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DUAL DIMENSION OF MODULES OVER NORMALIZING EXTENSIONS

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Abstract

Let $S = \sum_{i=1}^{n} Ra_i$ be a finite normalizing extension of R and suppose that ${}_SM$ is a left S-module. Denote by $\text{crk}(A)$ the dual Goldie dimension of the module A. We show that $\text{crk}({}_RM) \leq n \cdot \text{crk}({}_SM)$ if either ${}_SM$ is artinian or the group homomorphism $M \to a_iM$ given by $x \mapsto a_ix$ is an isomorphism.

1. Let R be a ring and let M be a left R-module. The *Goldie dimension* of M, defined as the cardinality of a maximal independent family of submodules of M, is denoted by $\text{rk}(M)$. A family $A_1, \dots, A_n$ of proper submodules of M is said to be *coindependent* if for each index i, $1 \leq i \leq n$, $A_i + \bigcap_{j \neq i} A_j = M$. A family $(A_i)_{i \in I}$ of submodules of M is said to be *coindependent* if each of its finite subfamilies is coindependent. The module M is said to be *hollow* if $M \neq 0$ and if every proper submodule of M is superfluous in M. Every family of submodules of M contains a maximal coindependent subfamily. The cardinality of a maximal coindependent family of submodules of M, denoted by $\text{crk}(M)$, is called the *dual Goldie dimension* of M. We shall need the following results, which can be found in [**2**], [**3**], [**6**].

1.1. If N is a proper submodule of M and if crk is finite then there exists a finite family of submodules $(A_i)_{i \in I}$ of M such that $\{N\} \cup \{A_i : i \in I\}$ is coindependent, M/A_i is hollow for each $i \in I$, and $N \cap \bigcap_{i \in I} A_i$ is superfluous in M.

1.2. $\text{crk}(M_1 \oplus M_2) = \text{crk}(M_1) + \text{crk}(M_2)$ for any modules M_1 and M_2.

1.3. If N is a submodule of M then

$$\text{crk}(M/N) \leq \text{crk}(M) \leq \text{crk}(M/N) + \text{crk}(N),$$

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and, when $\mathrm{crk}(M)$ is finite, $\mathrm{crk}(M/N) = \mathrm{crk}(M)$ if and only if N is superfluous in M.

1.4. It follows from 1.3 and the exact sequence

$$0 \to M/(N_1 \cap N_2) \to (M/N_1) \oplus (M/N_2) \to M/(N_1 + N_2) \to 0$$

for submodules N_1 and N_2 of M that if $\mathrm{crk}(M)$ is finite then

$$\mathrm{crk}(M) - \mathrm{crk}(M/(N_1 + N_2)) \leq \sum_{i=1}^{2}(\mathrm{crk}(M) - \mathrm{crk}(M/N_i)).$$

Let now $R \subset S$ be a finite normalizing extension, write $S = \sum_{i=1}^{n} Ra_i$ where $a_i R = Ra_i$ for each i, $1 \leq i \leq n$, and let ${}_sM$ be a left S-module. We fix this notation throughout this article. It was shown in Bit-David and Robson [1] that

$$\mathrm{rk}({}_sM) \leq \mathrm{rk}({}_RM) \leq n \cdot \mathrm{rk}({}_sM).$$

Since the proof of the second inequality appeals to Zorn's lemma, it is not clear that a formal dual of this result holds true. The purpose of this note is to show that under certain conditions, the inequality

$$(*) \qquad \mathrm{crk}({}_RM) \leq n \cdot \mathrm{crk}({}_SM)$$

is valid.

