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RELATIVE HERMITIAN MORITA THEORY
Part II: Hermitian Morita contexts

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Dedicated to the memory of Prof. Pere Menal

Abstract

We introduce the notion of a relative hermitian Morita context between torsion triples and show how these induce equivalences between suitable quotient categories of left and right modules.

Due to the lack of involutive bimodules, the induced Morita equivalences are not necessarily hermitian, however.

0. Introduction

In the first part of this paper [**12**], the authors essentially introduced and studied hermitian Morita theory for arbitrary involutive Grothendieck categories, generalizing previous work by Hahn [**9**] et al. Using the Gabriel-Popescu theorem [**5**], this study reduces to the study of categories of the form (A, σ)-mod, i.e., quotient categories of the category A-mod of left modules over some ring A by a localizing subcategory $\mathcal{T}_\sigma$, consisting of all left A-modules, which are torsion with respect to some idempotent kernel functor σ in A-mod.

More precisely, one starts in the aforementioned paper from the notion of a torsion triple (A, α, σ), where (A, α) is an algebra with involution over some fixed commutative ring R and where σ is an idempotent kernel functor in A-mod. The idempotent kernel functor σ then induces an idempotent kernel functor $\alpha(\sigma)$ in mod-A, defined by letting its Gabriel filter $\mathcal{L}(\alpha(\sigma))$ consist of all right A-ideals $\alpha(L)$ with $L \in \mathcal{L}(\sigma)$. In particular, one may then localize A with respect to both σ and $\alpha(\sigma)$, thus obtaining rings of quotients $Q_\sigma(A)$ resp. $Q_{\alpha(\sigma)}(A)$, which, in general,

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are *not* isomorphic, however. Let (B,β,τ) be a second torsion triple and let

$$(P,Q,\theta : P \to Q,\ \mu : Q_\tau(P\otimes_A Q) \to Q_\tau,\ \nu : Q_\sigma(Q\otimes_B P) \to Q_\sigma(A))$$

be a relative hermitian Morita equivalence, i.e., we assume P resp. Q to be a (τ,σ)-invertible B-A-bimodule resp. a (σ,τ)-invertible A-B-bimodule defined over some fixed commutative ring R, cf. [**12**], μ resp. ν to be isomorphisms of B resp. A-bimodules and θ to be an additive bijection, with the property that $\theta(bpa) = \alpha(a)\theta(p)\beta(b)$ for all $p \in P$, $a \in A$ and $b \in B$, these data satisfying some natural compatibility conditions, cf. [**12**]. In this case, P is a so-called *involutive* B-A-bimodule in the sense that there is a B-A-bimodule isomorphism $P \cong^\beta_B [Q, Q_\tau(B)]^\alpha$, where for any A-B-bimodule M, we define the B-A-bimodule ${}^\beta M^\alpha$ by $b\cdot m\cdot a = \alpha(a)m\beta(b)$, for all $a \in A$, $b \in B$ an $m \in M$. It follows that the existence of a relative hermitian Morita equivalence between the torsion triples (A,α,σ) and (B,β,τ) implies that $Q_{\alpha(\sigma)}(A) = Q_\sigma(A)$ and $Q_{\beta(\tau)}(B) = Q_\tau(B)$ up to canonical isomorphism.

Using this, the authors of [**12**] were able to derive a hermitian version of Morita I, (i.e., they showed that any relative hermitian Morita equivalence as above defines an equivalence between the categories $\mathcal{S}(A,\alpha,\sigma)$ and $\mathcal{S}(B,\beta,\tau)$ of so-called σ-sesquilinear left A-modules) as well as a hermitian version of Morita II.

In the present note, we introduce the notion of a relative hermitian Morita context between torsion triples. Using techniques, which are originally due to Mueller [**15**], we show that such Morita contexts induce equivalences between suitable quotient categories of left and right modules. However, the induced Morita equivalences are not necessarily hermitian, in general, mainly due to the lack of involutive bimodules.

We will see in this note, that relative hermitian Morita contexts exactly induce so-called relative hermitian Morita bieqivalences in the sense given below, thus providing the proper framework into which Mueller's results may be generalized in the hermitian case. The behaviour of relative sesquilinear modules in the present set-up and, more generally in that of Morita contexts between so-called linked pairs will be the object of [**13**].

1. Generalities

Notations and terminology will be as in [**12**]. For the reader's convenience, let us briefly recollect here some of the main features that will be used throughout.

(1.1) In this note, R will be a commutative ring and A, $B,\ldots$ will be R-algebras. We denote by A-mod resp. mod-A the category of left resp. right A-modules. If M, N are left resp. right A-modules, then we write ${}_A[M,N]$ resp. $[M,N]_A$ for the corresponding sets of A-linear morphisms. If A and B are both R-algebras, then we denote by A-mod-B the category of A-B-bimodules defined over R.

(1.2) An R-algebra with involution is a couple (A,α), where A is an R-algebra and $\alpha : A \to A$ an R-involution, i.e., an R-linear map with the property that $\alpha^2 = id_A$ and $\alpha(aa') = \alpha(a')\alpha(a)$ for all a, $a' \in A$. If we only wish to consider *rings* with involution, it suffices to let $R = Z$.

Let (A,α) be an R-algebra with involution. If M is a left A-module, then by definition, the right A-module M^α has the same underlying additive group as M and its right A-action is given by $m \cdot a = \alpha(a)m$, for any $m \in M$ and $a \in A$. Since any left A-linear map $u : M \to N$ induces a right A-linear map $u^\alpha : M^\alpha \to N^\alpha$, this actually yields an exact covariant functor $(-)^\alpha : A\text{-mod} \to \text{mod-}A$. For $M \in$ mod-A, we define ${}^\alpha M \in A$-mod similarly, thus yielding an exact covariant functor ${}^\alpha(-) : \text{mod-}A \to A\text{-mod}$.

