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ON THE SELF-INTERSECTION LOCAL TIME OF BROWNIAN MOTION-VIA CHAOS EXPANSION

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Abstract

We discuss the weak compactness problem related to the self-intersection local time of Brownian motion. We also propose a regular renormalization for self-intersection local time of higher dimensional Brownian motion.

1. Introduction

Let $B_t = (B_t^1, \dots, B_t^d)$, $0 \le t < \infty$ be the d-dimensional standard Brownian motion starting at 0 and let $(\Omega, \mathcal{F}, W)$ be the associated canonical probability space with the natural filtration $(\mathcal{F}_t)_{t\ge 0}$. Let $T > 0$ be fixed throughout this paper. The following formal expression is called the self-intersection local time of the Brownian motion:

$$\int_0^T \int_0^t \delta(B_t - B_s)\, ds\, dt, \tag{1.1}$$

where δ is the Dirac delta function on $\mathbb{R}^d$ and (1.1) is interpreted as the limit $\int_0^T \int_0^t P_\varepsilon(B_t - B_s)\, ds\, dt$ as $\varepsilon \to 0$, where P_ε is the heat kernel. There are many references on it. In recent years there have been some results on the smoothness and renormalization of (1.1). When $d = 2$, Nualart and Vives [**NV92**] (see also [**NV94**]) proved that

$$\int_0^T \int_0^t \delta(B_t - B_s)\, ds\, dt - \mathbb{E}\left\{\int_0^T \int_0^t \delta(B_t - B_s)\, ds\, dt\right\} \tag{1.2}$$

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is in $D_{\alpha,2}$ for all $\alpha < 1/2$, where $D_{\alpha,2}$ is the so-called Meyer-Watanabe distribution space. It was proved later by Imkeller, Pérez-Abreu and Vives [**IPV93**] that (1.2) is in $D_{\alpha,2}$ for all $\alpha < 1$. In [**AHZ95**], Albeverio, Hu and Zhou proved that (1.2) is not in $D_{\alpha,2}$ when $\alpha \geq 1$, giving a complete description of the smoothness of (1.2) for $d = 2$. Motivated by a result of Yor [**Yo85**], Imkeller, Pérez-Abreu and Vives [**IPV93**] also proved that when $d = 3$,

$$\frac{1}{\sqrt{\log(1/\varepsilon)}}\left\{\int_0^T\int_0^t P_\varepsilon(B_t - B_s)\,ds\,dt - \mathbb{E}\left(\int_0^T\int_0^t P_\varepsilon(B_t - B_s)\,ds\,dt\right)\right\} \tag{1.3}$$

is weakly compact in $D_{\alpha,2}$ for all $\alpha < 1/2$ and when $d \geq 4$,

$$\varepsilon^{\frac{d-4}{2}}\left\{\int_0^T\int_0^t P_\varepsilon(B_t - B_s)\,ds\,dt - \mathbb{E}\left(\int_0^T\int_0^t P_\varepsilon(B_t - B_s)\,ds\,dt\right)\right\} \tag{1.4}$$

is weakly compact in $D_{\alpha,2}$ for all $\alpha < \frac{4-d}{2}$. In this paper we shall prove that when $d = 3$ (1.3) is unbounded in $D_{\alpha,2}$ when $\alpha \geq 1/2$ and when $d \geq 4$, (1.4) is unbounded in $D_{\alpha,2}$ when $\alpha \geq \frac{4-d}{2}$. As a by-product, we also prove the result of Imkeller, Pérez-Abreu and Vives in a simpler manner (we believe). Since when $d \geq 4$, (1.4) is singular (the element of $D_{\alpha,2}$ is a distribution when $\alpha < 0$), we propose a new renormalization scheme (see (5.2) below) to replace (1.4).

In section 2, we give a quick derivation of the chaos expansion of (1.1) and its approximation that we are going to use throughout this paper. In section 3, we provide some necessary estimates. In section 4, we prove the weak compactness result. In section 5, we propose a new (more regular) renormalization formula.

