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ON MUCKENHOUPT AND SAWYER CONDITIONS FOR MAXIMAL OPERATORS

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Abstract

Let $M_s (0 \leq s < n)$ be the maximal operator

$$(M_s f)(x) = \sup \left\{ |Q|^{\left[\frac{s}{n}-1\right]} \|f \mathbf{1}_Q\|_{L^1(dy)};\ Q \text{ a cube with } Q \ni x \right\},$$

and $u(x)$ and $v(x)$ be weight functions on $\mathbb{R}^n$. For $1 < p \leq q < \infty$ and $[p^{-1} - q^{-1}] \leq (s/n)$, we prove the equivalence of the Sawyer condition

$$\|(M_s v^{-1/(p-1)} \mathbf{1}_Q) \mathbf{1}_Q\|_{L^q_u} \leq S \|\mathbf{1}_Q\|_{L^p_{v^{-1/(p-1)}}} \quad \text{for all cubes } Q$$

to the Muckenhoupt condition

$$|Q|^{\frac{s}{n}+\frac{1}{p}-\frac{1}{p}} \left(\frac{1}{|Q|} \int_Q u \right)^{1/q} \left(\frac{1}{|Q|} \int_Q v^{-1/(p-1)} \right)^{1-\frac{1}{p}} \leq$$
$$\leq A \text{ for all cubes } Q$$

whenever the measure $d\sigma = v^{-1/(p-1)}\, dx$ satisfies

$$\frac{|Q'|_\sigma}{|Q|_\sigma} \leq C \left(\frac{|Q'|}{|Q|} \right)^\nu \quad \text{for all cubes } Q,\ Q'$$
$$\text{with } Q' \subset Q \text{ and } 1 - (s/n) \leq \nu.$$

This growth condition is weaker than the A_∞ condition usually used to obtain such an equivalence.

0. Introduction

Let u, v weight functions on $\mathbb{R}^n$, $n \geq 1$ (i.e. nonnegative locally integrable functions). The Hardy-Littlewood maximal operator is given by

$$(Mf)(x) = \sup \left\{ |Q|^{-1} \|f \mathbf{1}_Q\|_{L^1(dy)};\ Q \text{ a cube with } Q \ni x \right\}.$$

Throughout this paper Q will denote a cube with sides parallel to the co-ordinate planes. It is fundamental in analysis to characterize the pairs of nonnegative weights (u, v) for which

$$(1) \quad \|Mf\|_{L^p_u} \leq C\|f\|_{L^p_v} \quad \text{for all functions } f (1 < p < \infty,\ C = C(n,p,u,v) > 0);$$

here $\|g\|_{L^r_w}$ denotes $\left(\int_{\mathbb{R}^n} |g|^r w\, dx\right)^{1/r}$, and dx the Lebesgue measure on $\mathbb{R}^n$. Muckenhoupt [**Mu**] showed that inequality (1) for $u = v$ holds if and only if

$$\left(\frac{1}{|Q|}\int_Q v\right)^{1/p}\left(\frac{1}{|Q|}\int_Q v^{-1/(p-1)}\right)^{1-\frac{1}{p}} \leq A \text{ for all cubes } Q.$$

We write $v \in A_p$. This condition can be viewed as a particular case of $(u, v) \in A(p)$, i.e.

$$\left(\frac{1}{|Q|}\int_Q u\right)^{1/p}\left(\frac{1}{|Q|}\int_Q v^{-1/(p-1)}\right)^{1-\frac{1}{p}} \leq A \text{ for all cubes } Q.$$

It is clear that $(u, v) \in A(p)$ is a necessary condition for (1), but in general it is not a sufficient condition (see [**Mu**] for a countrexample). A special case of a Sawyer's result [**Sa**^{2}] shows that (1) is in fact equivalent to $(u, v) \in S(p)$, i.e.

$$\|(Mv^{-1/(p-1)}\mathbf{1}_Q)\mathbf{1}_Q\|_{L^p_u} \leq S\|\mathbf{1}_Q\|_{L^p_{v^{-1/(p-1)}}} < \infty \text{ for all cubes } Q.$$

However for $u = v$, it is not obvious that $(v, v) \in A(p)$ implies $(v, v) \in S(p)$. This point was solved by Hunt-Kurtz-Neugebauer [**Hu-Ku-Ne**].

More generally the two weight norm inequality

$$(2) \quad \|M_s f\|_{L^q_u} \leq c\|f\|_{L^p_v}\ 1 < p \leq q < \infty,\ 0 < s < n,\ [p^{-1} - q^{-1}] \leq (s/n)$$

for the fractional maximal operator

$$(M_s f)(x) = \sup\left\{|Q|^{\left[\frac{s}{n}-1\right]}\|f\mathbf{1}_Q\|_{L^1(dy)};\ Q \text{ a cube with } Q \ni x\right\}$$

was characterized by Sawyer [**Sa**^{2}] by the condition $(u, v) \in S(s, p, q)$, i.e.

$$\|(M_s v^{-1/(p-1)}\mathbf{1}_Q)\mathbf{1}_Q\|_{L^q_u} \leq S\|\mathbf{1}_Q\|_{L^p_{v^{-1/(p-1)}}} < \infty \text{ for all cubes } Q.$$

A necessary condition for (2) is $(u,v) \in A(s,p,q)$, i.e.

$$|Q|^{\frac{s}{n}+\frac{1}{p}-\frac{1}{p}}\left(\frac{1}{|Q|}\int_Q u\right)^{1/q}\left(\frac{1}{|Q|}\int_Q v^{-1/(p-1)}\right)^{1-\frac{1}{p}} \leq A \text{ for all cubes } Q.$$

Although $(u,v) \in A(s,p,q)$ is not sufficient for (2), it is nevertheless a more easily verifiable condition. So for $d\sigma = v^{-1/(p-1)}\,dx \in A_\infty$ (i.e. $d\sigma \in A_r$ for some $r > 1$) Perez [**Pe**] (see also Sawyer [**Sa**^{1}]) proved that $(u,v) \in A(s,p,q)$ implies (2).

