $\widetilde{U}^{\circ w}$ to a unitary germ $U^{\circ w}$ (for $\mathrm{Re}(z) > x_0$), we have to assign *real* sums U^* and ${}^*U = (U^*)^{-1}$ to the real formal series $\widetilde{U}^*$ and ${}^*\widetilde{U}$. But J. Ecalle proved that $\widetilde{U}^*$ and ${}^*\widetilde{U}$ are generically divergent and resurgent (see [**2**]) and that their Borel transform $\hat{U}^*(\zeta)$ and ${}^*\hat{U}(\zeta)$ have ramified singularities over $\mathbf{Z}^*$. As in the previous resummation scheme, the difficulty stems from the singularities over $\mathbf{R}^+$, that obstruct the Laplace integration on the positive axis (which is a necessity to get real germs U^* and ${}^*U$ for $\mathrm{Re}(z) > x_0$), and from the faster-than-exponential growth which generically occurs on "often-crossing" paths.

We could apply the Laplace transform to the lateral determinations **mur** or **mul**, but this would yield imaginary parts and $U^{\circ w}$ would not be unitary. We could also try to use their half-sum $(\mathbf{mur} + \mathbf{mul})/2$ but

this average does not respect the convolution and then the sums that are obtained for $\widetilde{U}^*$ and ${}^*\widetilde{U}$ cannot be solutions of the *non-linear* equation:

$$U = {}^*U \circ T_{2i\pi} \circ U^* \quad ({}^*U \circ U^* = \mathrm{Id}). \tag{38}$$

In the same way, we cannot use the median average **mun** (which respects both convolution and realness), because the faster-than-exponential growth prevents us from carrying out the Laplace transform.

The Laplace transform can be applied only to "well-behaved" averages, which simultaneously respect convolution, realness and lateral growth:

For any given well-behaved average **m**, there exists a half plane $\mathrm{Re}(z) > x_0$ such that $\widetilde{U}^*$ and ${}^*\widetilde{U}$ can be resummed to *real* germs U^* and ${}^*U$ (for $\mathrm{Re}(z) > x_0$) and this is done by applying the Laplace transform to the averages $(\mathbf{m}\hat{U}^*)(\zeta)$ and $(\mathbf{m}^*\hat{U})(\zeta)$. Thus, for any real number w, it can be assigned a unitary germ $U^{\circ w}$ (unitary iterate of real order w) to the formal iterate $\widetilde{U}^{\circ w}$ (as soon as $\mathrm{Re}(z)$ is large enough).

Note that these germs depend on the choice of the average **m** (for details, see [**1**], [**4**]).

The second illustration of the need for well-behaved averages is the example of the real normalization of real vector fields.

5. Second illustration: real normalization of real vector fields

For the sake of simplicity, we will consider a real resonant vector field (saddle-node):

$$X = (x_1 + g(x_1, x_2))\partial_{x_1} - (x_2)^2\partial_{x_2} \quad (g(x_1, x_2) \in x_1x_2\mathbf{R}\{x_1, x_2\}) \tag{39}$$

which is formally conjugate to the vector field:

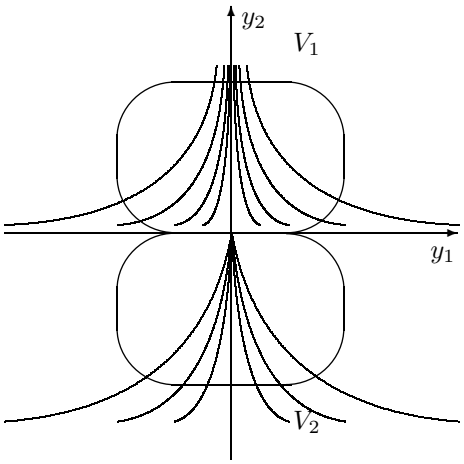
$$Y = y_1\partial_{y_1} - (y_2)^2\partial_{y_2}. \tag{40}$$

The real formal normalizing map:

$$(y_1, y_2) \mapsto (x_1 = \tilde{\varphi}(y_1, y_2), x_2 = y_2) \tag{41}$$

is generically divergent but also resurgent in the infinitely large variable $z = 1/y_2$ ($\tilde{\varphi}(y_1, y_2) = \tilde{\varphi}(y_1, z^{-1})$) and the formal Borel transform $z^{-1} \mapsto \zeta$ will generically produce singularities over $\mathbf{Z}$. The formal normalizing

map can be resummed (see [**3**]) in small enough neighborhoods V_1 and V_2:



but these first sectorial resummations are not real (because the averages **mul** and **mur** were applied), that is to say, if we consider X and Y as complex vector fields, these sectorial resummations won't preserve the real plane.