2. Chaos expansion

We will use the heat kernel to approximate the Dirac delta function δ on $\mathbb{R}^d$, where d is an integer ≥ 3. (We will concern with the case $d \geq 3$. But the approach of this section works also for $d = 1, 2$.) Denote

$$P_t(x) = (2\pi t)^{-d/2} e^{-\frac{|x|^2}{2t}}; \quad P_t(x,y) = P_t(x-y), \quad t > 0, \quad x, y \in \mathbb{R}^d,$$

$$\nabla_i f := \frac{\partial}{\partial x_i} f, \quad P_\varepsilon f(x) = \int_{\mathbb{R}^d} P_\varepsilon(x,y) f(y)\, dy.$$

Remark 2.1. We will make use of the following simple facts without further mention: For f sufficiently regular

$$\frac{\partial}{\partial t} P_t(x,y) = \frac{1}{2} \sum_{i=1}^{d} \nabla_i P_t(x,y); \quad \frac{\partial}{\partial x_i} P_t f = P_t \left\{ \frac{\partial}{\partial x_i} f \right\}, \quad i = 1, \dots, d.$$

$$\int_{\mathbb{R}^d} P_s(x,z) P_t(z,y)\, dz = P_{t+s}(x,y), \quad P_0 f(x) := \lim_{t \to 0} P_t f(x) = f(x).$$

We are going to deduce an explicit chaos expansion for the approximation of the self-intersection local time of Brownian motion. Denote

$$\mathcal{E}_\varepsilon(T) = \int_0^T \int_0^t P_\varepsilon(B_t - B_s)\, ds. \tag{2.1}$$

Given $u \geq s \geq 0$ and $f \in C^\infty$, consider the process

$$X_v := P_{u-v} f(B_v - B_s), \quad s \leq v \leq u.$$

Applying the Itô's formula to the above process, we obtain

$$f(B_u - B_s) = P_{u-s} f(0) + \int_s^u \nabla_i P_{u-u_1} f(B_{u_1} - B_s)\, dB^i_{u_1}, \tag{2.2}$$

where we used the Einstein's convention on summation for $i = 1, \dots, d$, i.e. $a_i b^i := \sum_{i=1}^d a_i b^i$. Taking $f = P_\varepsilon$ in (2.2), we have

$$P_\varepsilon(B_t - B_s) = P_{t-s+\varepsilon}(0) + \int_s^t \nabla_i P_{t-u_1+\varepsilon}(B_{u_1} - B_s)\, dB^i_{u_1}. \tag{2.3}$$

Applying (2.2) to the integrand $\nabla_i P_{t-u_1+\varepsilon}$ in (2.3), noting $\Delta p_t = 0$ and repeating this we obtain

Lemma 2.2.

$$\begin{aligned} P_\varepsilon(B_t - B_s) &= P_{t-s+\varepsilon}(0) + \int_s^t \nabla_i P_{t-u_1+\varepsilon}(0)\, dB^i_{u_1} + \cdots \\ &\quad + \int_{s \leq u_1 < \cdots < u_n \leq t} \nabla_{\alpha_1} \cdots \nabla_{\alpha_n} P_{t-s+\varepsilon}(0)\, dB^{\alpha_1}_{u_1} \cdots dB^{\alpha_n}_{u_n} \\ &\quad + \int_{s \leq u_1 < \cdots < u_{n+1} \leq t} \nabla_{\alpha_1} \cdots \nabla_{\alpha_{n+1}} P_{t-u_1+\varepsilon}(B_{u_1} - B_s) \\ &\qquad\qquad dB^{\alpha_1}_{u_1} \cdots dB^{\alpha_{n+1}}_{u_{n+1}} \\ &= P_{t-s+\varepsilon}(0) + I_1 + \cdots + I_n + I_{n+1}, \end{aligned} \tag{2.4}$$

where

$$I_j := \int_{s\le u_1<\cdots<u_j\le t} \nabla_{\alpha_1}\cdots\nabla_{\alpha_j} P_{t-s+\varepsilon}(0)\, dB_{u_1}^{\alpha_1}\cdots dB_{u_n}^{\alpha_j}, \quad j=1,2,\ldots,n;$$

and

$$I_{n+1} := \int_{s\le u_1<\cdots<u_{n+1}\le t} \nabla_{\alpha_1}\cdots\nabla_{\alpha_{n+1}} P_{t-u_1+\varepsilon}(B_{u_1}-B_s)\, dB_{u_1}^{\alpha_1}\cdots dB_{u_{n+1}}^{\alpha_{n+1}}.$$