If (B,β) is another R-algebra with involution, then for any $M \in A$-mod-B, we define ${}^\beta M^\alpha \in B$-mod-A by putting $b \cdot m \cdot a = \alpha(a)m\beta(b)$ for any $a \in A$, $b \in B$ and $m \in M$. If $(B,\beta) = (A,\alpha)$, then we write M_α for ${}^\alpha M^\alpha$.

In particular, if $M \in A$-mod-B and $N \in B$-mod-A, then morphisms of A-B-bimodules $M \to {}^\alpha N^\beta$ correspond bijectively to additive morphisms $\theta : M \to N$ with the property that $\theta(amb) = \beta(b)\theta(m)\alpha(a)$ for any $a \in A$, $b \in B$ and $m \in M$. We call such maps *A-B-antimorphisms.* If θ is invertible, then we speak of an *A-B-anti-isomorphism.* If $(A,\alpha) = (B,\beta)$, then we will also speak of antimorphisms and anti-isomorphisms of A-bimodules.

(1.3) If σ is an idempotent kernel functor in R-mod with associated localization functor Q_σ, then any R-algebra with involution (A,α) induces an R-algebra with involution $(Q_\sigma(A),\hat{\alpha})$, where $\hat{\alpha}$ is the extension $Q_\sigma(\alpha)$ of α to $Q_\sigma(A)$.

More generally, let (A,α) be an R-algebra with involution and let σ be an arbitrary idempotent kernel functor in A-mod, (we then call (A,α,σ) a (left) *torsion triple* - right torsion triples and related concepts are defined similarly), then σ induces an idempotent kernel functor $\alpha(\sigma)$ in mod-A, defined by its Gabriel filter $\mathcal{L}(\alpha(\sigma))$, which consists of all right ideals $\alpha(L)$ of A with $L \in \mathcal{L}(\sigma)$. In particular, for any right A-module M, it follows that $m \in M$ belongs to $\alpha(\sigma)M$ if and only if $m\alpha(L) = 0$ for some $L \in \mathcal{L}(\sigma)$. Let Q^l_σ resp. $Q^r_{\alpha(\sigma)}$ denote the localization functor at σ resp. $\alpha(\sigma)$ in A-mod resp. mod-A. It then has been proved in [**12**] that the involution $\alpha : A \to A$ extends to an anti-isomorphism $\hat{\alpha} : Q^l_\sigma(A) \to Q^r_{\alpha(\sigma)}(A)$, i.e., an additive bijection with the property that $\hat{\alpha}(qq') = \hat{\alpha}(q')\hat{\alpha}(q)$, for any $q, q' \in Q^l_\sigma(A)$. However, in general (in the absence of involutive bimodules, cf. [**12**, (3.7.)]), the R-algebras $Q^l_\sigma(A)$ and $Q^r_{\alpha(\sigma)}(A)$ are not isomorphic and α does not extend to an *involution* $\hat{\alpha} : Q^l_\sigma(A) \to Q^l_\sigma(A)$!

(1.4) We leave it as an exercise to the reader to verify that for any $M \in A$-mod resp. $N \in$ mod-A and any idempotent kernel functor σ in A-mod we have $\alpha(\sigma)M^\alpha = (\sigma M)^\alpha$ resp. $\sigma({}^\alpha N) = {}^\alpha(\alpha(\sigma)N)$. It is then fairly easy to deduce, for example, that for any A-bimodule M the A-bimodules $Q^r_{\alpha(\sigma)}(M)$ and $Q^l_\sigma(M_\alpha)_\alpha$ are isomorphic i.e., that there exists an anti-isomorphism $Q^l_\sigma(M_\alpha) \xrightarrow{\sim} Q^r_{\alpha(\sigma)}(M)$ of A-bimodules.

From this, it easily follows, for example, that the ring A is σ-closed if and only if A is $\alpha(\sigma)$-closed. Indeed, the involution α induces an isomorphism of A-bimodules $A \cong A_\alpha$. But then, if A is σ-closed, the A-bimodule isomorphism $A \cong Q^l_\sigma(A)$ induces an isomorphism

$$A \cong A_\alpha \cong Q^l_\sigma(A)_\alpha \cong Q^r_{\alpha(\sigma)}(A_\alpha),$$

which proves that A is $\alpha(\sigma)$-closed. The converse may be proved in a similar way.

2. Hermitian Biequivalences

(2.1) Let us fix a pair of R-algebras A and B and idempotent kernel functors σ resp. τ in A-mod resp. B-mod. Recall from [**18**], [**19**] that an A-B-bimodule Q is said to be (σ, τ)-*flat*, if it possesses the following two properties:

(2.1.1) if $M'' \in B$-mod is torsion with respect to τ, then $Q \otimes_B M'' \in A$-mod is torsion with respect to σ;

(2.1.2) for any monomorphism $i : M' \to M$ in B-mod, the kernel $\mathrm{Ker}(Q \otimes_B i) \in A$-mod is σ-torsion.

In particular, if Q is a (σ, τ)-flat A-B-bimodule, then it follows, that for any left B-module M there is a canonical isomorphism of left A-modules $Q^l_\sigma(Q \otimes_B M) \cong Q^l_\sigma(Q \otimes_B Q^l_\tau(M))$. Applying this to $M = B$, it follows that any σ-closed (σ, τ)-flat A-B-bimodule Q is automatically $Q^l_\sigma(A)$-$Q^l_\tau(B)$-bimodule.

Finally, recall that a σ-closed A-B-bimodule Q is said to be (σ, τ)-*invertible*, if it is (σ, τ)-flat and if there exists another τ-closed (τ, σ)-flat

B-A-bimodule P together with bimodule isomorphisms $\mu : Q^l_\tau(P \otimes_A Q) \xrightarrow{\sim} Q^l_\tau(B)$ resp. $\nu : Q^l_\sigma(Q \otimes_B P) \xrightarrow{\sim} Q^l_\sigma(A)$.

The right-handed version of these notions is defined similarly.

