ON THE EXISTENCE, UNIQUENESS AND PARAMETRIC DEPENDENCE ON THE COEFFICIENTS OF THE SOLUTION PROCESSES IN McSHANE'S STOCHASTIC INTEGRAL EQUATIONS

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Abstract

In this paper we use the Schauder fixed point theorem and methods of integral inequalities in order to prove a result on the existence, uniqueness and parametric dependence on the coefficients of the solution processes in McShane stochastic integral equations

1. Introduction

In this paper we consider the problem of the existence and uniqueness of the solution processes in the stochastic integral system

$$(1) \quad x^i(t) = \alpha^i(t) + \sum_{j=1}^{r} \int_0^t g_j^i(s, x(s)) \, dz_j(s)$$

$$+ \sum_{j,k=1}^{r} \int_0^t h_{jk}^i(s, x(s)) \, dz_j(s) \, dz_k(s), \quad 0 \le t \le a, \quad i = 1, \dots, n,$$

where the stochastic integrals involving dz_j are interpreted as McShane stochastic integrals.

Some previous results were given by McShane [13] in the special case in which α^i is not depending on time and the processes z_j are sample continuous and by Elworthy [8] which does not require these conditions but considers stronger hypotheses on g_j^i and h_{jk}^i. Recently, Angulo Ibañez and Gutiérrez Jaimez [1] proved (under weaker requirements than in [8] and [13]) an existence and uniqueness theorem for (1) assuming

that the functions g^i_j and h^i_{jk} satisfy a Lipschitz condition and the processes z_j satisfy a K-condition, using an adequate version of the Fomin-Kolmogorov fixed point theorem. They considered in [2] the problem of the convergence of the solution processes in McShane's stochastic integral equation systems with coefficients depending on a parameter under similar assumptions to [1].

Under the hypothesis of a weaker condition than the Lipschitz condition on g^i_j and h^i_{jk} we prove an existence and uniqueness result for the solution process of equation (1) applying Shauder's fixed point theorem [15] and methods of integro-differential inequalities (see [3], [4], [5], [7], [9]). Under the same hypothesis we prove also the continuity with respect to the initial condition and we consider the problem of parametric dependence of the solution processes on the coefficients, generalizing the results of [2].

2. Preliminaries

Let (Ω, F, P) be a complete probability space and let $\{F_t, 0 \le t \le a\}$ be a family of complete σ-subalgebras of F such that if $0 \le s \le t \le a$ then $F_s \subseteq F_t$.

Let L_2 be the space of all random variables $y : \Omega \to \mathbb{R}$ with finite L_2-norm $\|\cdot\|$ and let L_2^n be the space of all random variables $x : \Omega \to \mathbb{R}^n$ with finite norm $\|\cdot\|_n$,

$$\|x\|_n^2 = \sum_{i=1}^n \|x_i\|^2, \quad x = (x_1, \ldots, x_n) \in L_2^n.$$

We say that the real valued second order stochastic process z on $[0, a]$ satisfies a K-condition if z is adapted to the F_t (i.e. $z(t)$ is F_t-measurable for every $t \in [0, a]$) and

$$|E[(z(t) - z(s))^p / F_s]| \le K|t - s|$$

a.s. whenever $0 \le s \le t \le a$, $p = 1, 2, 4$. An example of such a process is a Wiener process with respect to the F_t, $0 \le t \le a$.

Let us define also for each second order process x on $[0, a]$ the norm

$$|||x||| = \sup_{t \in [0,a]} \{\|x(t)\|\}.$$

For the basic elements of the McShane stochastic calculus theory we refer to McShane [13] and Elworthy [8]. Let us remind only that if f :

$[0, a] \rightarrow L_2$ is a measurable process adapted to the F_t and if $t \rightarrow \|f(t)\|^2$ is Lebesgue integrable on $[0, a]$, then ([8], [9]), if z_1 and z_2 satisfy a K-condition, the McShane integrals

$$\int_0^a f(s)\, dz_1(s), \quad \int_0^a f(s)\, dz_1(s) dz_2(s)$$

exist, and the following estimates are true

$$\left\| \int_0^a f(s)\, dz_1(s) \right\| \leq C \left\{ \int_0^a \|f(s)\|^2\, ds \right\}^{1/2}$$

$$\left\| \int_0^a f(s)\, dz_1(s)\, dz_2(s) \right\| \leq C \left\{ \int_0^a \|f(s)\|^2\, ds \right\}^{1/2}$$

where $C = (2 + 8Ka)^{1/2}$.