For a fixed $0\le j\le n$, I_j is orthogonal to I_k, $k=0,1,\ldots,j-1$, $j+1,\ldots,n+1$, i.e. $\mathbb{E}\big(I_jI_k\big)=0$. So I_n is the n-th Itô-Wiener chaos of $P_\varepsilon(B_t-B_s)$. Now we compute $\nabla_{\alpha_1}\cdots\nabla_{\alpha_n}P_{t-s+\varepsilon}(0)$ in (2.4). First we define for $j=1,\ldots,d$ and $\alpha=(\alpha_1,\ldots,\alpha_{2n})$, we set $2n_j := 2n_j(\alpha) := \#\{\alpha_i;\alpha_i=j\}$ and $n=n_1+\cdots+n_d$. $n_j, j=1,\ldots,d$ are integer or half integer. Using the explicit expression for P_t

$$\begin{aligned} P_{t-s+\varepsilon}(x) &= [2\pi(t-s+\varepsilon)]^{-d/2} e^{-\frac{|x|^2}{2(t-s+\varepsilon)}} \\ &= [2\pi(t-s+\varepsilon)]^{-d/2} \sum_{n=0}^{\infty} \frac{(-1)^n |x|^{2n}}{2^n n!(t-s+\varepsilon)^n}, \end{aligned}$$

we have for n_j, $j=1,\ldots,d$ which are integer

$$\nabla_{\alpha_1}\cdots\nabla_{\alpha_{2n}} P_{t-s+\varepsilon}(0) = \frac{(-1)^n (2n_1)!\cdots(2n_d)!}{(2\pi)^{d/2}2^n n_1!\cdots n_d!(t-s+\varepsilon)^{n+d/2}}. \tag{2.5}$$

Other derivatives vanish at 0. We also say that $\emptyset$ is also a multi-index with length $|\alpha|=0$. We set when $|\alpha|=0$,

$$C_\emptyset = \begin{cases} \dfrac{1}{8\pi^2} & \text{when } d=4; \\ (2\pi)^{-d/2} & \text{when } d\neq 4 \end{cases}$$

and

$$f_\emptyset = \begin{cases} 2\varepsilon^{-1}T - \log\left(\dfrac{T+\varepsilon}{\varepsilon}\right) & \text{when } d=4; \\ 2(d-2)^{-1}\big[2(d-4)^{-1}\big(\varepsilon^{-d/2+2}-(T+\varepsilon)^{-d/2+2}\big) - \varepsilon^{-d/2+1}T\big] & \\ & \text{when } d\neq 4. \end{cases}$$

And we introduce for $|\alpha|\ge 1$,

$$C_\alpha = (-1)^n \frac{(2n_1)!\cdots(2n_d)!}{(2\pi)^{d/2}2^n n_1!\cdots n_d!}(n+d/2-1)^{-1}(n+d/2-2)^{-1} \tag{2.6}$$

and

$$\begin{aligned} f_\alpha^\varepsilon(T)(u_1,\ldots,u_{2n}) = [&-(u_{2n}+\varepsilon)^{-n-d/2+2} + (T+\varepsilon)^{-n-d/2+2} \\ &+ (u_{2n}-u_1+\varepsilon)^{-n-d/2+2} - (T-u_1+\varepsilon)^{-n-d/2+2}]. \end{aligned} \tag{2.7}$$

We will use $\sum_{|\alpha|=2n}$ to denote the sum over all $\alpha=(\alpha_1,\ldots,\alpha_{2n})$ such that $\#\{\alpha_i;\alpha_i=j\}$ is even.

Theorem 2.3. *Using the notations above, we have*

$$\mathcal{E}_\varepsilon(T) = \sum_{n=0}^{\infty} \sum_{|\alpha|=2n} C_\alpha J_\alpha(f_\alpha^\varepsilon(T)), \tag{2.8}$$

where $J_\alpha(f_\alpha^\varepsilon(T))$ *is defined as (see also* [**HM88**]*)*

$$J_\alpha(f_\alpha^\varepsilon(T)) := \int_{0\le u_1<\cdots<u_{2n}\le T} f_\alpha^\varepsilon(T)(u_1,\dots,u_{2n})\, dB_{u_1}^{\alpha_1}\cdots dB_{u_{2n}}^{\alpha_{2n}}. \tag{2.9}$$

Proof: It is obvious that the chaos of odd terms are zero. Let $0 \le u_1 < \cdots < u_{2n} \le T$. Then it is easy to show that

$$\begin{aligned}\int_0^T \int_0^t \int_{s\le u_1<\cdots<u_n\le t} \nabla_{\alpha_1}\cdots\nabla_{\alpha_n} P_{t-s+\varepsilon}(0)\, dB_{u_1}^{\alpha_1}\cdots dB_{u_n}^{\alpha_n} \\ = C_\alpha J_\alpha(f_\alpha^\varepsilon(T)).\end{aligned} \tag{2.10}$$