(2.2) As in [**18**], one defines a *relative (left) Morita equivalence* between (A, σ) and (B, τ) to be a tuple

$$(P, Q, \mu : Q^l_\tau(P \otimes_A Q) \to Q^l_\tau(B),\ \nu : Q^l_\sigma(Q \otimes_B P) \to Q^l_\sigma(A)),$$

where P and Q are (τ, σ) resp. (σ, τ)-invertible bimodules and where μ resp. ν are bimodule isomorphisms, all of these data being defined over R. Relative *right* Morita equivalences are defined similarly.

It has been proved in [**18**], that one may always modify the isomorphisms μ and ν in such a way that they satisfy for any $p, p' \in P$, $q, q' \in Q$, $a \in A$ and $b \in B$:

$$\mu(p \perp q)p' = p\nu(q \perp p') \text{ resp. } \nu(q \perp p)q' = q\mu(p \perp q'),$$

where, for example, $p \perp q$ denotes the image of $p \otimes q$ under the canonical map $P \otimes_A Q \to Q^l_\tau(P \otimes_A Q)$. So, without restriction of generality, we will assume this condition to be satisfied for any left resp. right Morita equivalence used in this text.

(2.3) With notations as in the previous paragraph, it has been proved in [**18**] that any relative Morita equivalence (P, Q, μ, ν) determines an equivalence between the quotient categories (A, σ)-mod an (B, τ)-mod, given by the functor

$$F = Q^l_\tau(P \otimes_A -) : (A, \sigma)\text{-mod} \to (B, \tau)\text{-mod}$$

and its adjoint

$$G = Q^l_\sigma(Q \otimes_B -) : (B, \tau)\text{-mod} \to (A, \sigma)\text{-mod}.$$

Conversely, if $(F, G) : (A, \sigma)\text{-mod} \approx (B, \tau)\text{-mod}$ is an equivalence of categories, then (F, G) determines a relative Morita equivalence (P, Q, μ, ν). Indeed, put $P = F(Q^l_\sigma(A))$ resp. $Q = G(Q^l_\tau(B))$ both endowed with their canonical bimodule structure. One may then show that (up to isomorphism) $F = Q^l_\tau(P \otimes_A -)$ resp. $G = Q^l_\sigma(Q \otimes_B -)$. We then let μ resp. ν denote the morphisms obtained by applying the natural isomorphisms $FG \xrightarrow{\sim} id_{(B,\tau)\text{-mod}}$ resp. $GF \xrightarrow{\sim} id_{(A,\sigma)}\text{-mod}$ to $Q^l_\tau(B)$ resp. $Q^l_\sigma(A)$. It may finally be shown that these data determine each other up to isomorphism.

(2.4) Let us define a *relative hermitian Morita biequivalence* between the torsion triples (A, α, σ) and (B, β, τ) to consist of a relative left Morita equivalence

$$(P_l, Q_l, \mu_l : Q^l_\tau(P_l \otimes_A Q_l) \to Q^l_\tau(B),\ \nu_l : Q^l_\sigma(Q_l \otimes_B P_l) \to Q^l_\sigma(A))$$

between (A, σ) and (B, τ) and a relative right Morita equivalence

$$(P_r, Q_r, \mu_r : Q^r_{\beta(\tau)}(P_r \otimes_A Q_r) \to Q^r_{\beta(\tau)}(B),$$
$$\nu_r : Q^r_{\alpha(\sigma)}(Q_r \otimes_B P_r) \to Q^r_{\alpha(\sigma)}(A))$$

between $(A, \alpha(\sigma))$ and $(B, \beta(\tau))$, linked by a B-A-anti-isomorphism

$$\theta' : P_l \xrightarrow{\sim} Q_r.$$

Of course, one then easily verifies that the map θ' also induces a B-A-anti-isomorphism $\theta'' : P_r \xrightarrow{\sim} Q_l$.
Note also that θ' induces bimodule isomorphisms

$$P_l \xrightarrow{\sim} {}^\beta_A[P_r, Q^r_{\alpha(\sigma)}(A)]^\alpha \text{ resp. } Q_l \xrightarrow{\sim} {}^\alpha_B[Q_r, Q^r_{\beta(\tau)}(B)]^\beta$$

and

$$P_r \xrightarrow{\sim} {}^\beta_B[P_l, Q^l_\tau(B)]^\alpha \text{ resp. } Q_r \xrightarrow{\sim} {}^\alpha_A[Q_l, Q^l_\sigma(A)]^\beta.$$

(2.5) On the other hand, define a *hermitian biequivalence* between torsion triples (A, α, σ) and (B, β, τ) to be a couple of equivalences

$$(F^l, G^l) : (A, \sigma)\text{-mod} \approx (B, \tau)\text{-mod}$$

resp.

$$(F^r, G^r) : \text{mod-}(A, \alpha(\sigma)) \approx \text{mod-}(B, \beta(\tau)),$$

linked through an isomorphism

$$\Theta' : F^l(-)^\beta \xrightarrow{\sim} F^r((-)^\alpha).$$

Since

$$G^l F^l \cong id_{(A,\sigma)\text{-mod}} \text{ resp. } G^r F^r \cong id_{\text{mod-}(A,\alpha(\sigma))},$$

it then also follows that there exists an isomorphism

$$\Theta'' : G^l(-)^\alpha \xrightarrow{\sim} G^r((-)^\beta).$$

Indeed, if $N \in (B,\tau)$-mod, say with $N = F^l M$ for $M \in (A,\sigma)$-mod, then, up to natural isomorphism

$$\begin{aligned} G^l(N)^\alpha &= (G^l F^l M)^\alpha = M^\alpha \\ &= G^r F^r(M^\alpha) = G^r((F^l M)^\beta) \\ &= G^r(N^\beta), \end{aligned}$$

which proves the assertion.