We will use in this paper the following result

Lemma (Bihari [3]). *Let v, h be positive, continuous functions on $0 \leq t \leq a$, and let q be a nonnegative constant; further let w be a positive nondecreasing function on $[0, \infty)$. Then the inequality*

$$v(t) \leq q + \int_0^t h(s) w(v(s))\, ds, \quad 0 \leq t \leq a,$$

implies the inequality

$$v(t) \leq G^{-1}\left(G(q) + \int_0^t h(s) \right) ds, \quad 0 \leq t \leq a',$$

where $G(u) = \int_1^u \frac{ds}{w(s)}$, $u > 0$, and a' is defined so that $G(q) + \int_0^t h(s)\, ds$ lies within the domain of definition of G^{-1} for $0 \leq t \leq a'$.

For proofs of the lemma, see [3] and [5].

3. Existence and uniqueness of solutions

Let $C[0, a]$ denote the space of all processes $x : [0, a] \rightarrow L_2^n$ which are continuous and adapted to the F_t. A solution to the equation (1) on $[0, a]$ is a process $x \in C[0, a]$ which satsifies (1) on $[0, a]$.

Let us assume that

(H_1) the noise processes z_j, $j = 1, \ldots, r$, satisfy on $[0, a]$ a K-condition;

(H$_2$) if f is any one of the functions g^i_j, h^i_{jk} : $[0,a] \times L^n_2 \to L_2$, $i = 1,\dots,n$; j, $k = 1,\dots,r$, then $f(s,x)$ is continuous in x on L^n_2 for every $s \in [0,a]$, and for any $x \in C[0,a]$, the process $t \to f(t,x(t))$ is measurable and F_t-adapted with $t \to \|f(t,x(t))\|^2$ bounded on $[0,a]$;

(H$_3$) there exists a continuous, nondecreasing function $w : \mathbb{R}_+ \to \mathbb{R}_+$ with $w(0) = 0$, $w(t) > 0$ for $t > 0$ and

$$\lim_{t \to 0} \int_t^1 \frac{ds}{w(s)} = \infty, \quad \lim_{t \to \infty} \int_1^t \frac{ds}{w(s)} = \infty$$

such that

$$\|f(t,x) - f(t,y)\|^2 \leq w(\|x-y\|^2_n), \quad t \in [0,a], \quad x,y \in L^n_2;$$

(H$_4$) the initial condition α belongs to $C[0,a]$.

Theorem 1. *Let us suppose that the hypotheses* (H$_1$)*-*(H$_4$) *are satisfied. Then there exists an unique solution of the equation* (1) *on* $[0,a]$.

Proof: Let us first prove the existence of a solution on $[0,a]$.
We define the operator $T : C[0,a] \to C[0,a]$ by

$$Tx(t) = \alpha(t) + \sum_{j=1}^r \int_0^t g_j(s,x(s))\, dz_j(s)$$

$$+ \sum_{j,k=1}^r \int_0^t h_{jk}(s,x(s))\, dz_j(s)\, dz_k(s), \quad 0 \leq t \leq a.$$

Let $M = 2n \sup_{t \in [0,a]\|} \|\alpha(t)\|^2_n + 4nC^2(r+r^2)^2 Ka$ where

$$K = \max \left\{ \sup_{\substack{t \in [0,a] \\ i=1,\dots,n;\, j=1,\dots,r}} \|g^i_j(t,0)\|^2, \quad \sup_{\substack{t \in [0,a] \\ i=1,\dots,n;\, j,k=1,\dots,r}} \|h^i_{jk}(t,0)\|^2 \right\}$$

and let us consider

$$G : (0,\infty) \to \mathbb{R}, \quad G(u) = \int_1^u \frac{ds}{w(s)}$$

(the hypotheses guarantee that G is a bijection from $(0,\infty)$ to $\mathbb{R}$).

We also consider the function

$$m(t) = G^{-1}(G(M) + 4nC^2(r + r^2)^2 t), \quad 0 \le t \le a.$$

This function has the property that

$$m(t) = M + 4nC^2(r + r^2)^2 \int_0^t w(m(s)) \, ds, \quad 0 \le t \le a,$$

being the solution of the differential equation

$$m'(t) = 4nC^2(r + r^2)^2 w(m(t)), \quad 0 \le t \le a,$$

with initial condition $m(0) = M$.