In this paper we give an analogous result (see Theorem I) for weights v such that $d\sigma \in B_\nu$ with $[1-(s/n)] \leq \nu$, i.e.:

$$\frac{|Q'|_\sigma}{|Q|_\sigma} \leq C\left(\frac{|Q'|}{|Q|}\right)^\nu \text{ for all cubes } Q,\, Q' \text{ with } Q' \subset Q;$$

here $|Q|_\sigma$ denotes $\int_Q \sigma\,dx$.

If $d\sigma \in A_\infty$ then $d\sigma \in B_\delta$ for some $\delta > 0$ i.e.

$$\frac{|E|_\sigma}{|Q|_\sigma} \leq C\left(\frac{|E|}{|Q|}\right)^\delta \text{ for all cubes } Q \text{ and all mesurable sets } E \text{ with } E \subset Q.$$

But, as we will see, there are measures $d\mu$ such that $d\mu \in B_\delta$ and $d\mu \notin A_\infty$. First it is known [**Ga-Fr**] that $d\sigma \in A_\infty$ implies $d\sigma \in D_\infty$ i.e.

$$|2Q|_\sigma \leq D|Q|_\sigma \text{ for all cubes } Q,\, D = D(\sigma) > 1;$$

$2Q$ is the cube with the same center as Q but with lenghts expanded two times. The condition $d\sigma \in D_\infty$ is equivalent to $d\sigma \in D_\varepsilon$ for some $\varepsilon \geq 1$ (see Proposition VIII below), i.e.

$$|tQ|_\sigma \leq Ct^{n\varepsilon}|Q|_\sigma \text{ for all cubes } Q \text{ and all } t \geq 1.$$

Also $d\sigma \in D_\infty$ implies $d\sigma \in RD_\nu$ for some $\nu \in]0,1]$ (see Proposition VIII below), i.e.

$$t^{n\nu}|Q|_\sigma \leq C|tQ|_\sigma \text{ for all cubes } Q \text{ and all } t \geq 1.$$

The condition RD_ν is weaker than the doubling condition D_∞ (for example if $w(x) = e^{|x|}$ then $w\,dx \in RD_\nu$ for some $\nu \in]0,1]$ but $w\,dx \notin D_\infty$). Hence if $d\sigma \in A_\infty$ then $d\sigma \in D_\infty \cap RD_\nu$ for some $\nu \in]0,1]$. But we can have $d\sigma \in D_\infty$ with $d\sigma \notin A_\infty$ (see [**Wi**] for an example). As we will see below, if $d\sigma \in B_\nu$ then $d\sigma \in RD_\nu$ and conversely $d\sigma \in D_\infty \cap RD_\nu$ implies $d\sigma \in B_\nu$. The condition $d\sigma \in D_\infty \cap RD_\nu$ is weaker than $d\sigma \in A_\infty$

and it is more verifiable than $d\sigma \in B_\nu$. So if $d\sigma \in D_\infty$ then $d\sigma \in B_\nu$ for ν small enough, while $d\sigma$ does not automatically belong to A_∞.

Contrary to the Perez's approach [**Pe**] (which consists to obtain (2) from $A(s,p,q)$ by exploiting properties of Calderon-Zygmund cubes) our method lies on the same philosophy as the Hunt-Kurtz-Neugebauer [**Hu-Ku-Ne**] results mentioned above. Using the condition $d\sigma \in B_\nu$ we directely derive the condition $S(s,p,q)$ from $A(s,p,q)$. For applications, the nature of our result leads to the following: "Let $d\sigma \in D_\infty$. For what reals ε, ν (with $\varepsilon \geq 1$ and $\nu \leq 1$) have we $d\sigma \in D_\varepsilon$ and $d\sigma \in RD_\nu$? Can we choose ε sufficiently small and ν big?".

In Section 1 we begin to state our main result (see Theorem I). Then we give growth conditions (see Proposition II) which are more useful than those used in our result. In Section 2 with the usual weights $u(x) = |x|^\beta$, $v(x) = |x|^\alpha$ we recall how to realize the $A(s,p,q)$ condition (see Proposition IV). In order to answer the above questions we reviewed how $A_p \Rightarrow D_\infty$ and $A_p \Rightarrow RD_\nu$ (see Proposition V), $D_\infty \Rightarrow RD_\nu$ (see Proposition VIII). By these, we bring out precise values of ε and ν (see Section 4). Proofs of main results are in Section 3.

1. The main result

To include classical maximal functions, we work with the operator

$$(M_\Phi f)(x) = \sup\left\{\Phi(Q)|Q|^{-1}\|f\mathbf{1}_Q\|_{L^1(dy)}; Q \text{ a cube with } Q \ni x\right\}$$

where Φ is a map defined on the set of cubes, taking its values in $]0,\infty[$ and satisfying the following growth conditions H:

1) $\Phi(Q_1) \leq C\Phi(Q_2)$ for all cubes Q_1, Q_2 with $Q_1 \subset Q_2$; $C = C(\Phi) > 0$.
2) There are C_1, $C_2 > 0$, λ, $\eta \in [0,1[$ such that

$$C_1 t^{n\lambda}\Phi(Q) \leq \Phi(tQ) \leq C_2 t^{n\eta}\Phi(Q) \text{ for all cubes } Q \text{ and all } t \geq 1.$$

When $\Phi(Q) = 1$ we obtain the Hardy-Littlewood maximal operator. The fractional maximal operator $M_s (0 < s < n)$ is given by $\Phi(Q) = |Q|^{s/n}$. Maximal operators connected to the Bessel potential (see [**Ke-Sa**]) are defined by $\Phi(Q) = \int_0^{|Q|^{1/n}} \varphi(s)\,ds$; and generally M_Φ arises in studies of other potential operators (see [**Ch-St-Wh**]).