Since $\int_0^T \int_0^t P_\varepsilon(B_t - B_s)\, ds\, dt$ is in L^2, it admits a chaos expansion according to the Wiener-Itô chaos expansion theorem. Since (2.10) is orthogonal to the remaining term obtained from (2.4), we see that (2.10) is the $2n$-th chaos expansion of $\int_0^T \int_0^t P_\varepsilon(B_t - B_s)\, ds\, dt$. ■

Remark 2.4. Letting $\varepsilon \to 0$, we get a formal expansion of the self-intersection of local time (1.1). This formula was obtained in [**FHSW94**], [**HWYY94**] using the so-called $\mathcal{S}$-transform. It was also obtained in [**AHZ95**] with another simpler technique when $d = 2$ and used to prove a non differentiability theorem. The obtention above seems to be the simplest. Let us also point out that the explicit chaos expansion of $\mathcal{E}_\varepsilon(T)$ is already known in [**NV92**], [**NV94**] and [**IPV93**] in terms of Hermite polynomials. The method used here appeared in [**Hu94**] to obtain the Isobe-Sato formula.

3. Some estimates

Let A_n and B_n, $n = 1, 2, \dots$ be two sequences of real numbers. We denote $A_n \approx B_n$ iff there are two positive constants $p > 0$ and $q > 0$ (independent of n and T) such that $pA_n \le B_n \le qA_n$.

The following result should be found in literature and is stated explicitly in ([**AHZ95**]) when $d = 2$. However, we still give a simple proof.

Lemma 3.1. *Let C_α be given by (2.6). Then*

$$\sum_{|\alpha|=2n} C_\alpha^2 \approx (2n)! n^{\frac{d}{2}-5}. \tag{3.1}$$

Proof: By the Stirling formula $n! = (2\pi)^{1/2} n^{n+1/2} e^{-n}(1+O(1/n))$, i.e. $n! \approx n^{n+1/2}e^{-n}$ we see that

$$\frac{(2n_1)!\cdots(2n_d)!}{2^{2n} n_1!^2 \cdots n_d!^2} \approx (n_1+1)^{-1/2}\cdots(n_d+1)^{-1/2}.$$

(We allow $n_1, \ldots, n_d$ to be 0.) On the other hand it is easy to have

$$\begin{aligned}
&\sum_{n_1+\cdots+n_d=n} (n_1+1)^{-1/2}\cdots(n_d+1)^{-1/2} \\
&\qquad \approx \int_{u_1,\ldots,u_{d-1}\geq 0} (u_1+1)^{-1/2}\cdots(n_{d-1}+1)^{-1/2} \\
&\qquad\qquad (n-u_1-\cdots-u_{d-1}+1)^{-1/2}\, du_1\cdots du_{d-1} \\
&\qquad \approx n^{d/2-1}.
\end{aligned}$$

Thus

$$\begin{aligned}
\sum_{|\alpha|=2n} C_\alpha^2 &\approx \sum_{n_1+\cdots+n_d=n} \frac{(2n)!}{(2n_1)!\cdots(2n_d)!}\left(\frac{(2n_1)!\cdots(2n_d)!}{2^n n_1!^2\cdots n_d!^2}\right)^2 n^{-4} \\
&\approx (2n)! \sum_{n_1+\cdots+n_d=n} (n_1+1)^{-1/2}\cdots(n_d+1)^{-1/2} \approx (2n)! n^{d/2-5}
\end{aligned}$$

proving the lemma. ■

We need the elementary computation

Lemma 3.2. *Let $d \geq 3$ and $n \geq 1$. Then when $d \geq 4$,*

$$\begin{aligned}
\lim_{K\to\infty} K^{-1} \int_{0\leq u<v\leq K} &(v-u+1)^{-2n-d+4}(v-u)^{2n-2}\, du\, dv \\
&= \int_0^\infty (x+1)^{-2n-d+4} x^{2n-2}\, dx;
\end{aligned} \tag{3.2}$$

$$\lim_{K\to\infty} K^{-1} \int_{0\leq u<v\leq K} (v+1)^{-2n-d+4}(v-u)^{2n-2}\, du\, dv = 0; \tag{3.3}$$

$$(3.4)\quad \lim_{K\to\infty} K^{-1}\int_{0\le u<v\le K}(K-u+1)^{-2n-d+4}(v-u)^{2n-2}\,du\,dv = 0;$$