2. If $a \in S$ is a normal element of S, that is, if $Ra = aR$, then for a submodule N of ${}_RM$, aN is a submodule of ${}_RM$. The map $K \mapsto a^{-1}K \cap N = \{x \in N \mid ax \in K\}$ is a one-to-one function that takes a family of coindependent submodules of ${}_R(aN)$ into a family of coindependent submodules of ${}_RN$, so $\mathrm{crk}(aN) \leq \mathrm{crk}(N)$. If $aM = M$ and a is not a zero divisor on M then the map $K/N \mapsto aK/aN$ becomes a one-to-one function that takes a coindependent family of R-submodules of M/N to a coindependent family of submodules of M/aN . It follows that in this case, $\mathrm{crk}(M/aN) \geq \mathrm{crk}(M/N)$. We shall find it necessary to introduce the set $\mathcal{N}$ of all submodules ${}_RN$ of ${}_RM$ such that $SN = M$.

Lemma 2.1. *If $N_1, \ldots, N_k$ is a coindependent family of submodules of ${}_RM$ such that $SN_i \neq M$ for $i = 1, \ldots, k$ then $SN_1, \ldots, SN_k$ is a coindependent family of submodules of ${}_SM$.*

Proof: This follows from the observation that $SN_i + \bigcap_{j \neq i} SN_j \supseteq S(N_i + \bigcap_{j \neq i} N_j)$. ■

Proposition 2.2. *If N is a minimal member of $\mathcal{N}$ then*

$$\operatorname{crk}({}_RN) \leq \operatorname{crk}({}_SM) \leq \operatorname{crk}({}_RM) \leq n \cdot \operatorname{crk}({}_RN) \leq n \cdot \operatorname{crk}({}_SM).$$

In particular, $(*)$ is true when either of the modules ${}_RM$ or ${}_SM$ is artinian.

Proof: The first inequality follows from Lemma 2.1 and the minimality of N. Since M is the homomorphic image of $\bigoplus_{i=1}^n a_iN$, we deduce from 1.3 and the remarks preceding Lemma 2.1 that $\operatorname{crk}{}_RM \leq n \cdot \operatorname{crk}{}_RN$. Note that by Lemanoire [4], ${}_RM$ is artinian if and only ${}_SM$ is artinian. ■

Corollary 2.3. *If ${}_SM$ is a module such that ${}_RM$ has a submodule N with $S \otimes_R N \cong {}_S(SN)$ then $\operatorname{crk}({}_RN) \leq \operatorname{crk}({}_SM)$. In particular, we have*

(i) $\operatorname{crk}({}_RR) \leq \operatorname{crk}({}_SS)$;

(ii) *if $S \otimes_R M \cong {}_SM$ then $\operatorname{crk}({}_RM) = \operatorname{crk}({}_SM)$.*

Proof: If K is a submodule of ${}_RN$ then the hypothesis implies that $S \otimes_R (N/K) \cong SN/SK$. It follows from Shamsuddin [5] that $SK \neq M$ if $K \neq N$. Lemma 2.1 now gives the inequality $\operatorname{crk}({}_RN) \leq \operatorname{crk}({}_SM)$. Observe that $S \otimes_R R \cong {}_SS$ and we always have $\operatorname{crk}({}_SM) \leq \operatorname{crk}({}_RM)$, so the last two statements follow. ■

Proposition 2.4. *Suppose that for each i, $1 \leq i \leq n$ the group homomorphism $M \to a_iM$ given by $x \mapsto a_ix$ is an isomorphism. Then*

$$\operatorname{crk}({}_SM) \leq \operatorname{crk}({}_RM) \leq n \cdot \operatorname{crk}({}_SM).$$