Let $1 < p \leq q < \infty$ and (u,v) be a pairwise of weights. We write $(u,v) \in S(\Phi,p,q)$ if for some constant $S > 0$

$$\|(M_\Phi v^{-1/(p-1)}\mathbf{1}_Q)\|_{L^q_u} \leq S\|\mathbf{1}_Q\|_{L^p_{v^{-1/(p-1)}}} < \infty \text{ for all cubes } Q.$$

Also we write $(u,v) \in A(\Phi,p,q)$ holds for some $A>0$ if

$$\Phi|Q|Q^{\frac{1}{q}-\frac{1}{p}}\left(\frac{1}{|Q|}\int_Q u\right)^{1/q}\left(\frac{1}{|Q|}\int_Q v^{-1/(p-1)}\right)^{1-\frac{1}{p}} \leq A \text{ for all cubes } Q.$$

In this paper we always adopt the convention $0\cdot\infty = 0$. From condition $A(\Phi,p,q)$ and the Lebesgue theorem whenever $u \neq 0$, we see that it is necessary to suppose

H3) $$\lim_{|Q|\to 0}\left(\Phi(Q)|Q|^{\frac{1}{q}-\frac{1}{p}}\right) \leq c.$$

For instance H3) is satisfied if $[p^{-1}-q^{-1}] \leq \lambda$. For $\Phi(Q)=1$ H3) implies $q \leq p$, and for $\Phi(Q)=|Q|^{s/n}$ it means $[p^{-1}-q^{-1}] \leq (s/n)$.

Let $\rho > 0$ and $d\sigma = \sigma dx$ be a weight function. As in Section 0, we write $d\sigma \in B_\rho$ if there is $B = B(\sigma) > 0$ such that

$$\frac{|Q'|_\sigma}{|Q|_\sigma} \leq B\left(\frac{|Q'|}{|Q|}\right)^\rho \text{ for all cubes } Q,\ Q' \text{ with } Q' \subset Q.$$

Also for a weight function u, then $d\sigma \in B_\rho(u)$ when

$$\frac{|Q'|_\sigma}{|Q|_\sigma} \leq B\left(\frac{|Q'|_u}{|Q|_u}\right)^\rho \text{ for all cubes } Q,\ Q' \text{ with } Q' \subset Q;\ B = B(\sigma,u) > 0.$$

Now we can state our main result:

Theorem I.

Let $1 < p \leq q < \infty$ and let Φ be a function which satisfies H1)-2-3.

A) *If $(u,v) \in S(\Phi,p,q)$ for a constant $S > 0$, then $(u,v) \in A(\Phi,p,q)$ for the constant $A = S$.*

B) *If $(u,v) \in A(\Phi,p,q)$ for a constant $A > 0$, then $(u,v) \in S(\Phi,p,q)$ whenever one of the following condition is satisfied:*

i) $d\sigma = v^{-1/(p-1)}\,dx \in B_\nu$ with $1-\lambda \leq \nu$

ii) $d\sigma = v^{-1/(p-1)}\,dx \in B_{(p/q)}(u)$.

If B is the constant in the condition on $d\sigma$ then the constant in $S(\Phi,p,q)$ takes the form $S = ABc(\Phi,n)$ in case of i), and $S = AB^{1/p}c(\Phi,n)$ in case of ii), here $c(\Phi,n) > 0$ depends only on Φ and n.

Proposition II.

A) *If $d\sigma \in B_\nu$ for some $\nu \in]0,\infty[$, then $d\sigma \in RD_\nu$. Conversely if $d\sigma \in D_\infty \cap RD_\nu$ then $d\sigma \in B_\nu$.*

B) *If $d\sigma \in B_{(p/q)}(u) \cap D_\infty$, there are $\varepsilon \in [1,\infty[$ and $\nu \in]0,1]$ such that $d\sigma \in RD_\nu$, $du \in D_\varepsilon$ and $\nu q \leq \varepsilon p$. Conversely if $d\sigma \in RD_\nu$ and $du \in D_\varepsilon$ for some $\varepsilon \in [1,\infty[$ and $\nu \in]0,1]$ with $\varepsilon p \leq \nu q$ then $d\sigma \in B_{(p/q)}(u)$.*

Consequently, for the case of the fractional maximal operator, we can state

Proposition III.
Let $1 < p \le q < \infty$, $0 \le s < n$, *and* $[p^{-1} - q^{-1}] \le (s/n)$. *Then* $(u, v) \in S(s,p,q)$ *is equivalent to* $(u, v) \in A(s,p,q)$ *if one of the following holds:*

i) $d\sigma = v^{-1/(p-1)}\,dx \in D_\infty \cap RD_\nu$ *with* $1 - (s/n) \le \nu$

ii) $d\sigma = v^{-1/(p-1)}\,dx \in RD_\nu$; $du \in D_\varepsilon$ *with* $\varepsilon p \le \nu q$.

2. Applications and furthers results

Assume the condition $A(s,p,q)$ holds for a constant $A > 0$. It is also equivalent to ask

(3)

$$|B|^{\frac{s}{n}+\frac{1}{q}-\frac{1}{p}}\left(\frac{1}{|B|}\int_B u\right)^{1/q}\left(\frac{1}{|B|}\int_B v^{-1/(p-1)}\right)^{1-\frac{1}{p}} \le A_1 \text{ for all balls } B$$

with $A_1 = Ac(s,n,p,q)$.

Let B be the ball $B(x_0, R) = \{y \in \mathbb{R}^n;\ |x - y| < R\}$.

If $|x_0| \le 2R$ then $B \subset B(0, 3R)$ and hence the first member of (3) is majorized by the quantity

$$c(s,n,p,q)R^{s+\frac{n}{q}-\frac{n}{p}}\left(\frac{1}{R^n}\int_{|y|<R} u\right)^{1/q}\left(\frac{1}{R^n}\int_{|y|<R} v^{-1/(p-1)}\right)^{1-\frac{1}{p}}$$

which can be easily computed mainly if u and v are radial functions.