If we want resummation to convert the real formal normalization into a *real* sectorial one, uniformizing averages are needed:

- On small enough neighborhoods of type V_1, the well-behaved averages are a necessity (because of the singularities on $\mathbf{R}^+$ in the ζ-plane) and each one yields a *real* sectorial normalization that conjugates the real vector fields X and Y on V_1.
- On small enough neighborhoods of type V_2, the structure of the real negative singularities is very simple and then any average **m** that preserves convolution and realness (but not necessarily the lateral growth) will yield a real sectorial resummation. These results are due to J. Ecalle (see [**1**]).

We can note that, once again, these real sectorial normalizations depend on the choice of the uniformizing average. But there is nothing wrong about this multiplicity of sums. The flow of Y is parameterized by:

$$\begin{cases} y_1 = ue^z \\ y_2 = 1/z \end{cases} \quad (u \in \mathbf{R};\ z \in \mathbf{R}^*). \tag{42}$$

On the upper half plane ($y_2 > 0$):

Let $\mathbf{m}_1$ and $\mathbf{m}_2$ be two well-behaved uniformizing averages and V_1 a small enough neighborhood such that $\mathbf{m}_1$ and $\mathbf{m}_2$ yield two real sectorial resummations of the formal normalizing map. Using the (z,u) parameterization of the flow of Y, these two sectorial normalizations give rise to some (z,u) real parameterizations of the flow of X. For $(ue^z, 1/z) \in V_1$ $(-\rho < ue^z < \rho;\ 0 < z^{-1} < \rho)$:

$$\mathbf{m}_1 : \begin{cases} x_1 = {}^{\mathbf{m}_1}x(z,u) \\ x_2 = 1/z \end{cases} \tag{43}$$

$$\mathbf{m}_2 : \begin{cases} x_1 = {}^{\mathbf{m}_2}x(z,u) \\ x_2 = 1/z. \end{cases} \tag{44}$$

Thus each average $\mathbf{m}_i$ yields a real local (z,u) parameterization of the flow of X. But there is no ambiguity: once $\mathbf{m}_1$ and $\mathbf{m}_2$ are fixed, there exists a germ $f(u)$ (depending on $\mathbf{m}_1$, $\mathbf{m}_2$ and X and $f(u) \in u+u^2\mathbf{R}\{u\}$) such that:

$${}^{\mathbf{m}_1}x(z,u) = {}^{\mathbf{m}_2}x(z,f(u)). \tag{45}$$

The choice of a well-behaved average simply implies a choice of the parameterization of the space of trajectories of X, for x_1, x_2 small enough and x_2 positive. Note that for each well-behaved average, the "middle trajectory" (corresponding to $y_1 = 0$, $y_2 = 1/z$, $z > 0$) remains unchanged:

$${}^{\mathbf{m}_1}x(z,0) = {}^{\mathbf{m}_2}x(z,0). \tag{46}$$

On the lower half plane $(y_2 < 0)$:

Once again, on a small enough neighborhood V_2, two averages $\mathbf{m}_1$ and $\mathbf{m}_2$ of type P1, P2 will yield two possibly different real sectorial resummations of the formal normalizing map. Using the same notations, each average $\mathbf{m}_i$ yields a real local (z,u) parameterization of the flow of X. But there is no ambiguity: once $\mathbf{m}_1$ and $\mathbf{m}_2$ are fixed, there exists a real constant c (depending on $\mathbf{m}_1$, $\mathbf{m}_2$ and X) such that:

$${}^{\mathbf{m}_1}x(z,u) = {}^{\mathbf{m}_2}x(z,u+c). \tag{47}$$

This simply means that, for the trajectories of X in the lower half plane, there is no clear candidate for the role of "middle trajectory" because we could have:

$${}^{\mathbf{m}_1}x(z,0) \neq {}^{\mathbf{m}_2}x(z,0). \tag{48}$$

6. Conclusion

These two illustrations are representative of the usefulness of the well-behaved uniformizing averages. These objects can be extended such that they act on more general algebras of resurgent functions. Their field of application is potentially quite vast, since it covers most of the situations characterized by a combination of (1) non-linearity, (2) divergence, (3) realness. The reader could refer to [**1**] for a complete exposition on the well-behaved averages and their applications to real resummation.

References

1. J. ECALLE, Well-behaved uniformizing averages and their applications to real resummation (to appear).
2. J. ECALLE, Les fonctions résurgentes, *Publ. Math. Orsay* **2** (1981).
3. J. ECALLE, Les fonctions résurgentes, *Publ. Math. Orsay* **3** (1985).
4. J. ECALLE AND F. MENOUS, Well-behaved convolution averages and the non-accumulation theorem for limit-cycles, in "*The stokes phenomenon and Hilbert's* 16^{th} *problem*," B. L. J. Braaksma, G. K. Immink, M. van der put, eds., World Scientific, 1996.
5. F. MENOUS, Les bonnes moyennes uniformisantes et leurs applications à la resommation réelle, PhD thesis (1996).

Laboratoire d'Analyse Harmonique
Université d'Orsay
Orsay
FRANCE

e-mail: menous@ens.fr

Primera versió rebuda el 30 de Novembre de 1996,
darrera versió rebuda el 5 de Febrer de 1997