When $d=3$, we have

$$(3.5)\quad \lim_{K\to\infty}(K\log K)^{-1}\int_{0\le u<v\le K}(v-u+1)^{-2n+1}(v-u)^{2n-2}\,du\,dv = 1;$$

$$(3.6)\quad \lim_{K\to\infty}(K\log K)^{-1}\int_{0\le u<v\le K}(v+1)^{-2n+1}(v-u)^{2n-2}\,du\,dv = 0;$$

$$(3.7)\quad \lim_{K\to\infty}(K\log K)^{-1}\int_{0\le u<v\le K}(K-u+1)^{-2n+1}(v-u)^{2n-2}\,du\,dv = 0.$$

Proof: Making the transformation $u=y$ and $v-u=x$, we have

$$\begin{aligned}\int_{0\le u<v<K}&(v-u+1)^{-2n-d+4}(v-u)^{2n-2}\,du\,dv\\ &=\int_0^K dx\int_x^K dy(x+1)^{-2n-d+4}x^{2n-2}\\ &=\int_0^K (K-x)(x+1)^{-2n-d+4}x^{2n-2}\,dx\\ &=K\int_0^K (x+1)^{-2n-d+4}x^{2n-2}\,dx\\ &\quad-\int_0^K (x+1)^{-2n-d+4}x^{2n-1}\,dx\\ &=: A_K+B_K.\end{aligned}$$

Let $K\to\infty$. When $d\ge 4$, we have

$$\lim_{K\to\infty} K^{-1}A_K=\int_0^\infty (x+1)^{-2n-d+4}x^{2n-2}\,dx$$

and $\lim_{K\to\infty} K^{-1}B_K = 0$. This gives (3.2). When $d=3$, one can see that $\lim_{K\to\infty}(K\log K)^{-1}A_K = 1$ and $\lim_{K\to\infty}(K\log K)^{-1}B_K = 0$. This proves (3.5). Now

$$\begin{aligned}&\int_{0\le u<v\le K}(v+1)^{-2n-d+4}(v-u)^{2n-2}\,du\,dv\\ &\le \frac{2}{2n-2+1}\int_0^K (v+1)^{-d+3}\,dv \le \frac{3}{2n-2+1}(K+1)^{-d+4}\log(K+1).\end{aligned}$$

This shows that when $d \geq 4$,

$$\lim_{K\to\infty} K^{-1} \int_{0\leq u<v\leq K} (v+1)^{-2n-d+4}(v-u)^{2n-2}\,du\,dv = 0,$$

proving (3.3). Similarly, we can prove that when $d = 3$, we have

$$\lim_{K\to\infty} (K\log K)^{-1} \int_{0\leq u<v\leq K} (v+1)^{-2n+1}(v-u)^{2n-2}\,du\,dv = 0,$$

proving (3.6). ■

Now we estimate $\int_{0\leq u_1<\cdots<u_{2n}\leq T} |f^\varepsilon_\alpha(T)(u_1,\dots,u_{2n})|^2\,du_1\cdots du_{2n}$. Denote for $n \geq 1$

$$g^\varepsilon_n(T)(u_1,\dots,u_{2n}) := (u_{2n}-u_1+\varepsilon)^{-n-d/2+2}$$

and

$$\begin{aligned} &G^\varepsilon_n(T)(u_1,\dots,u_{2n}) \\ &= -(u_{2n}+\varepsilon)^{-n-d/2+2} + (T+\varepsilon)^{-n-d/2+2} - (T-u_1+\varepsilon)^{-n-d/2+2} \end{aligned}$$

so that $f^\varepsilon_\alpha(T) = g^\varepsilon_n(T) + G^\varepsilon_n(T)$.