If $2R < |x_0|$ then $(1/2)|x_0| < |y| < (3/2)|x_0|$ for each $y \in B$ and hence the first member of (3) is now majorized by

$$c(s,n,p,q)R^{s+\frac{n}{q}-\frac{n}{p}}\left(\sup_{|y|\sim 2^j R} u(y)\right)^{1/q}\left(\sup_{|y|\sim 2^j R} v(y)^{-1/(p-1)}\right)^{1-\frac{1}{p}}$$

where $j \in \mathbb{N}^*$.

Also if each of functions u, $v^{-1/(p-1)}$ satisfies a growth condition as:

$$\left[\sup_{(1/4)R<|x|\le 4R} w(x)\right] \le \frac{c}{R^n}\left(\int_{c_1R<|y|\le c_2R} w(y)\,dy\right)$$

and if $[p^{-1} - q^{-1} \leq (s/n)$ then condition $(u,v) \in A(s,p,q)$ is equivalent to

$$R^{s+\frac{n}{q}-\frac{n}{p}} \left(\frac{1}{R^n} \int_{|y|<R} u \right)^{1/q} \left(\frac{1}{R^n} \int_{|y|<R} v^{-1/(p-1)} \right)^{1-\frac{1}{p}} \leq A_2,$$

$$A_2 = Ac(s,n,p,q).$$

Taking $u(x) = |x|^{\beta}$, $v(x) = |x|^{\alpha}$ we obtain

Proposition IV.

Assume

i) $1 < p \leq q < \infty$, $0 \leq s < n$, $[p^{-1} - q^{-1}] \leq (s/n)$;
ii) $-n < \alpha < n(p-1)$;
iii) $ps - n < \alpha$;
iv) $\beta = (q/p)(n+\alpha) - qs - n$;

and define $u(x) = |x|^{\beta}$, $v(x) = |x|^{\alpha}$. *Then* $(u,v) \in A(s,p,q)$.

The condition ii) is equivalent to $v \in A_p$. Now we recall a known result, yielding D_{ε} or RD_{ν} from the A_p condition.

Proposition V.

A) *Let* $1 < p < \infty$, *and* $w \in A_p$ *for a constant* $A > 0$. *Then* $w \in D_p$ *i.e.*

$|tQ|_w \leq Dt^{np}|Q|_w$ *for all cubes* Q *and all* $t \geq 1$; *here* $D = A^p$.

B) *Let* $1 < r < \infty$, *and* $w \in RH_{r/(r-1)}$ *i.e.*

$$\left(\frac{1}{|Q|} \int_Q w^{[r/(r-1)]} \right)^{1-\frac{1}{r}} \leq R \left(\frac{1}{|Q|} \int_Q w \right) \text{ for all cubes } Q,$$

$$R = R(w) > 0$$

then $w \in RD_{1/r}$ *with the constant* R.

If $w \in A_p$ then it is known (**[Ga-Fr]**) that $w \in RH_{1+\rho}$ for some $\rho > 0$ (which depends on n, p, w) and so $w \in RD_{\nu}$ for some $\nu \in]0,1[$. Proposition V can be merely seen by the use of the Hölder inequality.

Proposition VI.

Let $1 < r < \infty, \gamma \in \mathbb{R}$ *and* $w(x) = |x|^{\gamma}$. *If* $-n < \min(\gamma, \gamma r)$ *then* $w \in RH_r$ *and so* $w \in RD_{1-(1/r)}$.

From Propositions III-IV-VI we get

Proposition VII.

Assume

i) $1 < p \leq q < \infty$, $0 \leq s < n$, $[p^{-1} - q^{-1}] \leq (s/n)$;
ii) $-n < \alpha < s(p-1)$;
iii) $ps - n < \alpha$;
iv) $\beta = (q/p)(n+\alpha) - qs - n$;

and define $u(x) = |x|^{\beta}$, $v(x) = |x|^{\alpha}$. *Then there is* $c > 0$ *such that*

$$\|M_s f\|_{L^q_u} \leq c\|f\|_{L^p_v} \text{ for all nonnegative functions } f.$$

Finally we end with the fact that the D_∞ condition implies D_ε or RD_ν (for some ε and ν).

Proposition VIII.

A) *Let* $w \in D_\infty$: *i.e.* $|2Q|_w \leq D|Q|_w$ *for all cubes* Q, $D = D(w) > 1$.
Then $w \in D_\varepsilon$: *i.e.* $|tQ|_w \leq Dt^{n\varepsilon}|Q|_w$ *for all cubes* Q *and all* $t > 1$, *with* $\varepsilon = \frac{\ln D}{\ln 2^n}$.
In particular if $2^n \leq D$ *then* $\varepsilon \geq 1$.

B) *Let* $w \in D_\varepsilon$ *with a constant* $D > 1$.
Then $w \in RD_\nu$: *i.e.* $t^{n\nu}|Q|_w \leq 2^{n\varepsilon}D|tQ|_w$ *for all cubes* Q *and all* $t > 1$, *where* $\nu = \nu(\varepsilon, D, n) = \frac{1}{\ln 2^n}\ln\left[\frac{12^{n\varepsilon}D^2}{12^{n\varepsilon}D^2 - 1}\right]$.
In particular if $2 \leq 12^{n\varepsilon}D^2$ *then* $\nu \leq 1$.

