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A COUNTEREXAMPLE IN OPERATOR THEORY

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Abstract

The purpose of this note is to give an explicit construction of a bounded operator T acting on the Space $L^2[0,1]$ such that $|Tf(x)| \leq \int_0^1 |f(y)|\,dy$ for a.e. $x \in [0,1]$ and, nevertheless, $\|T\|_{S_p} = \infty$ for every $p < 2$. Here $\| \quad \|_{S_p}$ denotes the norm associated to the Schatten-von Neumann classes

A. Definitions and statement of the problem

The purpose of this note is to give an alternative and direct construction to the one presented in reference [**1**] and we shall follow closely the lines of introduction contained in that paper:

Let $(X, \mathcal{F}, \mu)$ be a measure space and let S, T be two bounded linear operators on $L^2(X, \mathcal{F}, \mu)$. The operator S dominates pointwise T if it happens that $|Tf(x)| \leq S(|f|)(x)$ a.e. x, for every function $f \in L^2$. For example: if $T = T_K$ is an integral operator associated to a $\mu \otimes \mu$-measurable kernel K then obviously $T_{|K|}$ dominates T_K. For an operator T on a Hilbert space H we have the singular numbers

$$S_n(T) := \inf\{\|T - T_n\|;\ \text{rank}(T_n) < n\}.$$

If T is compact then it is known that $S_n(T) = \lambda_n(\sqrt{T^*T})$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \dots$ denotes the sequence of eigenvalues of $\sqrt{T^*T}$ arranged in non-increasing order and repeated according to their multiplicities.

The Schatten-von Neumann classes $S_p = S_p(H)$ are defined by

$$S_p = \{T \text{ bounded} \mid \sum_{n=1}^{\infty} [S_n(T)]^p < \infty\} \text{ if } 0 < p < \infty$$

and

$$S_\infty = \{T \text{ bounded} \mid \lim_{n\to\infty} S_n(T) = 0\}.$$

Among them the more important ones are S_1, S_2, S_∞ which correspond, respectively, to nuclear, Hilbert-Schmidt and compact operators.

Suppose that we have the information that the operator $S \in S_p(H)$ pointwise dominates T. Does it follows necessarily that $T \in S_p$?.

This is a natural question whose answer is known to be YES when $p = 2n$ is an even natural number. In the paper quoted above a construction of a probabilistic nature is introduced to observe that the answer to our question is NO when $0 < p < 2$. Here we present an explicit example where such situation occurs.

B. The counterexample

In the following we shall consider $X = [0,1]$ and $\mu = dx$ is Lebesgue measure. Let us define the operator T by the formula:

$$Tf(x) = \int_0^1 e^{2\pi i(x-y)\cdot\nu(y)} f(y)\,dy$$

where $\nu(t) = [e^{\frac{1}{t}}]$ and $[x]$ denotes the integer part of the real number x i.e.

$$\nu(x) = n \text{ if } x \in I_n = \left(\frac{1}{\log(n+1)}, \frac{1}{\log(n)}\right]$$
$$n = 3,4,\ldots,\ I_2 = \left(\frac{1}{\log 3}, 1\right].$$

Then we have:

$$T^* f(x) = \int_0^1 e^{-2\pi i(y-x)\cdot\nu(x)} f(y)\,dy$$

and

$$T^*Tf(x) = \left[\int_{I_{\nu(x)}} e^{-2\pi i z\cdot\nu(x)} f(z)\,dz\right] e^{2\pi i x\nu(x)}.$$

Let us consider the family of functions

$$f_k(x) = e^{2\pi i k\cdot x}\chi_{I_k}(x)$$

where χ_{I_k} is the indicator function of the interval I_k i.e. $\chi_{I_k}(x) = 1$ if $x \in I_k$ and $\chi_{I_k}(x) = 0$ otherwise.

Then we have:

$$T^*Tf_k(x) = \mu(I_k)f_k(x)$$

i.e., f_k is an eigenfunction corresponding to the eigenvalue $\mu(I_k) = \frac{1}{\log k} - \frac{1}{\log(k+1)} \sim \frac{1}{k(\log k)^2}$.

Therefore $\sqrt{T^*T}$ has eigenvalues $\sqrt{\mu(I_k)} \sim \frac{1}{k^{1/2}\log k}$ and, by well known results, the decreasing sequence of singular values of T must satisfy

$$S_n(T) \geq \frac{1}{n^{1/2}\log n}$$

which implies $T \notin S_p$, if $p < 2$. On the other hand it is clear that

$$|Tf(x)| \leq \int_0^1 |f(y)|\,dy = S(|f|)$$

and $\operatorname{rank}(S) = 1$ which yields $S \in S_p$ for every $0 < p$.

Remark. There is nothing particularly special about the division points $\frac{1}{\log n}$ and the reader may consider the more general operator

$$Tf(x) = \sum_{n=1}^{\infty} \int_{x_{n-1}}^{x_n} e^{2\pi i(x-y)\cdot n} f(y)\,dy$$

where x_n is any increasing sequence with $x_0 = 0$ and x_n tending to 1 as $n \to \infty$. It has the kernel $\sum_{n=1}^{\infty} e^{2\pi in(x-y)}\chi_n(y)$ where χ_n denotes the characteristic function of the interval (x_{n-1}, x_n) while the adjoint kernel is

$$\sum_{n=1}^{\infty} e^{-2\pi in(x-y)}\chi_n(x).$$

This yields

$$T^*Tf(x) = \sum_{n=1}^{\infty} e^{-2\pi inx}\chi_n(x) \int_{x_{n-1}}^{x_n} e^{-2\pi iny}\chi_n(y)\,dy.$$

It follows then that the functions $f_n(x) = e^{-2\pi inx}\chi_n(x)$ are eigenfunctions with corresponding eigenvalues given by $\lambda_n = x_n - x_{n-1}$. And this yields the estimate

$$\|T\|_{S_p} \geq \left(\sum_{n=1}^{\infty} \lambda_n^{p/2}\right)^{1/p}.$$

References

1. F. Cobos and T. Kühn, On a conjecture of Barry Simon on trace ideals, *Duke Math. J.* **59(1)**, 295–299.
2. B. Simon, "*Trace ideals and their applications*," Cambridge Univ. Press, 1979.

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