First we have

(3.8)

$$\begin{aligned} &\int_{0\leq u_1<\cdots<u_{2n}\leq T} |g^\varepsilon_\alpha(T)(u_1,\dots,u_{2n})|^2\,du_1\cdots du_{2n} \\ &= \int_{0\leq u_1<\cdots<u_{2n}\leq T} (u_{2n}-u_1+\varepsilon)^{-2n-d+4}\,du_1\cdots du_{2n} \\ &= \frac{1}{(2n-2)!}\int_{0\leq u_1<u_{2n}\leq T} (u_{2n}-u_1)^{-2n-d+4}(u_{2n}-u_1+\varepsilon)^{2n-2}\,du_1\,du_{2n} \\ &= \frac{\varepsilon^{4-d}}{(2n-2)!}\int_{0\leq u<v\leq T/\varepsilon} (v-u+1)^{-2n-d+4}(v-u)^{-2n-2}\,du\,dv \\ &= \begin{cases} \frac{T}{(2n-2)!}\int_0^\infty (x+1)^{-2n-d+4}x^{2n-2}\,dx\cdot O(\varepsilon^{3-d}) & d\geq 4 \\ \frac{T}{(2n-2)!}\cdot O\left(\log\left(\frac{1}{\varepsilon}\right)\right), & d=3 \end{cases} \end{aligned}$$

as $\varepsilon \to 0$, where the last identity follows from (3.2) and (3.5). This gives the estimate for the L^2 norm of $g^\varepsilon_n(T)$. Now we have to estimate the L^2 norm of $G^\varepsilon_n(T)$. It is easy to see that

(3.9) $|G_n(u_1,\dots,u_{2n})| \leq \mu[(u_{2n}+\varepsilon)^{-n-d/2+2} + (T-u_1+\varepsilon)^{-n-d/2+2}]$

for some positive constant $0 < \mu < \infty$. We should dominate the two terms arising from (3.9). When $d \geq 4$, we have for $n \geq 1$

$$\begin{aligned}
&\int_{0\le u_1<\cdots<u_{2n}\le T} (u_{2n}+\varepsilon)^{-2n-d+4}\, du_1\cdots du_{2n} \\
&\quad= \int_{0\le u_1<u_{2n}\le T} (u_{2n}+\varepsilon)^{-2n-d+4}(u_{2n}-u_1)^{2n-2}\, du_1\, du_{2n} \\
&\quad= \frac{\varepsilon^{4-d}}{(2n-2)!}\int_{0\le u<v\le T/\varepsilon} (v+1)^{-2n-d+4}(v-u)^{-2n-2}\, du\, dv.
\end{aligned}$$

By (3.3), we see that when $d \geq 4$

$$\lim_{\varepsilon\to 0} \varepsilon^{d-3}\int_{0\le u_1<\cdots<u_{2n}\le T} (u_{2n}+\varepsilon)^{-2n-d+4}\, du_1\cdots du_{2n} = 0.$$

Similarly, we can prove by (3.6) that when $d = 3$,

$$\frac{1}{\log(1/\varepsilon)}\int_{0\le u_1<\cdots<u_{2n}\le T} (u_{2n}+\varepsilon)^{-2n+1}\, du_1\cdots du_{2n} = 0.$$

This gives the estimate arising from the first member of the RHS of (3.9). The same argument (using (3.4) and (3.7)) implies that the same conclusion holds for the second member of the RHS of (3.9). Thus we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \varepsilon^{d-3}\int_{0\le u_1<\cdots<u_{2n}\le T} |G_n^\varepsilon(T)|^2\, du_1\cdots du_{2n} = 0 \quad \text{when } d\geq 4 \text{ and} \\
&\frac{1}{\log(1/\varepsilon)}\int_{0\le u_1<\cdots<u_{2n}\le T} |G_n^\varepsilon(T)|^2\, du_1\cdots du_{2n} = 0 \quad \text{when } d=3.
\end{aligned}$$

Thus we obtain

Theorem 3.3. *We have*
1) when $d = 3$,

$$\lim_{\varepsilon\to 0}\frac{1}{\log(1/\varepsilon)}\int_{0\le u_1<\cdots<u_{2n}\le T} |f_\alpha^\varepsilon(T)(u_1,\ldots,u_{2n})|^2\, du_1\cdots du_{2n} = \frac{T}{(2n-2)!}. \tag{3.10}$$

2) When $d \geq 4$,

$$\lim_{\varepsilon\to 0}\varepsilon^{d-3}\int_{0\le u_1<\cdots<u_{2n}\le T} |f_\alpha^\varepsilon(T)(u_1,\ldots,u_{2n})|^2\, du_1\cdots du_{2n} = \frac{T}{(2n-2)!}\int_0^\infty (x+1)^{-2n-d+4}x^{2n-2}\, dx. \tag{3.11}$$

4. Renormalization I

We denote $\|F\|^2 := \int_\Omega |F(\omega)|^2 P(d\omega)$. We need the following

Definition 4.1. Let $F = \sum_{n=0}^{\infty} F_n$ be the chaos expansion of F, where F_n is the n-th chaos of F. We say that F is in $D_{\theta,2}$, $\theta \in \mathbb{R}$, iff

$$\sum_{n=0}^{\infty} (n+1)^\theta \|F_n\|^2 < \infty. \tag{4.1}$$

It is easy to see that $D_{\theta,2}$ is a Hilbert space. We will not discuss this space here. However, we refer to [**Wa84**] for more details.