Let $\theta > 0$, then $\theta \geq \varepsilon$ if and only if $D \leq 2^{n\theta}$ and $\theta \leq \nu$ if and only if $12^{n\varepsilon}D^2 \leq \left[\frac{2^{n\theta}}{2^{n\theta}-1}\right]$. From this proposition we see that if $w\,dx \in D_\infty$ with a doubling constant $D = D(w) > 1$ then $w \in RD_\nu$ with $\nu = \nu(D,n) = \frac{1}{\ln 2^n}\ln\left[\frac{D^c}{D^c - 1}\right]$ where $c = 4 + \frac{\ln 3}{\ln 2}$.

Part A can be easily obtained by induction. The next part was proved by Strömberg and Torchinsky [**St-To**], but here we include the proof since we need the precise value of ν.

3. Proofs of the main results

For each cube Q_0 we define the local maximal function

$$(M_{\Phi,Q_0}f)(x) = \sup\left\{\Phi(Q)|Q|^{-1}\|f\mathbf{1}_Q\|_{L^1(dy)};\, Q \ni x,\, Q \subset Q_0\right\}.$$

The proof of Theorem I is based on the following lemmas

Lemma 1.

There is $C = C(n, \Phi) > 0$ *such that for each cube* Q_0 *and for each function* f *locally integrable whose support is contained in* Q_0

$$(M_{\Phi,Q_0} f)(x) \leq (M_\Phi f)(x) \leq C(M_{\Phi,Q_0} f)(x) \text{ for all } x \in Q_0.$$

Lemma 2.

Suppose $(u, v)A(\Phi, p, q)$ *and* $d\sigma$ *satisfying one of i)-ii) as in part B of Theorem 1. Let* Q_0 *be a cube with* $0 < |Q_0|_\sigma < \infty$. *Then* $\sup_{z \in Q_0} (M_{\Phi,Q_0} \mathbf{1}_{Q_0} \sigma)(z) < A \frac{|Q_0|_\sigma^{1/p}}{|Q_0|_u^{1/q}} < \infty$.

Lemma 3.

With the same hypothesis as in Lemma 2, one can find a subcube Q_1 *of* Q_0 *such that* $(M_{\Phi,Q_0} \mathbf{1}_{Q_0} \sigma)(z) < 4 \left(\frac{\Phi(Q_1)}{|Q_1|} |Q_1|_\sigma \right)$ *for all* $z \in Q_0$.

We postpone the proofs below, and we first show how Theorem I is derived from these lemmas.

Proof of Theorem I:

Since

$$\left(\frac{\Phi(Q_0)}{|Q_0|} |Q_0|_\sigma \right) \mathbf{1}_{Q_0}(\cdot) \leq (M_{\Phi,Q_0} \mathbf{1}_{Q_0} \sigma)(\cdot) \mathbf{1}_{Q_0}(\cdot)$$

it is clear that if $(u, v) \in S(\Phi, p, q)$ for a constant $S > 0$, then $(u, v) \in A(\Phi, p, q)$ with the constant $A = S$.

Conversely let $(u, v) \in A(\Phi, p, q)$ for a constant $A > 0$, and let Q_0 be a cube. If $|Q_0|_\sigma = 0$ then it is trivial to have $(u, v) \in S(\Phi, p, q)$. Also (since $0 \cdot \infty = 0$) if $|Q_0|_\sigma = \infty$ then $(u, v) \in S(\Phi, p, q)$ because in this case $|Q_0|_u = 0$. So we can assume $0 < |Q_0|_\sigma < \infty$. From Lemmas 1 and 3 we first have

$$\begin{aligned} \|(M_\Phi \mathbf{1}_{Q_0} \sigma) \mathbf{1}_{Q_0}\|_{L_u^q} &\leq C \|(M_{\Phi,Q_0} \mathbf{1}_{Q_0} \sigma) \mathbf{1}_{Q_0}\|_{L_u^q} \qquad C = C(n, \Phi) \\ &\leq 4C \left(\frac{\Phi(Q_1)}{|Q_1|} |Q_1|_\sigma \right) |Q_0|_u^{1/q}. \end{aligned}$$

Now suppose $d\sigma = v^{-1/(p-1)}\, dx \in B_\nu$ with $1 - \lambda \leq \nu$. Then we get

$$\begin{aligned} \|(M_\Phi \mathbf{1}_{Q_0} \sigma) \mathbf{1}_{Q_0}\|_{L_u^q} &\leq C(\Phi, n) \left(\frac{|Q_1|}{|Q_0|} \right)^{\lambda - 1} \left(\frac{|Q_1|_\sigma}{|Q_0|_\sigma} \right) \left(\frac{\Phi(Q_0)}{|Q_0|} |Q_0|_\sigma \right) |Q_0|_u^{1/q} \\ &\leq C(\Phi, n) B \left(\frac{|Q_1|}{|Q_0|} \right)^{\lambda - 1 + \nu} \left(\frac{\Phi(Q_0)}{|Q_0|} |Q_0|_\sigma \right) |Q_0|_u^{1/q} \\ &\leq C(\Phi, n) BA |Q_0|_\sigma^{1/p} = C(\Phi, n) BA \|\mathbf{1}_{Q_0}\|_{L_\sigma^p}. \end{aligned}$$

Now suppose $d\sigma = v^{-1/(p-1)}\,dx \in B_{(p/q)}(u)$. Then we obtain

$$\begin{aligned}\|(M_\Phi \mathbf{1}_{Q_0}\sigma)\mathbf{1}_{Q_0}\|_{L^q_u} &\leq 4C\left(\frac{\Phi(Q_1)}{|Q_1|}|Q_1|_\sigma\right)|Q_1|_u^{1/q}\left(\frac{|Q_0|_u}{|Q_1|_u}\right)^{1/q}\\ &\leq 4CA\left(\frac{|Q_1|_\sigma}{|Q_0|_\sigma}\right)^{1/p}\left(\frac{|Q_0|_u}{|Q_1|_u}\right)^{1/q}|Q_0|_\sigma^{1/p}\\ &\leq 4CAB^{1/p}\|\mathbf{1}_{Q_0}\|_{L^p_\sigma}. \blacksquare\end{aligned}$$

Proof of Lemma 1:

Let Q_0 be a cube and let f be a function whose support is contained in Q_0. Firstly it is clear that

$$(M_{\Phi,Q_0}f)(x) \leq (M_\Phi f)(x) \text{ for all } x.$$

For the converse we use the growth properties H)1-2 of Φ. Let Q be a cube which contains x, with $x \in Q_0$. We suppose that Q_0 does not contain Q (otherwise there is nothing to prove). We distinguish two cases.

