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ON NONSINGULAR P-INJECTIVE RINGS

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Dedicated to the memory of Professor Hisao Tominaga

Abstract

A ring R is said to be *left p-injective* if, for any principal left ideal I of R, any left R-homomorphism I into R extends to one of R into itself. In this note left nonsingular left p-injective rings are characterized using their maximal left rings of quotients and the structure of semiprime left p-injective rings of bounded index is investigated.

A left R-module M is said to be *p-injective* if given any principal left ideal I and any R-homomorphism $\sigma : I \to M$, there exists an R-homomorphism $\hat{\sigma} : R \to M$ that extends σ. This notion was first introduced by Ikeda and Nakayama [**8**]. They proved that a ring R is left p-injective if and only if every principal right ideal of R is a right annihilator. In [**11**, Proposition 1], it was proved that a ring R without nonzero nilpotent elements is von Neumann regular if and only if R is left p-injective. However, in general, a semiprime left p-injective ring R need not be von Neumann regular. In this note, we give a characterization of a left nonsingular left p-injective ring using its maximal left ring of quotients and consider the structure of semiprime left p-injective rings of bounded index. We also construct a semiprime left and right p-injective PI-ring which is not von Neumann regular and a semiprime left p-injective PI-ring which is not right p-injective.

For a subset F of a ring R, $r_R(F)$ (resp. $l_R(F)$) denote the right (resp. left) annihilator of F in R. To state our theorem, we need the following definition.

Definition 1. Let R be a ring, and M a left R-module. A submodule P of M is said to be *R-pure* if $aM \bigcap P = aP$ for all $a \in R$.

Theorem 1. *Let R be a left nonsingular ring and let Q denote the maximal left quotient ring of R. Then the following statements are equivalent:*

1) R is left p-injective.

2) ${}_RR$ is R-pure in ${}_RQ$.

Proof: 1) $\Rightarrow$ 2). Let $a \in R$. By [**4**, Corollary 2.31], Q is a von Neumann regular ring. Hence, there exists an idempotent $e \in Q$ such that $aQ = eQ$. Then $l_R(a) = l_Q(a) \bigcap R = l_Q(eQ) \bigcap R = Q(1-e) \bigcap R$. By [**8**, Theorem 1], the p-injectivity of ${}_RR$ implies that every principal right ideal of R is a right annihilator ideal. Hence $aR = r_R l_R(a) = r_R(Q(1-e) \bigcap R) \supseteq eQ \bigcap R = aQ \bigcap R \supseteq aR$. This proves $aQ \bigcap R = aR$ for all $a \in R$.

2) $\Rightarrow$ 1). Let $a \in R$. Then there exists an idempotent $e \in Q$ such that $aQ = eQ$. First we claim that $r_R(Q(1-e) \bigcap R) = eQ \bigcap R$. Clearly we have that $r_R(Q(1-e) \bigcap R) \supseteq eQ \bigcap R$. To prove the converse inclusion, let $b \in r_R(Q(1-e) \bigcap R)$. Since Q is the maximal left ring of quotients of R, there exists an essential left ideal I of R such that $I(1-e) \subseteq R$. Then $I(1-e)b = 0$. Since ${}_RQ$ is nonsingular by [**4**, Proposition 2.32], this implies $(1-e)b = 0$. Therefore the converse inclusion also holds. Then $r_R l_R(a) = r_R(Q(1-e) \bigcap R) = eQ \bigcap R = aQ \bigcap R = aR$, because ${}_RR$ is R-pure in ${}_RQ$. By [**8**, Theorem 1], this implies the p-injectivity of ${}_RR$. ■

A ring R is said to be of bounded index (of nilpotence) if there is a positive integer n such that $a^n = 0$ for each nilpotent element a of R. If n is the least such integer we say R has index n. For example, it is well known that any semiprime ring satisfying a polynomial identity is of bounded index ([**9**, Theorem 10.8.2]). Recall that R is said to be π-regular if for each element a of R, there exists a positive integer m and an element x of R such that $a^m = a^m x a^m$. On the other hand, R is said to be strongly π-regular if for each element a of R, there exists a positive integer k such that $a^k R = a^{k+1} R$. By [**2**, Théorème 1] this definition is left-right symmetric, and hence such a ring is π-regular. In particular, every nonnil one-sided ideal of a strongly π-regular ring contains a nonzero idempotent.

Proposition 1. *Let R be a semiprime p-injective ring of bounded index. Then we have:*

(1) R is a strongly π-regular ring.

(2) The maximal left quotient ring Q of R is a finite direct product of matrix rings over strongly regular self-injective rings.