We consider the set

$$B = \{x \in C[0, a] : \|x(t)\|_n^2 \le m(t), \quad 0 \le t \le a\}.$$

This is a closed, bounded and convex subset of the Banach space $C[0, a]$ and we will show that $T(B) \subset B$.

Let $x \in B$. We have then that

$$\left\| \int_0^t g_j^i(s, x(s)) \, dz_j(s) \right\| \le C \left\{ \int_0^t \|g_j^i(s, x(s))\|^2 \, ds \right\}^{1/2}$$

$$\le C \left\{ \int_0^t (\|g_j^i(s, x(s)) - g_j^i(s, 0)\| + \|g_j^i(s, 0)\|)^2 \, ds \right\}^{1/2}$$

$$\le C \left\{ \int_0^t (2\|g_j^i(s, x(s)) - g_j^i(s, 0)\|^2 + 2\|g_j^i(s, 0)\|^2) \, ds \right\}^{1/2}$$

$$\le C \left\{ 2 \int_0^t w(\|x(s)\|_n^2) \, ds + 2Kt \right\}^{1/2}, \quad 0 \le t \le a,$$

and similarly

$$\left\| \int_0^t h_{jk}^i(s, x(s)) \, dz_j(s) \, dz_k(s) \right\|$$

$$\le C \left\{ 2 \int_0^t w(\|x(s)\|_n^2) \, ds + 2Kt \right\}^{1/2}, \quad 0 \le t \le a.$$

We deduce that

$$\|Tx(t)\|_n \le \sqrt{n} \sup_{t \in [0,a]} \|\alpha(t)\|_n$$

$$+ \sqrt{2n}(r + r^2) C \left\{ \int_0^t w(\|x(s)\|_n^2) \, ds + Kt \right\}^{1/2}, \quad 0 \le t \le a,$$

thus (since $x \in B$)

$$\|Tx(t)\|_n^2$$

$$\leq \left\{ \sup_{t \in [0,a]} \sqrt{n}\|\alpha(t)\|_n + \sqrt{2n}(r + r^2)C \left\{ \int_0^t w(m(s))\,ds + Ka \right\}^{1/2} \right\}^2$$

$$\leq 2n \sup_{t \in [0,a]} \|\alpha(t)\|_n^2 + 4nC^2(r + r^2)^2 \int_0^t w(m(s))\,ds + 4nC^2(r + r^2)^2 Ka$$

$$= M + 4nC^2(r + r^2)^2 \int_0^t w(m(s))\,ds = m(t), \quad 0 \leq t \leq a.$$

Hence $T(B) \subset B$.

In a similar way we prove that if $x \in B$, then

$$\|Tx(t) - Tx(s)\|_n^2 \leq 4nC^2(r + r^2)^2 \int_s^t w(m(u))\,du + 2\|\alpha(t) - \alpha(s)\|_n^2$$

$$+ 4nC^2(r + r^2)^2(t - s)K, \quad 0 \leq s \leq t \leq a,$$

from where we conclude that the set $T(B)$ is equicontinuous.

On the other hand we have for $x,\, y \in B$ that

$$\|Tx^i(t) - Ty^i(t)\| \leq C\sum_{j=1}^r \left\{ \int_0^t \|g_j^i(s, x(s)) - g_j^i(s, y(s))\|^2\,ds \right\}^{1/2}$$

$$+ C\sum_{j,k=1}^r \left\{ \int_0^t \|h_{jk}^i(s, x(s)) - h_{jk}^i(s, y(s))\|^2\,ds \right\}^{1/2}, \quad 0 \leq t \leq a.$$

From (H_3) and the continuity of $g_j^i(s, x)$ and $h_{jk}^i(s, x)$ in x we deduce by the Lebesgue convergence theorem that T is continuous. An application of Schauder's fixed point theorem enables us to deduce that T has a fixed point in B, thus equation (1) has a solution on $[0, a]$.

Let us now prove the uniqueness of solutions for equation (1).

Suppose that there exist two different solutions $x,\, y \in C[0, a_1]$ of equation (1) on some interval $[0, a_1]$ with $0 < a_1 \leq a$. Then there exist points $0 < t \leq a_1$ with $\|x(t) - y(t)\|_n > 0$. Let b be the lower bound of these t. We have then that $\|x(b) - y(b)\|_n = 0$, but $\|x(t) - y(t)\|_n > 0$ for $b < t \leq b + \beta$ with a certain number $\beta > 0$.