1) For $|Q_0| \leq |Q|$:

Let Q_1 be a cube with the same center as Q_0 but with the lengths $3|Q|^{1/n}$. Since $\eta \leq 1$ we first have

$$\begin{aligned}\frac{\Phi(Q)}{|Q|} &\leq \frac{|Q_0|}{|Q|}\frac{\Phi(Q_1)}{|Q_0|}\\ &\leq C(\Phi,n)\left(\frac{|Q_0|}{|Q|}\right)^{1-\eta}\frac{\Phi(Q_0)}{|Q_0|}\\ &\leq C(\Phi,n)\frac{\Phi(Q_0)}{|Q_0|}.\end{aligned}$$

It results that

$$\begin{aligned}\frac{\Phi(Q)}{|Q|}\|(f\mathbf{1}_{Q_0})\,\mathbf{1}_Q\|_{L^1} &\leq C(\Phi,n)\frac{\Phi(Q_0)}{|Q_0|}\|f\mathbf{1}_{Q_0}\|_{L^1}\\ &\leq C(\Phi,n)(M_{\Phi,Q_0}f)(x).\end{aligned}$$

2) For $|Q| \le |Q_0|$:

One can find a cube $Q_2 \subset Q_0$ such that $|Q| = |Q_2|$, $Q \cap Q_0 \subset Q_2$ and $Q \subset 3Q_2$. Hence we get

$$\begin{aligned}\frac{\Phi(Q)}{|Q|} \|(f\mathbf{1}_{Q_0})\,\mathbf{1}_Q\|_{L^1} &\le \frac{\Phi(3Q_2)}{|Q_2|}\|f\mathbf{1}_{Q_2}\|_{L^1}\\ &\le C(\Phi,n)\frac{\Phi(Q_2)}{|Q_2|}\|f\mathbf{1}_{Q_2}\|_{L^1}\\ &\le C(\Phi,n)(M_{\Phi,Q_0}f)(x). \ \blacksquare\end{aligned}$$

Proof of Lemma 2:

Let $z \in Q_0$ and Q a subcube of Q_0 such that $Q \ni z$. Using one of hypothesis in part B of Theorem I we have to show

$$(\$) \qquad \left(\frac{\Phi(Q)}{|Q|}|Q|_\sigma\right) \le A\frac{|Q_0|_\sigma^{1/p}}{|Q_0|_u^{1/q}}.$$

This implies: $\sup_{z\in Q_0}(M_{\Phi,Q_0}\mathbf{1}_{Q_0}\sigma)(z) < \infty$. And so to obtain ($) it suffices to consider $\left(\frac{\Phi(Q)}{|Q|}|Q|_\sigma\right)|Q_0|_u^{1/q}$ and to estimate this with $A|Q_0|_\sigma^{1/p}$ as we have done in the proof of Theorem I. ■

Proof of Lemma 3:

Since $\sup_{z\in Q_0}(M_{\Phi,Q_0}\mathbf{1}_{Q_0}\sigma)(z) < \infty$ there is one $y \in Q_0$ such that

$$(M_{\Phi,Q_0}\mathbf{1}_{Q_0}\sigma)(x) < 2(M_{\Phi,Q_0}\mathbf{1}_{Q_0}\sigma)(y) \text{ for all } x \in Q_0.$$

Again, there is a subcube Q_1 of Q_0 which contains y such that

$$(M_{\Phi,Q_0}\mathbf{1}_{Q_0}\sigma)(y) < 2\left(\frac{|\Phi(Q_1)}{|Q_1|}|Q_1|_\sigma\right)$$

and so

$$\sup_{z\in Q_0}(M_{\Phi,Q_0}\mathbf{1}_{Q_0}\sigma)(z) \le 4\left(\frac{\Phi(Q_1)}{|Q_1|}|Q_1|_\sigma\right). \quad \blacksquare$$

Proof of Proposition II:

Part A

Let $d\sigma \in B_\nu$ for some $\nu \in]0,\infty[$ i.e.

$$\frac{|Q_1|_\sigma}{|Q_0|_\sigma} \le B\left(\frac{|Q_1|}{|Q_0|}\right)^\nu \text{ for all cubes } Q_0,\ Q_1 \text{ with } Q_1 \subset Q_0.$$

Let Q be a cube and $t \geq 1$. Taking $Q_1 = Q$ and $Q_0 = tQ$ we obtain

$$t^{n\nu}|Q|_\sigma \leq R|tQ|_\sigma, \text{ where } R = B$$

which means $d\sigma \in RD_\nu$.

Converserly let $d\sigma \in RD_\nu$ for a constant $R > 0$. Also if $d\sigma \in D_\infty$ then for $Q_1 \subset Q_0$ we have

$$|Q_1|_\sigma \leq R\left(\frac{|Q_1|}{|Q_0|}\right)^\nu |Q_2|_\sigma$$

where Q_2 has the same center as Q_1 and $|Q_2| = |Q_0|$

$$\leq \left(\frac{|Q_1|}{|Q_0|}\right)^\nu |3Q_0|_\sigma$$

$$\leq RD\left(\frac{|Q_1|}{|Q_0|}\right)^\nu |Q_0|_\sigma$$

where D depends on the constant which is in the doubling condition for $d\sigma$. So it appears that $d\sigma \in B_\nu$ with the constant $B = RD$.