(2.6) Consider a relative hermitian Morita biequivalence

$$((F^l, G^l), (F^r, G^r), \Theta' : F^l(-)^\beta \xrightarrow{\sim} F^r((-)^\alpha))$$

between the torsion triples (A, α, σ) and (B, β, τ) and let (C, γ) be a third algebra with involution. Let M be a σ-closed A-C-bimodule. Clearly, right multiplication by elements of C is left A-linear, hence this action canonically endows $F^l M$ with a C-B-bimodule structure. From this, it easily follows that the isomorphism Θ' extends to an isomorphism

$$\Theta' : {}^\gamma F^l(-)^\beta \xrightarrow{\sim} F({}^\gamma(-)^\alpha),$$

when restricted to A-C-bimodules in (A,σ)-mod.

(2.7) Lemma. *Let (A, α, σ) and (B, β, τ) be torsion triples, let P be a B-A-bimodule and Q an A-B-bimodule and let $\theta : P \to Q$ be an A-B-anti-isomorphism. Then for any left A-module M, there is a natural isomorphism*

$$Q^l_\tau(P \otimes_A M)^\beta \xrightarrow{\sim} Q^r_{\beta(\tau)}(M^\alpha \otimes_A Q)$$

of right B-modules.

Proof: First note that there is a natural isomorphism

$$\psi_M : (P \otimes_A M)^\beta \xrightarrow{\sim} M^\alpha \otimes_A Q : p \otimes m \mapsto m \otimes \theta(p)$$

of right B-modules. So,

$$\begin{aligned} Q^l_\tau(P \otimes_A M)^\beta = Q^l_\tau({}^\beta((P \otimes_A M)^\beta))^\beta &= Q^r_{\beta(\tau)}((P \otimes_A M)^\beta) \\ &\overset{(*)}{\to} Q^r_{\beta(\tau)}(M^\alpha \otimes_A Q), \end{aligned}$$

where the isomorphism $(*)$ is just $Q^r_{\beta(\tau)}(\psi_M)$. This proves the assertion. ■

We may now prove:

(2.8) Theorem. *Let (A, α, σ) and (B, β, τ) be torsion triples. Then there is a bijective correspondence between*

(2.8.1) *(isomorphism classes of) relative hermitian Morita bieqiuvalences between (A, α, σ) and (B, β, τ);*

(2.8.2) *(isomorphism classes of) hermitian biequivalences between (A, α, σ) and (B, β, τ).*

Proof: First, let us consider a relative hermitian Morita biequivalence between the torsion triples (A, α, σ) and (B, β, τ), determined by the relative Morita equivalences

$$(P_l, Q_l, \mu_l : Q^l_\tau(P_l \otimes_A Q_l) \to Q^l_\tau(B),\ \nu_l : Q^l_\sigma(Q_l \otimes_B P_l) \to Q^l_\sigma(A))$$

resp.

$$(P_r, Q_r, \mu_r : Q^r_\tau(P_r \otimes_A Q_r) \to Q^r_\tau(B),\ \nu_r : Q^r_\sigma(Q_r \otimes_B P_r) \to Q^r_\sigma(A))$$

and the B-A-anti-isomorphism $\theta' : P_l \xrightarrow{\sim} Q_r$. Then we may associate to it a hermitian biequivalence

$$((F^l, G^l), (F^r, G^r),\ \Theta' : F^l(-)^\beta \xrightarrow{\sim} F^r((-)^\alpha))$$

between the torsion triples (A, α, σ) and (B, β, τ), with $F^l = Q^l_\tau(P_l \otimes_A -)$, $G^l = Q^l_\sigma(Q_l \otimes_B -)$, $F^r = Q^r_{\alpha(\sigma)}(- \otimes_A P_r)$ resp. $G^r = Q^r_{\beta(\tau)}(- \otimes_B Q_r)$, and with

$$\Theta' : F^l(-)^\beta = Q^l_\tau(P_l \otimes_A -)^\beta \xrightarrow{\sim} Q^r_{\alpha(\sigma)}((-)^\alpha \otimes_A Q_r) = F^r((-)^\alpha)$$

determined by the B-A-anti-isomorphism $\theta' : P_l \xrightarrow{\sim} Q_r$, using (2.7).

Conversely, consider a hermitian biequivalence between the torsion triples (A, α, σ) and (B, β, τ) given by the equivalences

$$(F^l, G^l) : (A, \sigma)\text{-mod} \approx (B, \tau)\text{-mod}$$

and

$$(F^r, G^r) : \text{mod-}(A, \alpha(\sigma)) \approx \text{mod-}(B, \beta(\tau)),$$

linked through the isomorphism

$$\Theta' : F^l(-)^\beta \xrightarrow{\sim} F^r((-)^\alpha).$$

By the remarks made in (2.3), we then obtain a relative left Morita equivalence

$$(P_l = F^l(Q^l_\sigma(A)),\ Q_l = G^l(Q^l_\tau(B)), \mu_l, \nu_l),$$

where μ_l and ν_l are induced by the isomorphisms $F^lG^l \xrightarrow{\sim} id_{(B,\tau)\text{-mod}}$ resp. $G^lF^l \xrightarrow{\sim} id_{(A,\sigma)\text{-mod}}$, and a relative right Morita equivalence

$$(P_r = G^r(Q^r_{\beta(\tau)}(B)),\, Q_r = F^r(Q^r_{\alpha(\sigma)}(A)), \mu_r, \nu_r),$$

where μ_r and ν_r are defined similarly.