Proof: Let R is of index n and let $a \in R$. Then $l_R(a^n) = l_R(a^{n+1})$ by [**5**, Proposition 2]. Hence we have $a^nR = r_Rl_R(a^n) = r_Rl_R(a^{n+1}) = a^{n+1}R$. This proves that R is a strongly π-regular ring. Since R is a semiprime ring of bounded index, by virtue of [**6**, Lemma 1.1], every nonzero one-sided ideal of R contains a nonzero idempotent. Hence, the assertion (2) follows from [**5**, Theorem 9 and (2) in Remarks]. ■

Assume that R is a left p-injective ring without nonzero nilpotent elements. Then, by Proposition 1, R is a strongly π-regular ring of index 1, that is, R is a strongly regular ring. Hence we obtain [**11**, Proposition 1]. Also we have the following

Corollary 1. *Let R be a semiprime p-injective ring of bounded index. Then R is von Neumann regular if and only if the union of any chain of semiprime ideals of R is a semiprime ideal. In consequence, a semiprime p-injective ring R which is finitely generated as a module over its center is von Neumann regular.*

Proof: By Proposition 1, R is strongly π-regular, and so every prime factor ring of R is regular by [**7**, Proposition 2]. Now the result follows from [**3**, Theorem 1.1]. If R is finitely genereted over its center, then R satisfies a polynomial identity, and hence R is of bounded index. Also, by the proof of [**1**, Theorem 1], we know that the union of any chain of semiprime ideals of R is a semiprime ideal of R. ■

Now we shall generalize the construction technique of semiprime rings used in [**10**].

Definition 2. Let S be a ring and let T be a subring of S. For an infinite set I, $(S|T)^I$ denotes the subring of the direct product S^I of I's copies of S consisting of all $s = (s_i)$, for which $s_i \in T$ for all but a finite number of $i \in I$. $(S|T)^{(I)}$ denotes the subring of S^I consisting of all $s = (s_i)$, for which $s_i = t$ for some $t \in T$ for all but a finite number of $i \in I$.

Lemma 1. *Let S be a semisimple Artinian ring, and let I be an infinite set. Let R be a subring of S^I containing $S^{(I)}$, the direct sum of I's copies of S. Then R is a nonsingular semiprime ring and its maximal left quotient ring Q is S^I.*

Proof: Let $\mathfrak{a} = S^{(I)}$. Clearly $\mathfrak{a}$ is an ideal of both R and S^I. It is easy to see that $bab \neq 0$ for any nonzero $b \in R$. Hence R is a semiprime ring. Clearly R is of bounded index. Hence, by [**5**, Proposition 4], the semiprime ring R of bounded index is a left (and right) nonsingular ring.

Let K be a nonzero submodule of ${}_RS^I$. Then we can easily see that $0 \neq \mathfrak{a}K \subseteq K \cap R$. Hence ${}_RS^I$ is an essential extension of ${}_RR$. Clearly S^I is a regular, left self-injective ring. Hence, by [**4**, Proposition 2.11 and Corollary 2.31], we conclude that $Q = S^I$. ■

Proposition 2. *Let S be a semisimple Artinian ring, let T be a subring of S and let I be an infinite set. Then the following statements are equivalent:*

1) $(S|T)^I$ is a left p-injective ring.

2) $(S|T)^{(I)}$ is a left p-injective ring.

3) ${}_TT$ is T-pure in ${}_TS$.

Proof: 1) $\Longleftrightarrow$ 3). Let $R = (S|T)^I$. By Lemma 1, the maximal left ring Q of quotients of R is S^I. Assume first that ${}_TT$ is T-pure in ${}_TS$. Let $(a_i) \in R$ and let $(c_i) \in (a_i)Q \bigcap R$. Then $(c_i) = (a_i)(b_i)$ for some $(b_i) \in Q$. By the definition of R, there exist $i_1, i_2, \dots, i_n \in I$ such that $a_i, c_i \in T$ for all $i \in I - \{i_1, i_2, \dots, i_n\}$. Since ${}_TT$ is T-pure in ${}_TS$, we have $c_i \in a_iS \bigcap T = a_iT$ for all $i \in I - \{i_1, i_2, \dots, i_n\}$. Hence we can write $c_i = a_id_i$ with some $d_i \in T$ for each $i \in I - \{i_1, i_2, \dots, i_n\}$. Now define (x_i) by $x_i = b_i$ for $i = i_1, i_2, \dots, i_n$ and $x_i = d_i$ for each $i \in I - \{i_1, i_2, \dots, i_n\}$. Then $(x_i) \in R$, and hence $(c_i) = (a_i)(x_i) \in (a_i)R$. This proves that ${}_RR$ is R-pure in ${}_RQ$. Hence R is left p-injective by Theorem 1.

Conversely, assume that R is left p-injective. Take an arbitrary $a \in T$ and set $a_i = a$ for all i. Then, by Theorem 1, we have $(a_i)Q \cap R = (a_i)R$. Now, let $as \in aS \cap T$ where $s \in S$, and set $s_i = s$ for all i. Then $(a_i)(s_i) \in (a_i)Q \cap R = (a_i)R$. Hence there exists $(c_i) \in R$ such that $(a_i)(s_i) = (a_i)(c_i)$. By the definition of R, $c_i \in T$ for almost all i. So, let $a_N \in T$. Then $as = aa_N \in aT$. This proves that $aS \cap T = aT$ for all $a \in T$. Therefore ${}_TT$ is T-pure in ${}_TS$.