Part B

Let $d\sigma \in B_{(p/q)}(u)$, i.e.

$$\frac{|Q_1|_\sigma}{|Q_0|_\sigma} \leq B\left(\frac{|Q_1|_u}{|Q_0|_u}\right)^{p/q} \quad \text{for all cubes } Q_0,\, Q_1 \text{ with } Q_1 \subset Q_0.$$

Suppose also $d\sigma \in D_\infty$. Let Q be a cube and $t \geq 1$. Taking $Q_1 = Q$ and $Q_0 = tQ$ and using the fact that $d\sigma \in D_{\varepsilon'}$ for some $\varepsilon' \geq 1$ (see Proposition VIII) we obtain

$$\frac{1}{t^{n\varepsilon'}}D^{-1} \leq \frac{|Q|_\sigma}{|tQ|_\sigma} \leq B\left(\frac{|Q|_u}{|tQ|_u}\right)^{p/q}$$

that is

$$|tQ|_u \leq (DB)^{q/p} t^{n\varepsilon' q/p}|Q|_u$$

which means $du \in D_\varepsilon$ with $\varepsilon = \varepsilon'(q/p) \geq 1$. Also since $u\,dx \in RD_{\nu'}$ for some $\nu' \in]0,1]$ (see Proposition VIII) we get

$$\frac{|Q|_\sigma}{|tQ|_\sigma} \leq B\left(R\frac{1}{t^{n\nu'}}\right)^{p/q}$$

that is

$$t^{n\nu' p/q}|Q|_\sigma \leq B(R)^{p/q}|tQ|_\sigma$$

which means $d\sigma \in RD_\nu$ with $\nu = \nu'(p/q) \leq 1$. On other hand we must have for all $t \geq 1$

$$1 \leq DB(R)^{p/q} t^{n[\varepsilon' - \nu'(p/q)]}$$

hence $0 \leq \varepsilon' - \nu'(p/q)$, or $\nu q \leq \varepsilon p$.

Conversely let $d\sigma \in RD_\nu$, $du \in D_\varepsilon$ for some $\varepsilon \in [1, \infty[$ and $\nu \in]0,1]$ with $\varepsilon p \leq \nu q$. For all cubes Q_1, Q_0 with $Q_1 \subset Q_0$ we have

$$\frac{|Q_1|_\sigma}{|Q_0|_\sigma} \left(\frac{|Q_0|_u}{|Q_1|_u}\right)^{p/q} \leq RD \left(\frac{|Q_1|}{|Q_0|}\right)^{\nu - \varepsilon(p/q)} \leq RD$$

which implies $d\sigma \in B_{(p/q)}(u)$ for constant $B = (RD)^{(q/p)}$. ■

4. Proofs of further results

Proof of Proposition IV:

Let $R > 0$. The condition ii) implies that v and $v^{-1/(p-1)}$ are locally integrable functions and

$$\int_{|y|<R} v^{-1/(p-1)}\, dy = \int_{|y|<R} |y|^{-\alpha/(p-1)}\, dy \sim R^{n-[\alpha/(p-1)]}.$$

From iii) and iv) we have $\beta = (q/p)(n+\alpha) - qs - n > -n$, and so

$$\int_{|y|<R} u\, dy = \int_{|y|<R} |y|^\beta\, dy \sim R^{[(n+\alpha)(q/p)-qs]}([(n+\alpha)(q/p) - qs] > 0).$$

Since $[p^{-1} - q^{-1}] \leq (s/n)$ we only have to estimate

$$R^{s+\frac{n}{q}-\frac{n}{p}} \left(\frac{1}{R^n}\int_{|y|<R} u\right)^{1/q} \left(\frac{1}{R^n}\int_{|y|<R} v^{-1/(p-1)}\right)^{1-\frac{1}{p}} \quad \text{(see Section 2).}$$

Using the two equivalences aboev this last quantity is equivalent to

$$R^{[s+(n/q)-(n/p)]}(R^{[(n+\alpha)(q/p)-qs-n]})^{1/q}(R^{[-\alpha/(p-1)]})^{1-\frac{1}{p}} = \\ = R^{[s+(n/q)-(n/p)+(n/p)+(\alpha/p)-s-(n/q)-(\alpha/p)]} = 1. \ ■$$

Proof of Proposition VI:

Let $-n < \min(\gamma, \gamma r)$, and let B be the ball $B(x_0, R)$.

1) If $|x_0| \le 2R$ then $B \subset B(0,3R)$ and $B(0,R) \subset 3B$. Hence

$$\left(\frac{1}{|B|}\int_B w^r\right) \le \left(\frac{c(n)}{R^n}\int_{|y|<3R} |y|^{\gamma r}\, dy\right) \sim R^{\gamma r}$$

and since $-n < \gamma$ then, by Propositions IV-V, $w\, dx \in D_\infty$ and it follows

$$\begin{aligned}\left(\frac{1}{|B|}\int_B w\right) &\ge D(\gamma)\left(\frac{1}{|B|}\int_{3B} w\right)\\ &\ge D'(\gamma)\left(\frac{c(n)}{R^n}\int_{|y|<R} |y|^{\gamma}\, dy\right) \sim R^{\gamma}.\end{aligned}$$

2) If $2R < |x_0|$ then $(1/2)|x_0| < |y| < (3/2)|x_0|$ for each $y \in B$ and it results

$$\left(\frac{1}{|B|}\int_B w^r\right) \sim (2^j R)^{\gamma r} \text{ with } j \in \mathbb{N}^*, \text{ and } \left(\frac{1}{|B|}\int_B w\right) \sim (2^j R)^{\gamma}.$$

In all cases, since $w\, dx \in D_\infty$, we get

$$\left(\frac{1}{|Q|}\int_Q w^r\right)^r \le D(n,\gamma)\left(\frac{1}{|Q|}\int_Q w\right) \text{ for all cubes } Q$$

and hence $w\, dx \in RH_r$. ■

Proof of Proposition VII:

Let $\sigma(x) = v^{-1/(p-1)}(x) = |x|^{-[\alpha/(p-1)]}$. Note that $d\sigma \in A_p$ and so $d\sigma \in D_\infty$ (see Proposition V). If $\alpha \le 0$, then $-n < \gamma = -\alpha/(p-1) < \gamma r$ for all $r > 1$. Choose $r > 1$ with $(n/s) \le r$ then from Proposition VI: $d\sigma \in D_\infty \cap RD_{1-(1/r)}$ with $[1-(s/n)] \le \nu = [1-(1/r)]$. To obtain the same conclusion for $\alpha > 0$, we choose $r > 1$ such that $(n/s) \le r < [n(p-1)/\alpha]$, and so $-n < \gamma r \le \gamma$.