Finally, if we restrict to A-bimodules, then the remarks made in (2.6) show that the isomorphism Θ' induces an isomorphism

$${}^{\alpha}F^l(-)^{\beta} \xrightarrow{\sim} F^r({}^{\alpha}(-)^{\alpha}) = F^r((-)_{\alpha}),$$

so, in particular, this yields an isomorphism

$${}^{\alpha}F^l(Q^l_{\sigma}(A))^{\beta} \xrightarrow{\sim} F^r(Q^l_{\sigma}(A)_{\alpha}).$$

With $Q_r = F^r(Q^r_{\alpha(\sigma)}(A)) \in \text{mod-}(B, \beta(\tau))$, we clearly obtain a second isomorphism

$$\begin{aligned} F^r(Q^l_{\sigma}(A)_{\alpha}) &= F^r(Q^r_{\alpha(\sigma)}(A_{\alpha})) = Q^r_{\beta(\tau)}(Q^r_{\alpha(\sigma)}(A_{\alpha}) \otimes_A Q_r) \\ &= Q^r_{\beta(\tau)}(A_{\alpha} \otimes_A Q_r) \xrightarrow{\sim} Q^r_{\beta(\tau)}(Q_r) = Q_r, \end{aligned}$$

where the last isomorphism of A-B-bimodules is induced by

$$A_{\alpha} \otimes_A Q_r \to Q_r : a \otimes q \mapsto \alpha(a)q.$$

Composing the previous isomorphisms (with $P_l = F^lQ^l_{\sigma}(A)$), we then obtain an isomorphism of A-B-bimodules ${}^{\alpha}P_l{}^{\beta} \xrightarrow{\sim} Q_r$, i.e., a B-A-anti-isomorphism $\theta' : P_l \to Q_r$ defined over R.

It is now easy to verify that the isomorphism

$$F^l(-)^{\beta} = Q^l_{\tau}(P_l \otimes_A -)^{\beta} \xrightarrow{\sim} Q^r_{\beta(\tau)}((-)^{\alpha} \otimes_A Q_r) = F^r((-)^{\alpha})$$

is actually induced by $\theta' : P_l \to Q_r$, exactly as in (2.7), proving that the previous constructions are mutually inverse to each other. ∎

(2.9) Note. With notations as before, let us call a relative hermitian Morita biequivalence *involutive*, if $P_l = P_r$ (up to isomorphism). The anti-isomorphisms $\theta' : P_l \to Q_r$ and $\theta'' : P_r \to Q_l$ may then be used to show that $Q_l = Q_r$ as well. But then, it follows that θ' and θ'' induce anti-isomorphisms $\theta' : P_l \to Q_l$ resp. $\theta'' : P_r \to Q_r$. In particular, it follows that the relative Morita equivalences determined by (P_l, Q_l) resp. (P_r, Q_r) are actually relative *hermitian* Morita equivalences, in the sense of [**12**]. The theory developed in [**12**] then shows that the involution α resp. β on A resp. B uniquely extends to an involution $\hat{\alpha}$ resp. $\hat{\beta}$ on $Q^l_{\sigma}(A) = Q^r_{\alpha(\sigma)}(A)$ resp. $Q^l_{\tau}(B) = Q^r_{\beta(\tau)}(B)$. Moreover, $P_l = P_r$ resp. $Q_l = Q_r$ is then canonically endowed with a $Q^l_{\tau}(B)$-$Q^l_{\sigma}(A)$ resp. $Q^l_{\sigma}(A)$-$Q^l_{\tau}(B)$-bimodule structure.

3. Hermitian Morita contexts

(3.1) For any pair of rings (A, B), we define a Morita context between A and B to be a tuple

$$\mathcal{M} = (P, Q, \mu : P \otimes_A Q \to B,\ \nu : Q \otimes_B P \to A),$$

where P is a B-A-bimodule, Q an A-B-bimodule and where μ resp. ν are morphisms of B- resp. A-bimodules, satisfying for any p, $p' \in P$, q, $q' \in Q$,

$$\mu(p \otimes q)p' = p\nu(q \otimes p') \text{ resp. } \nu(q \otimes p)q' = q\mu(p \otimes q').$$

In other words, we want the following diagrams to commute:

$$\begin{array}{ccc} P \otimes_A Q \otimes_B P & \xrightarrow{\mu \otimes P} & B \otimes_B P \\ {\scriptstyle P\otimes\nu}\downarrow & & \downarrow \\ P \otimes_A A & \longrightarrow & P \end{array}$$

$$\begin{array}{ccc} Q \otimes_B P \otimes_A Q & \xrightarrow{\nu \otimes Q} & A \otimes_A Q \\ {\scriptstyle Q\otimes\mu}\downarrow & & \downarrow \\ Q \otimes_B B & \longrightarrow & Q \end{array}$$

The images of ν resp. μ are twosided ideals of A resp. B and will be denoted by J_A resp. J_B; they are called the *trace ideals* of the Morita context $\mathcal{M}$.

(3.2) Let us now assume (A, α) and (B, β) to be R-algebras with involution. We then define a *hermitian Morita context* between (A, α) and (B, β) to be a tuple

$$\mathcal{H} = (P, Q, \theta : P \to Q,\ \mu : P \otimes_A Q \to B,\ \nu : Q \otimes_B P \to A)$$

where P is a B-A-bimodule, Q an A-B-bimodule (both defined over R) θ an additive bijection and where μ resp. ν are morphisms of B- resp. A-bimodules.

Moreover, these data should satisfy the following requirements for any p, $p' \in P$, q, $q' \in Q$, $a \in A$ and $b \in B$:

(3.2.1) $\theta(bpa) = \alpha(a)\theta(p)\beta(b)$;
(3.2.2) $\mu(p \otimes q)p' = p\nu(q \otimes p')$ resp. $\nu(q \otimes p)q' = q\mu(p \otimes q')$;
(3.2.3) $\mu(p \otimes \theta(p')) = \beta(\mu(p' \otimes \theta(p)))$ resp. $\nu(\theta(p) \otimes p') = \alpha(\nu(\theta(p') \otimes p))$.

(Of course, the first condition just says that the bijection $\theta : P \to Q$ is a B-A-anti-isomorphism).

(3.3) Examples. It is easy to see that a hermitian Morita equivalence in the sense of [**3**], [**9**] determines a hermitian Morita context, in the obvious way.