The proof of 2) $\Longleftrightarrow$ 3) is quite similar to that of 1) $\Longleftrightarrow$ 3), and so we omit it. ■

The following example shows that a semiprime right and left p-injective ring satisfying a polynomial identity need not be von Neumann regular.

Example 1. Let K be a field and let T be the subring of the $n \times n$

full matrix ring $M_n(K)$ over K consisting of all matrices of the form

$$A = \begin{pmatrix} a_1 & a_2 & \cdots & \cdots & a_n \\ & a_1 & a_2 & & \vdots \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & a_2 \\ \mathbf{0} & & & & a_1 \end{pmatrix}$$

with $a_1, a_2, \ldots, a_n \in K$. Then ${}_T T$ is T-pure in ${}_T M_n(K)$. In fact, let A be the matrix in (0.1) and assume that $a_1 = a_2 = \cdots = a_{m-1} = 0$ and $a_m \neq 0$. Suppose that $AB = \begin{pmatrix} c_1 & \cdots & c_n \\ & \ddots & \vdots \\ \mathbf{0} & & c_1 \end{pmatrix} \in T$ for some $B \in M_n(K)$. Then we can write

$$\begin{pmatrix} a_m & \cdots & a_n \\ & \ddots & \vdots \\ \mathbf{0} & & a_m \end{pmatrix}^{-1} \begin{pmatrix} c_m & \cdots & c_n \\ & \ddots & \vdots \\ \mathbf{0} & & c_m \end{pmatrix} = \begin{pmatrix} d_m & \cdots & d_n \\ & \ddots & \vdots \\ \mathbf{0} & & d_m \end{pmatrix}$$

in $M_{n-m+1}(K)$ with $d_m, \ldots, d_n \in K$. Therefore if we set

$$X = \begin{pmatrix} d_m & \cdots & d_n & & \mathbf{0} \\ & d_m & \ddots & \ddots & \\ & & \ddots & \ddots & d_n \\ & & & \ddots & \vdots \\ \mathbf{0} & & & & d_m \end{pmatrix},$$

then $X \in T$ and $AB = AX$. Thus ${}_T T$ is T-pure in ${}_T M_n(K)$. Similarly we can prove that T_T is T-pure in $M_n(K)_T$. By Proposition 2, $(M_n(K)|T)^{\mathbf{N}}$ and $(M_n(K)|T)^{(\mathbf{N})}$ are semiprime right and left p-injective rings satisfying a polynomial identity. However, we can easily see that these are not von Neumann regular for $n \geq 2$.

The following example shows that a semiprime left p-injective ring satisfying a polynomial identity need not be right p-injective.

Example 2. Let K be a field and consider the subring

$$T = \left\{ \begin{pmatrix} a & 0 & 0 \\ b & a & 0 \\ c & 0 & a \end{pmatrix} \mid a, b, c \in K \right\}$$

of $M_3(K)$. Let $A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then it is easy to see that $BA \in T \cap M_3(K)A$, but $BA \notin TA$. Hence T_T is not T-pure in $M_3(K)_T$. Next, suppose that $AX = B \in T$ with $A \in T$ and $X \in M_3(K)$. If $\det(A) \neq 0$, then $X = A^{-1}B \in T$. So, assume that $\det(A) = 0$ and let x be the (1,1)-component of X. Then we can easily see that $B = A(xE) \in AT$, where E denotes the identity matrix in $M_3(K)$. Thus, $AM_3(K) \cap T = AT$. This implies ${}_TT$ is T-pure in ${}_TM_3(K)$. By Proposition 2, semiprime PI-rings $(M_3(K)|T)^{\mathbf{N}}$ and $(M_3(K)|T)^{(\mathbf{N})}$ are left p-injective, but not right p-injective.

The following shows that there are semiprime π-regular PI-rings which are neither right nor left p-injective.

Example 3. Let K be a field and let T be the algebra of upper triangular $n \times n$ matrices over K, where $n > 1$. Then ${}_TT$ is not T-pure in ${}_TM_n(K)$. In fact, if $\{e_{ij}\}$ denotes the set of matrix units of $M_n(K)$, then $e_{1n}M_n(K) \cap T = Ke_{11} + \cdots + Ke_{1n} \neq Ke_{1n} = e_{1n}T$. Hence, by Proposition 2, $(M_n(K)|T)^{\mathbf{N}}$ and $(M_n(K)|T)^{(\mathbf{N})}$ are neither right p-injective nor left p-injective. However we can easily see that these are semiprime π-regular rings.

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