Publicacions Matemàtiques, Vol **37** (1993), 127–132.

ON THE THREE-SPACE PROPERTY: NON-CONTAINMENT OF l_p, $1 \leq p < \infty$, OR c_0 SUBSPACES

J. C. Díaz

Abstract

The main result in this paper is the following: Let E be a Fréchet space having a normable subspace X isomorphic to l_p, $1 \leq p < \infty$, or to c_0. Let F be a closed subspace of E. Then either F or E/F has a subspace isomorphic to X.

In this paper we prove the following result: *Let E be a Fréchet space with a normable subspace X isomorphic to l_p, $1 \leq p < \infty$ or to c_0. Let F be a closed subspace of E. Then either F or E/F contains a copy of X*, i.e. we prove that "having no subspace isomorphic to l_p for some $1 \leq p < \infty$, or to c_0" is a three-space property.

We recall that a certain property (P) is said to be a *three-space property* for Fréchet spaces if whenever E is a Fréchet space and F is a closed subspace of E such that both F and E/F enjoy (P) then E also has the property (P).

Many authors have been concerned with studying whether the usual locally convex properties are or not three-space properties, e.g. the three-space problem has a positive answer for the following properties:

(i) being "Montel", "nuclear", "quasinormable", "reflexive" (see [**12**]);
(ii) "being a quojection", "having a continuous norm" (cf. [**10**]);
(iii) "(DN)", "(Ω)" ([**14**], [**15**]).

On the other hand it has a negative answer for the following ones:

(i) "being a product of Banach spaces" ([**10**]);
(ii) "having an unconditional basis" (see [**6**] for Banach spaces, [**9**] and [**11**] for nuclear spaces. Also see [**8**, 1.b and 1.g] for related results);
(iii) "being distinguished", "having the density condition" ([**2**]).

Diestel (see [**3**, lemma 8 and note]) has already proved that "having no subspace isomorphic to l_1" is a three-space property for Banach spaces. The proof quoted there is heavily based on the Rosenthal's l_1-theorem so we need a quite different approach to establish the general case. Indeed we first obtain a result on stability of basic sequences (proposition 2) that could be of independent interest. As a consequence we give an extension to Fréchet spaces of a theorem of Rosenthal on sums of totally incomparable Banach spaces; this is done in proposition 4 from where our theorem follows.

Throughout this note, given a Fréchet space E we denote by $(V_k)_{k\in\mathbb{N}}$ a basis of absolutely convex and closed 0-neighbourhoods, with $V_{k+1} \subset V_k (k \in \mathbb{N})$, moreover $(\|\cdot\|_k)_{k\in\mathbb{N}}$ denotes the sequence of associated seminorms.

The next lemma can be readily checked. ($B(l_\infty)$ denotes the closed unit ball of l_∞).

1. Lemma. *Let $(y_n)_{n\in\mathbb{N}}$ be an absolutely summing sequence in a Fréchet space E (i.e. $\sum_{n=1}^{\infty} \|y_n\|_k < \infty, \forall k \in \mathbb{N}$). Then the set $\{\sum_{n=1}^{\infty} \delta_n y_n;\ \delta = (\delta_n) \in B(l_\infty)\}$ is relatively compact.*

We first need a result on stability of basic sequences.

Let $(x_n)_{n\in\mathbb{N}}$ be a basic sequence in a Fréchet space E, set $F := \overline{sp}\{x_n;\ n \in \mathbb{N}\}$ and denote by $(f_n)_{n\in\mathbb{N}}$ the sequence of associated coefficient functionals (i.e. $f_n \in F'$ and $f_n(x_m) = \delta_{nm}$, for all $n, m \in \mathbb{N}$). By a result of Weill $(x_n)_{n\in\mathbb{N}}$ is bounded away from zero (i.e there is a continuous seminorm $p(\cdot)$ on E such that $\inf\{p(x_n);\ n \in \mathbb{N}\} > 0$) if and only if $(f_n)_{n\in\mathbb{N}}$ is equicontinuous. By Hahn-Banach we can extend f_n to E such that the sequence of extensions $(f_n^*)_{n\in\mathbb{N}}$ is still equicontinuous in E'. From now on if $(x_n)_{n\in\mathbb{N}}$ is a bounded away from zero basic sequence in E, then $(f_n^*)_{n\in\mathbb{N}}$ denotes the (equicontinuous) extension to E of the sequence of coefficient functionals.