We have that

$$\|x^i(t) - y^i(t)\| \le \sum_{j=1}^{r} C \left\{ \int_0^t \|g(s, x(s)) - g_j^i(s, y(s))\|^2 \, ds \right\}^{1/2}$$

$$+ \sum_{j,k=1}^{r} C \left\{ \int_0^t \|h_{jk}^i(s, x(s)) - h_{jk}^i(s, y(s))\|^2 \, ds \right\}^{1/2}$$

$$\le (Cr + Cr^2) \left\{ \int_0^t w(\|x(s) - y(s)\|_n^2) \, ds \right\}^{1/2}, \quad 0 \le t \le a_1,$$

thus

$$\|x(t) - y(t)\|_n^2 \le n(Cr + Cr^2)^2 \int_0^t w(\|x(s) - y(s)\|_n^2) \, ds, \quad 0 \le t \le a_1.$$

Since $w(0) = 0$ and $\|x(t) - y(t)\|_n = 0$ on $[0, b]$ we obtain that

$$\|x(t) - y(t)\|_n^2 \le n(Cr + Cr^2)^2 \int_b^t w(\|x(s) - y(s)\|_n^2) \, ds, \quad b \le t \le b + \beta,$$

thus (denoting $v(t) = \|x(t) - y(t)\|_n^2$, $b \le t \le b + \beta$) for every $\varepsilon > 0$ we have

$$v(t) \le \varepsilon + n(Cr + Cr^2)^2 \int_b^t w(v(s)) \, ds, \quad b \le t \le b + \beta.$$

Let $V_\varepsilon(t) = \varepsilon + n(Cr + Cr^2)^2 \int_b^t w(v(s)) \, ds$, $b \le t \le b + \beta$. We have then that

$$V_\varepsilon'(t) = n(Cr + Cr^2)^2 w(v(t)) \le n(Cr + Cr^2)^2 w(V_\varepsilon(t)), \quad b \le t \le b + \beta,$$

thus (since $V_\varepsilon(t) > 0$ on $[b, b + \beta]$)

$$\frac{V'(t)}{w(V_\varepsilon(t))} \le n(Cr + Cr^2)^2, \quad b \le t \le b + \beta.$$

An integration yields

$$G(V_\varepsilon(t)) - G(V_\varepsilon(b)) \le n(Cr + Cr^2)^2(t - b), \quad b \le t \le b + \beta,$$

thus

$$G(V_\varepsilon(t)) \le G(V_\varepsilon(b)) + n(Cr + Cr^2)^2(t - b)$$
$$\le G(\varepsilon) + n(Cr + Cr^2)^2, \quad b \le t \le b + \beta.$$

Since $0 < v(t) \leq V_\varepsilon(t)$ for $b < t \leq b + \beta$, we deduce that

$$G(v(t)) \leq G(\varepsilon) + n(Cr + Cr^2)^2 \beta, \quad b < t \leq b + \beta,$$

for every $\varepsilon > 0$.

This leads to a contradiction since $v(t) > 0$ for $t \in (b, b + \beta]$ thus $G(v(t))$ is a real number for $b < t \leq b + \beta$, but $\lim_{\varepsilon \to 0} G(\varepsilon) = -\infty$.

Thus equation (1) has an unique solution on $[0, a]$.

This completes the proof of Theorem 1. ∎

Remark 1. Theorem 1 is not only an existence and uniqueness result, it provides also a bound for the solution $x \in C[0, a]$:

$$\|x(t)\|_n^2 \leq m(t), \quad 0 \leq t \leq a.$$

Remark 2. If the hypothesis (H$_3$) is satisfied with $w(t) = Lt$ on $\mathbb{R}_+$ ($L > 0$ being a real constant) we obtain the existence and uniqueness theorem of Angulo Ibañez and Gutiérrez Jaimez [1] and as a special case (when α is not depending on time t and the processes z_j are sample continuous) of this we obtain the existence and uniqueness result of McShane [13]. Our requirements are weaker in some aspects with respect to those in [8] (Elworthy requires in [8] Lipschitz conditions on the functions g_j^i and h_{jk}^i).