More generally, recall that a relative hermitian Morita equivalence between torsion triples (A, α, σ) and (B, β, τ) is defined in [**12**] to be a tuple $\mathcal{H}_o$ of the form

$$(P, Q, \theta : P \to Q, \mu : Q^l_\tau(P \otimes_A Q) \to Q^l_\tau(B),\ \nu : Q^l_\sigma(Q \otimes_B P) \to Q^l_\sigma(A)),$$

where P resp. Q is a (τ, σ)-invertible B-A-bimodule resp. a (σ, τ)-invertible A-B-bimodule defined over R, where θ is a B-A-anti-isomorphism and where μ resp. ν is an isomorphism of B- resp. A-bimodules, these data satisfying relations as in (3.2), but replacing tensor products with *relative* tensor products, i.e., considering for example $Q^l_\tau(P \otimes_A Q)$ instead of $P \otimes_A Q$. It may be shown that the existence of a relative hermitian Morita equivalence between (A, α, σ) and (B, β, τ) implies the involutions α resp. β to extend to involutions $\hat{\alpha}$ resp. $\hat{\beta}$ on $Q^l_\sigma(A)$ resp. $Q^l_\tau(B)$, cf. [**12**, (3.7.)]. Writing $j_{P\otimes Q} : P \otimes_A Q \to Q^l_\tau(P \otimes_A Q)$ resp. $j_{Q\otimes P} : Q \otimes_B P \to Q^l_\sigma(Q \otimes_B P)$ for the canonical morphisms, it then follows that $\mathcal{H}_o$ induces a hermitian Morita context

$$\mathcal{H} = (P, Q, \theta : P \to Q,\ \hat{\mu} : P \otimes_A Q \to Q^l_\tau(B),\ \hat{\nu} : Q \otimes_B P \to Q^l_\sigma(A))$$

between $(Q^l_\sigma(A), \hat{\alpha})$ and $(Q^l_\tau(B), \hat{\beta})$, with $\hat{\mu} = \mu \circ j_{P\otimes Q}$ resp. $\hat{\nu} = \nu \circ j_{Q\otimes P}$.

(3.4) Consider a hermitian Morita context

$$\mathcal{H} = (P, Q, \theta : P \to Q,\ \mu : P \otimes_A Q \to B,\ \nu : Q \otimes_B P \to A)$$

between the R-algebras with involution (A, α) and (B, β) and let J_A and J_B be the trace ideals of the associated Morita context

$$\mathcal{M} = (P, Q, \mu : P \otimes_A Q \to B,\ \nu : Q \otimes_B P \to A).$$

We will then also call J_A and J_B the trace ideals of the hermitian Morita context $\mathcal{H}$. It is then easy to see that $\alpha(J_A) = J_A$ and $\beta(J_B) = J_B$. Indeed, if $j \in J_A$, then $j = \nu(\sum_i q_i \otimes p_i)$, for some $p_i \in P$ and some $q_i \in Q$. For any index i, pick $p'_i \in P$ with $q_i = \theta(p'_i)$, then

$$\alpha(j) = \alpha\left(\nu\left(\sum_i q_i \otimes p_i\right)\right) = \sum_i \alpha(\nu(\theta(p'_i) \otimes p_i)) = $$
$$= \sum_i \nu(\theta(p_i) \otimes p'_i) \in J_A.$$

So, $\alpha(J_A) \subseteq J_A$, whence equality as $J_A = \alpha(\alpha(J_A)) \subseteq \alpha(J_A)$ as well. A similar argument shows that $\beta(J_B) = J_B$.

(3.5) For any twosided ideal I of a ring A and any left resp. right A-module M, let us denote by $Ann^l_M(I)$ resp. $Ann^r_M(I)$ the set of all $m \in M$ with the property that $Im = 0$ resp. $mI = 0$. Then, as in **[15]**, for any twosided ideal I of a ring A, the set of all $M \in A$-mod with the property that $Ann^l_M(I) = 0$ is the torsionfree class of an idempotent kernel functor in A-mod, denoted by σ^l_I. The corresponding Gabriel filter $\mathcal{L}(\sigma^l_I)$ consists of all left ideals L of A with the property that $Ann^l_M(I) = 0$ implies that $Ann^l_M(L) = 0$ for all $M \in A$-mod. The quotient category (A, σ^l_I)-mod of A-mod associated to σ^l_I consists of all $M \in A$-mod with the property that the canonical map $M = {}_A[A, M] \to {}_A[I, M]$ is bijective.

Note also that if A is left noetherian, then σ^l_I is just the usual *symmetric* idempotent kernel functor σ_I in A-mod associated to I, with Gabriel filter generated by the positive powers I^n of I. Indeed, clearly $\sigma_I \geq \sigma^l_I$. On the other hand, if $\sigma^l_I M = 0$ and if $m \in \sigma_I M$, then $I^n m = 0$ for some positive integer n. Since $Ann^l_I M = 0$, it follows that $m = 0$, hence $\sigma_I M = 0$ as well. So, $\sigma^l_I = \sigma_I$, indeed.

The idempotent kernel functor σ^r_I associated to I in mod-A is defined similarly, using $Ann^r_M(I)$ instead of $Ann^l_M(I)$.

(3.6) Lemma. *Let (A, α) be an R-algebra with involution and let I be a twosided ideal of A, then $\alpha(\sigma^l_I) = \sigma^r_{\alpha(I)}$.*

Proof: The idempotent kernel functor $\alpha(\sigma^l_I)$ is defined through its Gabriel filter $\mathcal{L}(\alpha(\sigma^l_I))$, which consists of all right A-ideals $\alpha(L)$ with $L \in \mathcal{L}(\sigma^l_I)$. So, let $L \in \mathcal{L}(\sigma^l_I)$ and let $N \in$ mod-A be such that $Ann^r_N(\alpha(I)) = 0$. If $M = {}^\alpha N$, then $Ann^l_M(I) = 0$, as one easily checks. So, by definition, $Ann^l_M(L) = 0$ as well. But then, it also follows that $Ann^r_N(\alpha(L)) = 0$, so $\alpha(L) \in \mathcal{L}(\sigma^r_{\alpha(I)})$. We thus find that $\alpha(\sigma^l_I) \leq \sigma^r_I$, hence $\alpha(\sigma^l_I) = \sigma^r_{\alpha(I)}$, by symmetry. ■

(3.7) To the trace ideals J_A and J_B of a hermitian Morita context

$$\mathcal{H} = (P, Q, \theta : P \to Q,\ \mu : P \otimes_A Q \to B,\ \nu : Q \otimes_B P \to A),$$

we may associate, by the foregoing, idempotent kernel functors $\sigma^l_A = \sigma^l_{J_A}$ resp. $\sigma^l_B = \sigma^l_{J_B}$ in A-mod resp. B-mod and $\sigma^r_A = \sigma^r_{J_A}$ resp. $\sigma^r_B = \sigma^r_{J_B}$ in mod-A resp. mod-B. Moreover, as $\alpha(J_A) = J_A$ and $\beta(J_B) = J_B$, it follows that $\alpha(\sigma^l_A) = \sigma^r_A$ and $\beta(\sigma^l_B) = \sigma^r_B$. The associated localization functors will be denoted by Q^l_A resp. Q^l_B and Q^r_A resp. Q^r_B.