Proof: We show first that if $\operatorname{crk}({}_SM)$ is finite then so is $\operatorname{crk}({}_RM)$. By induction on the integer k, $1 \leq k \leq n$, we show that if ${}_RM$ has an infinite coindependent family of submodules then there exists an infinite coindependent family $(M_i)_{i\in\mathbb{N}}$ of submodules of ${}_RM$ such that the family $(\bigcap_{j=1}^k a_j^{-1}M_i)_{i\in\mathbb{N}}$ is coindependent. We may assume that $a_1 = 1$, so the base case of the induction is clear. Let $1 \leq k < n$ and assume that $(\bigcap_{j=1}^k a_j^{-1}M_i = T_i)_{i\in\mathbb{N}}$ is coindependent. Put $a = a_{k+1}$ and observe that $(a^{-1}M_i)_{i\in\mathbb{N}}$ is coindependent. If $a(\bigcap_{i=r}^\infty T_i) + \bigcap_{i=r}^\infty M_i = M$ for some $r \in \mathbb{N}$ then the family $(T_i \cap a^{-1}M_i)_{i\geq r}$ is coindependent and we are then done. Otherwise, $\mathbb{N}$ partitions into disjoint non-empty finite subsets A_i such that for each $j \in \mathbb{N}$, $N_j = a(\bigcap_{i\in A_j} T_i) + \bigcap_{i\in A_j} M_i$ is a proper submodule of M and so the family $(N_j)_{j\in\mathbb{N}}$ is coindependent. But $\bigcap_{i\in A_j} T_i \subseteq \bigcap_{i=1}^{k+1} a_i^{-1}N_j$ and because $(\bigcap_{i\in A_j} T_i)_{i\in\mathbb{N}}$ is coindependent, we conclude that $(\bigcap_{i=1}^{k+1} a_i^{-1}N_j)_{j\in\mathbb{N}}$ is also coindependent. Since the submodules $\bigcap_{i=1}^{n} a_i^{-1} M_j$ are actually S-submodules of ${}_S M$, we deduce that ${}_S M$ has infinite dual Goldie dimension.

Next we show that

$$\operatorname{crk}({}_R M) \leq n \cdot \operatorname{crk}({}_S M).$$

It is possible to choose a member $N \in \mathcal{N}$ such that $\operatorname{crk}(M/N)$ is as large as possible. By 1.1, there exists a family $H_1, \dots, H_r$ of submodules of M such that $N, H_1, \dots, H_r$ is coindependent, $N \cap H_1 \cap \cdots \cap H_r$ is superfluous in ${}_R M$ and each M/H_i is hollow. Since $M/(N \cap H_i) \cong M/N \oplus M/H_i$, we have $\operatorname{crk}(M/(N \cap H_i)) > \operatorname{crk}(M/N)$, hence $S(N \cap H_i) \neq (N \cap H_i)$. Lemma 2.1 now implies that $r \leq \operatorname{crk}({}_S M)$. It follows from 1.1 and 1.3 that $\operatorname{crk}({}_R M) = \operatorname{crk}({}_R(M/N)) + r$, so $\operatorname{crk}({}_R M) - \operatorname{crk}({}_R(M/N)) \leq \operatorname{crk}({}_S M)$. Using 1.4 and the observation that $\operatorname{crk}(M/a_i N) \geq \operatorname{crk}(M/N)$ we now conclude that

$$\operatorname{crk}({}_R M) \leq \sum_{i=1}^{n} \big(\operatorname{crk}(M) - \operatorname{crk}(M/(a_i N)) \big) \leq n \cdot \operatorname{crk}({}_S M). \quad \blacksquare$$

References

1. J. BIT-DAVID AND J. C. ROBSON, "*Title book?*," Lecture Notes in Mathematics **825**, Springer-Verlag, Berlin, New York, pp. 1–5.
2. P. GRZESZCZUCK AND E. R. PUCZYLOWSKI, On Goldie and dual Goldie dimensions, *J. Pure and Appl. Alg.* **31** (1984), 47–54.
3. A. HANNA AND A. SHAMSUDDIN, Duality in the category of modules. Applications, *Algebra Berichte* **49** (1984).
4. B. LEMANOIRE, Dimension de Krull et codéviation. Application au théoreme d'Eakin, *Communications in Algebra* **6** (1978), 1647–1665.
5. A. SHAMSUDDIN, Finite normalizing extensions, *Jour. Alg.* **151** (1992), 218–220.
6. K. VARADARAJAN, Dual Goldie dimension, *Comm. Alg.* **7** (1979), 565–610.

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