4. Continuity with respect to the initial condition

We proved in the preceding section that if the conditions (H$_1$)-(H$_3$) are satisfied, the stochastic integral equations

$$(2.k) \quad x(t) = \alpha_k(t) + \sum_{j=1}^r \int_0^t g_j(s, x(s)) \, dz_j(s)$$

$$+ \sum_{j,k=1}^r \int_0^t h_{jk}(s, x(s)) \, dz_j(s) \, dz_k(s), \quad 0 \leq t \leq a,$$

have an unique solution $x_k \in C[0, a]$ if $\alpha_k \in C[0, a]$, $k \geq 1$.

Theorem 2. *If the hypotheses* (H$_1$)-(H$_3$) *are satisfied and if* $\alpha_1, \alpha_2 \in C[0, a]$, *then*

$$\||x_1 - x_2\||_n^2 \leq G^{-1}(G(K_1 \||\alpha_1 - \alpha_2\||_n^2) + K_2)$$

where $K_1 = 1 + r + r^2$ and $K_2 = nK_1(C^2r + C^2r^2)a$.

Proof: In a similar way to the proof of Theorem 1 we can state

$$\|x_1^i(t) - x_2^i(t)\|^2$$

$$\leq \left\{ \|\alpha_1^i(t) - \alpha_2^i(t)\| + \sum_{j=1}^{r} C \left\{ \int_0^t \|g_j^i(s, x_1(s)) - g_j^i(s, x_2(s))\|^2 \, ds \right\}^{1/2} \right.$$

$$\left. + \sum_{j,k=1}^{r} C \left\{ \int_0^t \|h_{jk}^i(s, x_1(s)) - h_{jk}^i(s, x_2(s))\|^2 \, ds \right\}^{1/2} \right\}^2$$

$$\leq (1 + r + r^2) \left\{ \|\alpha_t^i(t) - \alpha_2^i(t)\|^2 + \sum_{j=1}^{r} C^2 \int_0^t w(\|x_1(s) - x_2(s)\|_n^2) \, ds \right.$$

$$\left. + \sum_{j,k=1}^{r} C^2 \int_0^t w(\|x_1(s) - x_2(s)\|_n^2) \, ds \right\}, \quad 0 \leq t \leq a,$$

thus (taking supremes on $[0, a]$ to the initial condition term)

$$\|x_1(t) - x_2(t)\|_n^2 \leq (1 + r + r^2)\|\|\alpha_1 - \alpha_2\|\|_n^2$$
$$+ n(1 + r + r^2)(C^2r + C^2r^2) \int_0^t w(\|x_1(s) - x_2(s)\|_n^2) \, ds, \quad 0 \leq t \leq a.$$

By Bihari's inequality we obtain that

$$\|x_1(t) - x_2(t)\|_n^2 \leq G^{-1}(G((1 + r + r^2)\|\|\alpha_1 - \alpha_2\|\|_n^2$$
$$+ n(1 + r + r^2)(C^2r + C^2r^2)t), \quad 0 \leq t \leq a,$$

so that

$$\|\|x_1 - x_2\|\|_n^2 \leq G^{-1}(G(K_1\|\|\alpha_1 - \alpha_2\|\|_n^2) + K_2). \quad \blacksquare$$

Theorem 3. *If the hypotheses* (H$_1$)-(H$_3$) *are satisfied,* $\alpha, \alpha_n \in C[0, a]$ *and* $\lim_{k \to \infty} \|\|\alpha_k - \alpha\|\|_n = 0$, *we have that*

$$\lim_{k \to \infty} \|\|x_k - x\|\|_n = 0$$

where $x \in C[0, a]$ *is the solution of equation (1) and* x_k *is the solution of (2.k) with initial condition* $\alpha_k \in C[0, a]$, $k \geq 1$.

Proof: By Theorem 2 we have that

$$\|\|x_k - x\|\|_n^2 \leq G^{-1}(G(K_1\|\|\alpha_k - \alpha\|\|^2) + K_2)$$

thus
$$G(|||x_k - x|||_n^2) \leq G(K_1|||\alpha_k - \alpha|||^2) + K_2.$$

Since $\lim\limits_{u \to 0} G(u) = -\infty$ and $\lim\limits_{k \to \infty} |||\alpha_k - \alpha|||_n^2 = 0$ we deduce that