First we discuss the case $d = 3$. Let

$$\begin{aligned} \Phi_\varepsilon(T) &:= \frac{1}{\sqrt{\log(1/\varepsilon)}} \{\mathcal{E}_\varepsilon(T) - \mathbb{E}\mathcal{E}_\varepsilon(T)\} \\ &= \sum_{n=1}^{\infty} \sum_{|\alpha|=2n} C_\alpha J_\alpha\big(f_\alpha^\varepsilon(T)/\sqrt{\log(1/\varepsilon)}\big). \end{aligned} \tag{4.2}$$

By Theorem 3.3, we have

$$\lim_{\varepsilon \to 0} \|J_\alpha\big(f_\alpha^\varepsilon(T)/\sqrt{\log(1/\varepsilon)}\big)\|^2 = \frac{T}{(2n-2)!}.$$

Thus

$$\begin{aligned} \left\| \sum_{|\alpha|=2n} C_\alpha J_\alpha\big(f_\alpha^\varepsilon(T)/\sqrt{\log(1/\varepsilon)}\big) \right\|^2 &\sim \sum_{|\alpha|=2n} C_\alpha^2 \|J_\alpha\big(f_\alpha^\varepsilon(T)/\sqrt{\log(1/\varepsilon)}\big)\|^2 \\ &\sim \frac{T}{(2n-2)!} \sum_{|\alpha|=2n} C_\alpha^2 \\ &\approx \frac{T}{(2n-2)!} (2n)! n^{3/2-5} \approx T n^{-3/2}, \end{aligned}$$

where $A_\varepsilon \sim B_\varepsilon$ means that A_ε and B_ε have the same limit when $\varepsilon \to 0$. Thus we see that for $\theta < 1/2$, $\Phi_\varepsilon(T)$ is bounded hence weakly compact in the Hilbert space $D_{\theta,2}$ as $\varepsilon \to 0$. And when $\theta \geq 1/2$, $\Phi_\varepsilon(T)$ is unbounded in $D_{\theta,2}$ as $\varepsilon \to 0$. Namely, we have

Theorem 4.2. *Let $d = 3$. When $\theta < 1/2$, $\Phi_\varepsilon(T)$ is weakly compact in $D_{\theta,2}$ as $\varepsilon \to 0$. When $\theta \geq 1/2$, $\Phi_\varepsilon(T)$ is unbounded in $D_{\theta,2}$ as $\varepsilon \to 0$.*

When $d \geq 4$, let

$$\begin{aligned}\Psi_\varepsilon(T) &:= \varepsilon^{\frac{d-3}{2}}\Big\{\mathcal{E}_\varepsilon(T) - \mathbb{E}\mathcal{E}_\varepsilon(T)\Big\} \\ &= \sum_{n=1}^{\infty}\sum_{|\alpha|=2n} C_\alpha J_\alpha\big(f_\alpha^\varepsilon(T)\varepsilon^{\frac{d-3}{2}}\big).\end{aligned} \tag{4.3}$$

By Theorem 3.3, we obtain

$$\begin{aligned}&\Bigg\|\sum_{|\alpha|=2n} C_\alpha J_\alpha\big(f_\alpha^\varepsilon(T)\varepsilon^{\frac{d-3}{2}}\big)\Bigg\|^2 = \sum_{|\alpha|=2n} C_\alpha^2 \|J_\alpha\big(f_\alpha^\varepsilon(T)\varepsilon^{\frac{d-3}{2}}\big)\|^2 \\ &\sim \frac{T}{(2n-2)!}\sum_{|\alpha|=2n} C_\alpha^2 \int_0^\infty (x+1)^{-2n-d+4}x^{2n-2}\,dx \approx Tn^{d/2-3}.\end{aligned} \tag{4.4}$$