Finally using Propositions IV-III and the Sawyer theorem [**Sa**[2]] then

$$\|M_s f\|_{L^q_u} \le c\|f\|_{L^p_v} \text{ for all functions } f. \quad ■$$

Proof of part B of Proposition VIII:

We need the following lemmas whose proofs will be given below.

Lemma 4.

Let $w\, dx \in D_\varepsilon$ for some $\varepsilon \in [1,\infty[$ and with a constant $D = D(w) > 1$. Then $\left|\frac{1}{2}Q\right|_w \le 6^{n\varepsilon} D \left|Q \setminus \left(\frac{1}{2}Q\right)\right|_w$ for each cube Q.

Lemma 5.

Let $w\,dx \in D_\varepsilon$ for some $\varepsilon \in [1,\infty[$ and with a constant $D = D(w) > 1$. Then $\left|\frac{1}{2}Q\right|_w \leq \beta|Q|_w$ for each cube Q, with $\beta = \frac{12^{n\varepsilon}D^2-1}{12^{n\varepsilon}D^2}$, and so $\beta \in]0,1[$.

The part B can be derived from Lemma 5. Indeed if Q is a cube then

$$|Q|_w \leq \beta^m |2^m Q|_w \text{ for each } m \in \mathbb{N}^*.$$

Let $t > 1$. There is $k = k(t) \in \mathbb{N}^*$ such that $2^{k-1} < t \leq 2^k$ (so $[(\ln t)/(\ln 2)] \leq k$). It results

$$\begin{aligned}
|Q|_w &\leq \beta^k |2^k Q|_w \\
&\leq 2^{n\varepsilon} D \beta^k |tQ|_w \\
&= 2^{n\varepsilon} D e^{[(\ln\beta)/\ln 2]\ln t} |tQ|_w \\
&\leq 2^{n\varepsilon} D e^{[(\ln\beta)/\ln 2]\ln t} |tQ|_w \\
&= 2^{n\varepsilon} D \left[\frac{1}{t}\right]^{-(\ln\beta)/\ln 2} |tQ|_w
\end{aligned}$$

and

$$t^{n\nu}|Q|_w \leq 2^{n\varepsilon} D |tQ|_w \text{ with } \nu = \frac{\ln\frac{1}{\beta}}{\ln 2^n}.$$

If $2 \leq 12^{n\varepsilon}D^2$ we get

$$(12^{n\varepsilon}D^2 + 2^n) \leq (2^{n-1}12^{n\varepsilon}D^2 + 2^{n-1}12^{n\varepsilon}D^2) = 2^n 12^{n\varepsilon} D^2$$

or $12^{n\varepsilon}D^2 \leq 2^n(12^{n\varepsilon}D^2 - 1)$ which implies $\frac{1}{\beta} \leq 2^n$ and so $\nu = \frac{\ln\frac{1}{\beta}}{\ln 2^n} \leq 1$.

Proof of Lemma 4:

In the proof of the Theorem I we have already used the following geometric argument:

"Let Q_1, Q_2 two cubes such that $Q_1 \cap Q_2 \neq \varnothing$ and $|Q_1|^{[1/n]} \leq |Q_2|^{[1/n]}$; then $Q_1 \subset 3Q_2$." Let Q be a cube and Q_0 a subcube of $(Q\backslash(2^{-1}Q))$ with lenghts $(1/4)|Q|^{[1/n]}$ and let $Q_1 = (2^{-1}Q)$. Then $(2^{-1}Q) \cap 2Q_0 \neq \varnothing$ and $|2^{-1}Q)|^{[1/n]} \leq |2Q_0|^{[1/n]}$. Using this argument we obtain $(2^{-1}Q) \subset 3(2Q_0) = 6Q_0$ and then

$$\begin{aligned}
\left|\frac{1}{2}Q\right|_w &\leq |6Q_0|_w \\
&\leq 6^{n\varepsilon} D |Q_0|_w \\
&\leq 6^{n\varepsilon} D \left|Q\backslash\left(\frac{1}{2}Q\right)\right|_w . \ \blacksquare
\end{aligned}$$

Proof of Lemma 5:
Let Q be a cube. By hypothesis

$$2^{-n\varepsilon}|Q|_w \leq D\left|\frac{1}{2}Q\right|_w, \qquad D = D(w) > 1.$$

So using Lemma 4 we get

$$\begin{aligned} 2^{-n\varepsilon}|Q|_w &\leq 6^{n\varepsilon}D^2\left|Q\backslash\left(\frac{1}{2}Q\right)\right|_w \\ &\leq 6^{n\varepsilon}D^2\left[|Q|_w - \left|\frac{1}{2}Q\right|_w\right]. \end{aligned}$$

It results that

$$\left|\frac{1}{2}Q\right|_w \leq \beta|Q|_w$$

with

$$\beta = \frac{6^{n\varepsilon}D^2 - 2^{-n\varepsilon}}{6^{n\varepsilon}D^2} = \frac{12^{n\varepsilon}D^2 - 1}{12^{n\varepsilon}D^2}, \text{ and so } \beta \in]0,1[. \quad \blacksquare$$

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