2. Proposition. *Let $(x_n)_{n\in\mathbb{N}}$ be a bounded away from zero basic sequence in a Fréchet space E. Let $k_0 \in \mathbb{N}$ such that $f_n^* \in V_{k_0}^\circ$. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in E such that $\sum_{n=1}^{\infty} \|y_n - x_n\|_k < \infty$ for every $k \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \|y_n - x_n\|_{k_0} = \rho < 1$. Then $(y_n)_{n\in\mathbb{N}}$ is a basic sequence equivalent to $(x_n)_{n\in\mathbb{N}}$.*

Proof: We define

$$T : E \longrightarrow E, \qquad x \longrightarrow \sum_{n=1}^{\infty} f_n^*(x)(y_n - x_n).$$

Our aim is to show that $T+I$ is an isomorphism from E onto E where I denotes the identity map. To do this we shall use the next Riesz's theorem (cf. [**7**, p. 207]): *Let E be a linear topological space and T a continuous linear mapping of E into itself that maps some 0-neighbourhood into a compact set. (Such an operator is termed completely continuous in* [**7**] *and compact in some other texts, e.g. see* [**5**].*) Then a non-zero scalar λ is not an eigenvalue of T if and only if $T-\lambda I$ is an isomorphism from E onto E.*

It is obvious that T is well defined, moreover for every $k \in \mathbb{N}$ we have

$$\left\| \sum_{n=1}^{\infty} f_n^*(x)(y_n - x_n) \right\|_k \leq \|x\|_{k_0} \left(\sum_{n=1}^{\infty} \|y_n - x_n\|_k \right),$$

whence T is continuous. By lemma 1 and by using once more that $f_n^* \in V_{k_0}^\circ (n \in \mathbb{N})$ we have that $T(V_{k_0})$ is relatively compact. Our claim is proved if we show that $T-(-I)$ is injective (i.e. $\lambda = -1$ is not an eigenvalue of T). Indeed, let $x \in E$ such that

$$T + I(x) = \sum_{n=1}^{\infty} f_n^*(x)(y_n - x_n) + x = 0.$$

Two cases can happen: (i) $f_n^*(x) = 0$, $\forall n \in \mathbb{N}$. Then x is obviously equal to 0. (ii) $\exists n \in \mathbb{N}$ such that $f_n^*(x) \neq 0$. Then $\|x\|_{k_0} > 0$ and we have a contradiction since

$$\|x\|_{k_0} \leq \left\| \sum_{n=1}^{\infty} f_n^*(x)(y_n - x_n) + x \right\|_{k_0} + \left\| \sum_{n=1}^{\infty} f_n^*(x)(y_n - x_n) \right\|_{k_0} \leq$$
$$\leq \sum_{n=1}^{\infty} |f_n^*(x)| \, \|y_n - x_n\|_{k_0} \leq \rho \|x\|_{k_0} < \|x\|_{k_0}.$$

Therefore $T+I$ is an isomorphism. We conclude by noting that $T+I(x_n) = y_n$ for every $n \in \mathbb{N}$. ■