Therefore, we have

Theorem 4.3. *Let $d \geq 4$. When $\theta < \frac{4-d}{2}$, $\Psi_\varepsilon(T)$ is weakly compact in $D_{\theta,2}$ as $\varepsilon \to 0$. And when $\theta \geq \frac{4-d}{2}$, $\Psi_\varepsilon(T)$ is unbounded in $D_{\theta,2}$ as $\varepsilon \to 0$.*

5. Renormalization II

In [**Yo85**], Yor showed that (1.3) (when $d = 3$) converges in distribution to a Brownian motion which is independent of the original Brownian motions. In this section we will exclusively discuss the case $d \geq 4$. It is unknown what is the limit of (1.4) (when $d \geq 4$). In fact from Theorem 4.3, we see that (1.4) is unbounded in any space $D_{\theta,2}$ for $\theta \geq 0$. Therefore (1.4) is not *regular* (i.e. in the sense that it is not in the Meyer-Watanabe test functional space). For any $\theta \in \mathbb{R}$, let us introduce

$$L^{-\theta}\Psi_\varepsilon(T) := \sum_{n=1}^{\infty} n^{-\theta}\sum_{|\alpha|=2n} C_\alpha J_\alpha(f_\alpha^\varepsilon(T)). \tag{5.1}$$

¿From the estimates in section 3, we see easily that $\varepsilon^{\frac{d-3}{2}} L^{-\theta}\Psi_\varepsilon(T)$ is weakly compact in $D_{\theta-d/2+2-\rho,2}$ for any $\rho > 0$. Thus if $\theta > d/2-2$, then $\varepsilon^{\frac{d-3}{2}} L^{-\theta}\Psi_\varepsilon(T)$ is a weakly compact in the Meyer-Watanabe test functional space $D_{\alpha,2}$ for some $\alpha > 0$. Hence we think that $\varepsilon^{\frac{d-3}{2}} L^{-\theta}\Psi_\varepsilon(T)$ would be a better renormalization scheme for the self-intersection local time of higher dimensional Brownian motion ($d \geq 4$). Using the operator $\Gamma(e^{-t})$ of second quantization of e^{-t}, we have

$$L^{-\theta}\Psi_\varepsilon(T) = \int_0^\infty t^{\theta+1}\Gamma(e^{-t})\Big\{\mathcal{E}_\varepsilon(T) - \mathbb{E}\Big(\mathcal{E}_\varepsilon(T)\Big)\Big\}\, dt.$$

Now using the Mehler formula, we have

$$\begin{aligned}\Gamma(e^{-t})P_\varepsilon(B_t - B_s) &= \mathbb{E}'\Big\{P_\varepsilon(e^{-u}(B_t - B_s) + \sqrt{1-e^{-2u}}(B'_t - B'_s))\Big\}\\ &= P_{\varepsilon+(t-s)(1-e^{-2u})}(e^{-u}(B_t - B_s)),\end{aligned}$$

where B' is a d dimensional Brownian motion independent of B and $\mathbb{E}'$ means the expectation with respect to B'. Therefore

$$L^{-\theta}\Psi_\varepsilon(T) = \psi_{\varepsilon,\theta}(t-s, B_t - B_s),$$

where

$$\psi_{\varepsilon,\theta}(\nu, x) = \int_0^\infty P_{\varepsilon+\nu(1-e^{-2u})}\big(e^{-u}x\big)u^{\theta+1}\, du. \tag{5.2}$$

Summarizing the above we have

Theorem 5.1. *Let $\psi_{\varepsilon,\theta}(\nu,x)$ be defined by (5.2). Then for any $\theta \in \mathbb{R}$,*

$$\begin{aligned}\tilde\Psi_{\varepsilon,\theta} := \varepsilon^{\frac{d-3}{2}}\Bigg\{&\int_0^T\int_0^t \psi_{\varepsilon,\theta}(t-s, B_t - B_s)\, ds\, dt\\ &-\mathbb{E}\left(\int_0^T\int_0^t \psi_{\varepsilon,\theta}(t-s, B_t - B_s)\, ds\, dt\right)\Bigg\}\end{aligned} \tag{5.3}$$

is weakly compact in $D_{\theta-d/2+4,2}$.

It is easy to see that $\|\tilde\Psi_{\varepsilon,\theta}\|^2 \to \mu T$ for some constant μ. Motivated by the result of Yor, we may propose the following conjecture:

Conjecture. There is a $\theta_0 > d/2-2$ such that $\tilde\Psi_{\varepsilon,\theta_0}$ converges in distribution to a Brownian motion which is independent of the original Brownian motion B.

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