$$\lim_{k \to \infty} G(|||x_k - x|||_n^2) = -\infty$$

thus $\lim\limits_{k \to \infty} |||x_k - x|||_n = 0.$ ∎

Remark 3. It seems (see the Introduction to [2]) that the first who considered the problem of the continuous dependence of the solution process with respect to the initial condition in the context of McShane's stochastic integrals were Angulo Ibañez and Gutiérrez Jaimez [2]. Our hypotheses do require a weaker condition than the Lipschitz condition on g_j^i and h_{jk}^i. If $w(x) = Lx$ on $\mathbb{R}_+$ ($L > 0$ being a real constant) we obtain from Theorem 2 that

$$|||x_1 - x_2|||_n^2 \leq K_1 e^{LK_2}|||\alpha_1 - \alpha_2|||_n^2.$$

This proves the Lipschitz character of the solutions with respect to the initial condition in the case of a Lipschitz condition on g_j^i and h_{jk}^i.

5. Parametric dependence
of the solution processes on the coefficients

In this section we consider the problem of the convergence of the solution processes in McShane's stochastic integral equation systems with coefficients depending on a parameter.

Let us consider families of stochastic integral equation systems

$$(3.\lambda) \quad x(t) = \alpha_\lambda(t) + \sum_{j=1}^{r} \int_0^t g_{j,\lambda}(s, x(s)) \, dz_j(s)$$

$$+ \sum_{j,k=1}^{r} \int_0^t h_{jk,\lambda}(s, x(s)) \, dz_j(s) \, dz_k(s), \quad 0 \leq t \leq a,$$

where for each $\lambda \in \Lambda$, Λ being an open and bounded subset of $\mathbb{R}^m$ ($m \in \mathbb{N}$), the hypotheses (H$_1$)-(H$_4$) are satisfied. By Theorem 1 we have that for every $\lambda \in \Lambda$, equation (3.λ) has an unique solution $x_\lambda \in C[0, a]$.

Let λ_0 be a fixed point in Λ.

We assume that for every process $x \in C[0, a]$ we have

$$(\text{H}_5) \qquad\qquad f_\lambda(t, x(t)) \xrightarrow{P} f_{\lambda_0}(t, x(t))$$

as $\lambda \to \lambda_0$ (P expresses the convergence in probability) if f_λ is any one of the functions $g^i_{j,\lambda}$ and $h^i_{jk,\lambda}$.

By Lemma I and the remark of [2] we have that condition (H_5) implies that for every $x \in C[0, a]$,

$$\lim_{\lambda \to \lambda_0} \int_0^a \|f_\lambda(s, x(s)) - f_{\lambda_0}(s, x(s))\|^2 \, ds = 0.$$

Theorem 4. *If the hypotheses* (H_1)-(H_5) *are satisfied, then* $\lim_{\lambda \to \lambda_0} |||\alpha_\lambda - \alpha_{\lambda_0}|||_n = 0$ *implies* $\lim_{\lambda \to \lambda_0} |||x_\lambda - x_{\lambda_0}|||_n = 0.$

Proof: The conditions of the theorem allow to state that

$$\|x^i_\lambda(t) - x^i_{\lambda_0}(t)\|^2 \le (1 + r + r^2) \left\{ \|\alpha^i_\lambda(t) - \alpha^i_{\lambda_0}(t)\|^2 \right.$$

$$+ \sum_{j=1}^r C^2 \int_0^t \|g^i_{j,\lambda}(s, x_\lambda(s)) - g^i_{j,\lambda_0}(s, x_{\lambda_0}(s))\|^2 \, ds$$

$$\left. + \sum_{j,k=1}^r C^2 \int_0^t \|h^i_{jk,\lambda}(s, x_\lambda(s)) - h^i_{jk,\lambda_0}(s, x_{\lambda_0}(s))\|^2 \, ds \right\}$$

$$\le (1 + r + r^2) \left\{ \|\alpha^i_\lambda(t) - \alpha^i_{\lambda_0}(t)\|^2 \right.$$

$$+ 2 \sum_{j=1}^r C^2 \int_0^t \|g^i_{j,\lambda}(s, x(s)) - g^i_{j,\lambda}(s, x_{\lambda_0}(s))\|^2 \, ds$$

$$+ 2 \sum_{j=1}^r C^2 \int_0^t \|g^i_{j,\lambda}(sx_{\lambda_0}(s)) - g^i_{j,\lambda_0}(s, x_{\lambda_0}(s))\|^2 \, ds$$