3. Remark. We have been inspired by the paper [**16**] of Weill where he shows a similar result for bases instead of basic sequences. Our proof is simpler since we just prove that we need. However, following step by step the proof of Weill and bearing in mind the technique of our proposition 2, one can check the next stronger result: *Let $(x_n)_{n\in\mathbb{N}}$ and $(f_n^*)_{n\in\mathbb{N}}$ be as in the statement of Proposition 2. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in E such that the series $\sum_{n=1}^{\infty}(x_n - y_n)$ is unconditionally convergent and $\sum_{n=1}^{\infty} \|y_n - x_n\|_{k_0} = \rho < 1$. Then $(y_n)_{n\in\mathbb{N}}$ is a basic sequence equivalent to $(x_n)_{n\in\mathbb{N}}$.*

The second preliminary result is an extension to Fréchet spaces of a well known theorem of Rosenthal [**13**]. (We must observe that our proposition below is a particular case of [**4**, Theorem 4.1] but we prefer to sketch a proof which is similar to the one of Rosenthal). Let us recall that two Fréchet spaces E and F are said to be *totally incomparable* if no infinite dimensional subspace of E is isomorphic to any subspace of F.

4. Proposition. *Let E be a Fréchet space. Let F and X be totally incomparable subspaces of E such that X is normable. Then $F+X$ is closed in E.*

Proof: Fix $\|\cdot\|$ a norm defining the topology of X and $(\|\cdot\|_k)_{k\in\mathbb{N}}$ a sequence of seminorms defining the topology of E. Our hypothesis implies that $F\cap X$ is finite dimensional whence we may assume that $F\cap X=\{0\}$. Suppose that $F+X$ is not closed. This is the same as to say that the operator

$$J: X\times F \longrightarrow E, \qquad (x,y)\longrightarrow x-y,$$

is not open onto its image, i.e. there is $k\in\mathbb{N}$ such that $J(\{(x,y);\ \|x\|+\|y\|_k\leq 1\})$ is not a 0-neighbourhood in $X+F$ or equivalently the origin is a cluster point of $\{x-y;\ x\in X,\ y\in F,\ \|x\|+\|y\|_k=1\}$. It is readily checked that this is equivalent to the fact that there is $\rho>0$ such that

$$(1) \qquad 0\in\overline{\{x-y;\ x\in X,\ y\in F,\ \|x\|=1,\ \|y\|_k\leq\rho\}}.$$

Now the proof becomes very similar to the one of Rosenthal and we just give some details. We assume that X is separable and choose a linear isometry $T: X\to C([0,1])$. Let $(z_n)_{n\in\mathbb{N}}$ be a basis for $C([0,1])$ and $(z_n^*)_{n\in\mathbb{N}}$ the sequence of associated coefficient functionals. On account of (1), we take as in [**13**] a sequence $(x_n)_{n\in\mathbb{N}}\subset X$, with $\|x_n\|=1$ for every $n\in\mathbb{N}$, such that $z_i^*(T(x_n))=0$ for every i, $1\leq i\leq n$, and

$$(2) \qquad \inf\{\|x_n-y\|_n;\ y\in F\}\leq 1/2^n.$$

From [**8**, 1.a] $(x_n)_{n\in\mathbb{N}}$ has a subsequence $(x_{\sigma(n)})_{n\in\mathbb{N}}$ such that $(T(x_{\sigma(n)}))_{n\in\mathbb{N}}$, and thus $(x_{\sigma(n)})_{n\in\mathbb{N}}$, is a basic sequence (bounded away from zero by construction). We use (2) to select $y_{\sigma(n)}$ such that $\|x_{\sigma(n)}-y_{\sigma(n)}\|_{\sigma(n)}\leq 2^{-\sigma(n)}(n\in\mathbb{N})$. To finish it is obvious that we can choose a subsequence of $(x_{\sigma(n)})_{n\in\mathbb{N}}$ satisfying the hypotheses of Proposition 2. Therefore X and F contain isomorphic subspaces, a contradiction. ■

5. Remark. The next particular and simple case of Proposition 4 has been observed in [**1**]: *Let E be a Fréchet space and let F and X be closed subspaces of E such that F is Montel and X is normable. Then $F+X$ is closed in E.*