(3.8) Recall from [**15**], [**18**] that any Morita context

$$\mathcal{M} = (P, Q, \mu : P \otimes_A Q \to B,\ \nu : Q \otimes_B P \to A)$$

induces an equivalence between the quotient categories (A, σ^l_A)-mod and (B, σ^l_B)-mod through the functor

$$F^l = {}_A[Q, -] : (A, \sigma^l_A)\text{-mod} \to (B, \sigma^l_B)\text{-mod}$$

and its adjoint

$$G^l = {}_B[P, -] : (B, \sigma^l_B)\text{-mod} \to (A, \sigma^l_A)\text{-mod}.$$

From [**15**], we also retain that, up to canonical isomorphism, we have

$${}_B[P, Q^l_B(B)] = Q^l_A(Q) \text{ resp. } {}_A[Q, Q^l_A(A)] = Q^l_B(P).$$

Indeed, for any $M \in (A, \sigma^l_A)$-mod, we have

$$\begin{aligned} {}_A[{}_B[P, Q^l_B(B)], M] &= {}_A[{}_B[P, Q^l_B(B)], {}_B[P, {}_A[Q, M]] \\ &= {}_B[Q^l_B(B), {}_A[Q, M]] \\ &= {}_A[Q, M] = {}_A[Q^l_A(Q), M], \end{aligned}$$

so ${}_B[P, Q^l_B(B)] = Q^l_A(Q)$. The other isomorphism may be derived in a similar way.

In view of the general theory developed in [**18**], it then follows that

$$F^l = {}_A[Q, -] = Q^l_B(Q^l_B(P) \otimes_A -) = Q^l_B(P \otimes_A -)$$

and similarly

$$G^l = {}_B[P, -] = Q^l_A(Q^l_A(Q) \otimes_B -) = Q^l_A(Q \otimes_B -).$$

It also follows that the maps μ and ν extend to bimodule isomorphisms

$$\mu_l : Q^l_B(P \otimes_A Q) \to Q^l_B(B)$$

and

$$\nu_l : Q^l_A(Q \otimes_B P) \to Q^l_A(A).$$

In particular, we thus obtain that the Morita context $\mathcal{M}$ induces a relative left Morita equivalence

$$(Q^l_B(P), Q^l_A(Q), \mu_l : Q^l_B(P \otimes_A Q) \to Q^l_B(B), \nu_l : Q^l_A(Q \otimes_B P) \to Q^l_A(A))$$

between (A,σ) and (B,τ), which we will usually denote by $\mathcal{M}^l$.

(3.9) Of course, in a completely similar way, one may derive that $\mathcal{M}$ induces an equivalence between the categories mod-(A,σ_A^r) and mod-(B,σ_B^r) through the functor

$$F^r = [P,-]_A : \text{mod-}(A,\sigma_A^r) \to \text{mod-}(B,\sigma_B^r)$$

and its adjoint

$$G^r = [Q,-]_B : \text{mod-}(B,\sigma_B^r) \to \text{mod-}(A,\sigma_A^r).$$

Moreover, we then also have that

$$F^r = [P,-]_A = Q_B^r(- \otimes_A Q_B^r(Q)) = Q_B^r(- \otimes_A Q)$$

and that

$$G^r = [Q,-]_B = Q_A^r(- \otimes_B Q_A^r(P)) = Q_A^r(- \otimes_B P).$$

Again, we thus obtain a relative right Morita equivalence $\mathcal{M}^r$ given by

$$(Q_B^r(P), Q_A^r(Q), \mu_r : Q_B^r(P\otimes_A Q) \to Q_B^r(B), \nu_r : Q_A^r(Q\otimes_B P) \to Q_A^r(A))$$

The previous discussion then implies:

(3.10) Theorem. *Any hermitian Morita context*

$$\mathcal{H} = (P, Q, \theta : P \to Q,\ \mu : P \otimes_A Q \to B,\ \nu : Q \otimes_B P \to A)$$

between the R-algebras with involution (A,α) and (B,β) induces a relative hermitian biequivalence, given by

$$(Q_B^l(P \otimes_A -), Q_A^l(Q \otimes_B -)) : (A,\sigma_A^l)\text{-mod} \approx (B,\sigma_B^l)\text{-mod}$$

and

$$(Q_B^r(- \otimes_A Q), Q_A^r(- \otimes_B P)) : \text{mod-}(A,\sigma_A^r) \approx \text{mod-}(B,\sigma_B^r),$$

and where the isomorphism

$$\Theta' : Q_B^l(P \otimes_A -) \xrightarrow{\sim} Q_B^r((-)^\alpha \otimes_A Q)^\beta$$

is induced by θ, as in (2.7).