$$+ 2 \sum_{j,k=1}^r C^2 \int_0^t \|h^i_{jk,\lambda}(s, x(s)) - h^i_{jk,\lambda}(s, x_{\lambda_0}(s))\|^2 \, ds$$

$$\left. + 2 \sum_{j,k=1}^r C^2 \int_0^t \|h^i_{jk,\lambda}(s, x_{\lambda_0}(s)) - h^i_{jk,\lambda_0}(s, x_{\lambda_0}(s))\|^2 \, ds \right\}, \quad 0 \le t \le a,$$

since if f_λ is any one of the functions $g^i_{j,\lambda}$, $h^i_{jk,l\lambda}$ we have

$$\int_0^t \|f_\lambda(s, x_\lambda(s)) - f_{\lambda_0}(s, x_{\lambda_0}(s))\|^2 \, ds$$

$$\leq \int_0^t (\|f_\lambda(s, x_\lambda(s)) - f_\lambda(s, x_{\lambda_0}(s))\| + \|f_\lambda(s, x_{\lambda_0}(s)) - f_{\lambda_0}(s, x_{\lambda_0}(s))\|)^2 \, ds$$

$$\leq \int_0^t (2\|f_\lambda(s, x_\lambda(s)) - f_\lambda(s, x_{\lambda_0}(s))\|^2 + 2\|f_\lambda(s, x_{\lambda_0}(s)) - f_{\lambda_0}(s, x_{\lambda_0}(s))\|^2) \, ds$$

for $0 \leq t \leq a$.

We denote by $J^i_{1,\lambda}(t)$, $J^i_{2,\lambda}(t)$, $J^i_{3,\lambda}(t)$ and $J^i_{4,\lambda}(t)$, $0 \leq t \leq a$, the last four terms in the last member of the previous inequalities.

The hypothesis (H$_3$) enables us to state that

$$\|x^i_\lambda(t) - x^i_{\lambda_0}(t)\|^2 \leq (1 + r + r^2) \left\{ \|\alpha^i_\lambda(t) - \alpha^i_{\lambda_0}(t)\|^2 + J^i_{2,\lambda}(t) + J^i_{4,\lambda}(t) \right.$$

$$\left. + 2(r + r^2)C^2 \int_0^t w(\|x_\lambda(s) - x_{\lambda_0}(s)\|^2_n) \, ds \right\}, \quad 0 \leq t \leq a.$$

Denoting

$$M_1(\lambda) = (1 + r + r^2) \left\{ \||\alpha_\lambda - \alpha_{\lambda_0}|\|^2_n + \sum_{i=1}^n J^i_{2,\lambda}(a) + \sum_{i=1}^n J^i_{4,\lambda}(a) \right\}$$

$$M_2 = 2n(1 + r + r^2)(r + r^2)C^2$$

we see that by the hypothesis (H$_5$) we have

$$\lim_{\lambda \to \lambda_0} \sum_{i=1}^n (J^i_{2,\lambda}(a) + J^i_{4,\lambda}(a)) = 0$$

and so, since $\lim_{\lambda \to \lambda_0} \||\alpha_\lambda - \alpha_{\lambda_0}|\|^2_n = 0$, we deduce that $\lim_{\lambda \to \lambda_0} M_1(\lambda) = 0$.

On the other hand, we have that

$$\|x_\lambda(t) - x_{\lambda_0}(t)\|^2_n \leq M_1(\lambda) + M_2 \int_0^t w(\|x_\lambda(s) - x_{\lambda_0}(s)\|^2_n) \, ds, \quad 0 \leq t \leq a.$$

Applying Bihari's inequality, we obtain

$$\|x_\lambda(t) - x_{\lambda_0}(t)\|^2_n \leq G^{-1}(G(M_1(\lambda)) + M_2 t), \quad 0 \leq t \leq a,$$

thus
$$|||x_\lambda - x_{\lambda_0}|||_n^2 \le G^{-1}(G(M_1(\lambda)) + M_2 a).$$

Since $\lim_{\lambda \to \lambda_0} M_1(\lambda) = 0$ we deduce in a similar way to the proof of Theorem 3 that $\lim_{\lambda \to \lambda_0} |||x_\lambda - x_{\lambda_0}|||_n^2 = 0$. ∎

Remark 4. The problem of the parametric dependence of the solution processes on the coefficients was studied in [2], in the case of a Lipschitz condition on the functions g_j^i and h_{jk}^i.

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