6. Lemma. *Let E be a Fréchet space. Let F and G be subspaces of E such that $F \cap G = \{0\}$ and $F + G$ is closed in E. Then the quotient mapping $\pi : E \to E/F$ restricted to G is an isomorphism onto $\pi(G)$.*

7. Theorem. *Let E be a Fréchet space having a normable subspace X isomorphic to l_p for some $1 \leq p < \infty$, or to c_0. Let F be a closed subspace of E. Then either F or E/F has a subspace isomorphic to X.*

Proof: Assume that F has no copy of X. Let us note that every closed subspace of X contains a subspace isomorphic to X (e.g. see [**8**, 2.a.2]). Therefore F can not have any infinite dimensional subspace of X, hence F and X are totally incomparable. By proposition 4, $F + X$ is closed in E. Moreover, by removing a finite dimensional subspace if it is necessary we can assume that $F \cap X = \{0\}$. We then aply lemma 6 to conclude that E/F has a subspace isomorphic to X. ■

Finally we observe that with the same proof of theorem 7 (and by the remark 5) we obtain the next result: *Let E be a Fréchet space and let F and X be closed subspaces of E such that F is Montel and X is normable. Then E/F has a subspace isomorphic to a finite-codimensional subspace of X.*

I thank Professor Bonet for reading and improving the manuscript.

References

1. J. Bonet and J. C. Díaz, Subspaces of Fréchet spaces without the density condition, preprint, 1991.
2. J. Bonet, S. Dierolf and C. Fernández, On the three-space-problem for distinguished Fréchet spaces, *Bull. Soc. Roy. Sci. Liège* **59** (1990), 301–306.
3. J. Diestel, A survey of results related to the Dunford-Pettis property, *Contemporary Math.* **2** (1980), 15–60.
4. L. Drewnowski, On minimally subspace-comparable F-spaces, *J. Funct. Anal.* **26** (1977), 315–332.
5. H. Jarchow, "*Locally convex spaces*," Teubner, Stuttgart, 1981.
6. N. J. Kalton and N. T. Peck, Twisted sums of sequence spaces and the three-space problem, *Trans. Amer. Math. Soc.* **255** (1979), 1–30.
7. J. L. Kelley and I. Namioka, "*Linear topological spaces*," Springer, 1963.

8. J. LINDENSTRAUSS AND L. TZAFRIRI, "*Classical Banach spaces I*," Springer, 1977.
9. G. METAFUNE, Quojections and finite-dimensional decompositions, *Arch. Math.* **57** (1991), 609–616.
10. G. METAFUNE AND V. B. MOSCATELLI, On the three-space property for locally convex spaces, *Collect. Math.* **37** (1986), 287–296.
11. G. METAFUNE AND V. B. MOSCATELLI, Nuclear Fréchet spaces with basis do not have the three-space property, *Arch. Math.* **50** (1988), 369–370.
12. W. ROECLKE AND S. DIEROLF, On the three-space-problem for topological vector spaces, *Collect. Math.* **32** (1981), 3–25.
13. H. P. ROSENTHAL, On totally incomparable Banach spaces, *J. Funct. Analysis* **4** (1969), 167–175.
14. D. VOGT, Subspaces and quotients of (s), in "*Functional Analysis: surveys and recent results*," K. D. Bierstedt, B. Fuchssteiner (eds.), North-Holland Math. Studies, Amsterdam, 1977, pp. 167–187.
15. D. VOGT, On two classes of (F)-spaces, *Arch. Math.* **45** (1985), 255–266.
16. L. J. WEILL, Stability of bases in complete barrelled spaces, *Proc. Amer. Math. Soc.* **18** (1967), 1045–1050.

Departamento de Matematicas
E.T.S.I. Agrónomos
Universidad de Córdoba
14004 Córdoba
SPAIN

Rebut el 10 de Juny de 1992