(3.11) Note. Starting from the hermitian Morita context

$$\mathcal{H} = (P, Q, \theta : P \to Q,\ \mu : P \otimes_A Q \to B, \nu : Q \otimes_B P \to A),$$

we may also construct directly from it a relative hermitian Morita biequivalence (equivalent to the hermitian biequivalence determined by the previous theorem), given by

(3.11.1) the relative left Morita equivalence

$$\mathcal{M}^l = (P_l, Q_l, \mu_l \colon Q^l_B(P_l \otimes_A Q_l) \to Q^l_B(B),\ \nu_l \colon Q^l_A(Q_l \otimes_B P_l) \to Q^l_A(A)),$$

where $P_l = Q^l_B(P)$, $Q_l = Q^l_A(Q)$ and where μ_l is the morphism

$$Q^l_B(\mu) \colon Q^l_B(P_l \otimes_A Q_l) \overset{(*)}{\cong} Q^l_B(Q^l_B(P_l) \otimes_A Q^l_A(Q)) = Q^l_B(P \otimes_A Q) \to Q^l_B(B)$$

and where, similarly, ν_l is the morphism

$$Q^l_A(\nu) \colon Q^l_A(Q_l \otimes_B P_l) \overset{(*)}{\cong} Q^l_A(Q^l_A(Q_l) \otimes_B Q^l_B(P)) = Q^l_A(Q \otimes_B P) \to Q^l_A(A)$$

and where the isomorphisms $(*)$ hold by the relative flatness of P and Q with respect to σ^l_A and σ^l_B;

(3.11.2) the relative right Morita equivalence $\mathcal{M}^r$ given by

$$(P_r, Q_r, \mu_r : Q^r_B(P_r \otimes_A Q_r) \to Q^r_B(B),\ \nu_r : Q^r_A(Q_r \otimes_B P_r) \to Q^r_A(A)),$$

defined in a similar way;

(3.11.3) the B-A-anti-isomorphism $\theta' : P_l \to Q_r$, obtained by applying the following Lemma to the B-A-anti-isomorphism $\theta : P \to Q$:

(3.12) Lemma. *Let (A, α) resp. (B, β) be R-algebras with involution and le σ be an idempotent kernel functor in A-mod. Denote by $\alpha(\sigma)$ the induced idempotent kernel functor in mod-A. Let M be an A-B-bimodule, N a B-A-bimodule and let $\theta : M \to N$ be an A-B-antimorphism. Then θ induces an A-B-antimorphism $\theta' : Q^l_\sigma(M) \to Q^r_{\alpha(\sigma)}(N)$.*

Proof: The morphism θ induces an A-B-linear morphism $\hat{\theta} : M \to {}^\alpha N^\beta$, which localizes to an A-B-linear morphism $Q^l_\sigma(\hat{\theta}) : Q^l_\sigma(M) \to Q^l_\sigma({}^\alpha N^\beta)$. From [(**2.6.**), 12], it follows that

$$Q^l_\sigma({}^\alpha N^\beta) \cong {}^\alpha Q^r_{\alpha(\sigma)}(N)^\beta$$

in B-mod-A, so $Q^l_\sigma(\hat{\theta})$ may be viewed as a morphism $\theta' : Q^l_\sigma(M) \to Q^r_{\alpha(\sigma)}(N)$, satisfying the above requirements. ■

(3.13) Consider a hermitian Morita context

$$\mathcal{H} = (P, Q, \theta : P \to Q,\ \mu : P \otimes_A Q \to B,\ \nu : Q \otimes_B P \to A)$$

between the algebras with involution (A, α) and (B, β). We say that $\mathcal{H}$ is *right normalized* if the underlying Morita context

$$(P, Q, \mu : P \otimes_A Q \to B,\ \nu : Q \otimes_B P \to A)$$

is right normalized, in the sense of [**15**], i.e., that the natural maps $A \to [Q, B]_B$, $Q \to [P, A]_A$, $A \to [Q, Q]_B$ and $B \to [P, P]_A$ are isomorphisms. (Recall that *any* Morita context canonically induces a right normalized context, cf. [**15**, (2.4.)]). From [**15**, Theorem 7], it then follows that the natural maps $A \to [J_A, A]_A$ and $P \to [J_A, P]_A$ are bijective, i.e., that A and P are σ_A^r-closed. So, $A = Q_A^r(A)$ and $P = P_r$.

On the other hand, since the equivalence mod-$(A, \sigma_A^r) \approx$ mod-(B, σ_B^r) is given by $F^r = [P, -]_A$ and its adjoint $G^r = [Q, -]_B$, it follows that $B \cong [P, P]_A = F^r(P)$ is σ_B^r-closed. From the remarks made in (1.4), it thus follows that A resp. B is σ_A^l resp. σ_B^l-closed, as well.

Next, it follows from the existence of the B-A-anti-isomorphism $\theta : P \to Q$ and the induced $\theta'' : P_r \to Q_l$, that $Q \cong Q_l$ as A-B-bimodules. But then, up to isomorphism of A-B-bimodules, we also obtain that

$$Q_r = Q_B^r(Q) = Q_B^r({}^{\alpha}P^{\beta}) = {}^{\alpha}Q_B^l(P)^{\beta} = {}^{\alpha}P^{\beta} = Q,$$

and similarly that $P_l = P$.

We thus have proved that the hermitian Morita biequivalence induced by $\mathcal{H}$ is involutive in the sense of (2.9), hence:

(3.14) Proposition. *Every right normalized hermitian Morita context*

$$\mathcal{H} = (P, Q, \theta : P \to Q,\ \mu : P \otimes_A Q \to B,\ \nu : Q \otimes_B P \to A)$$

between the algebras with involution (A, α) and (B, β) induces a relative left hermitian Morita equivalence

$$\mathcal{H}^l = (P, Q, \theta : P \to Q,\ \mu_l : Q_B^l(P \otimes_A Q) \to B, \nu_r : Q_A^l(Q \otimes_B P) \to A)$$

between the left torsion triples (A, α, σ) and (B, β, τ) and a relative right hermitian Morita equivalence

$$\mathcal{H}^r = (P, Q, \theta : P \to Q,\ \mu_r : Q_B^r(P \otimes_A Q) \to B, \nu_r : Q_A^r(Q \otimes_B P) \to A)$$

between the right torsion triples $(A, \alpha, \alpha(\sigma))$ and $(B, \beta, \beta(\tau))$.

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