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STRONGLY GRADED LEFT FTF RINGS*

JOSÉ GÓMEZ AND BLAS TORRECILLAS

This paper is dedicated to the memory of Professor Pere Menal

Abstract

An associative ring R with identity is said to be a left FTF ring when the class of the submodules of flat left R-modules is closed under injective hulls and direct products. We prove (Theorem 3.5) that a strongly graded ring R by a locally finite group G is left FTF if and only if R_e is left FTF, where e is the neutral element of G. This provides new examples of left FTF rings. Some consequences of this Theorem are given.

1. Introduction

In the papers [**5**], [**6**] we started the study of the rings R with the property that the class of the submodules of flat left modules, $\mathcal{F}_0$, is closed under direct products and injective hulls. For these rings, $\mathcal{F}_0$ is the torsionfree class for some hereditary torsion theory on R-Mod. Thus we can use techniques of localization relative to this torsion theory to investigate these rings. This class of rings is large and we call them left FTF rings. Clearly, QF and regular rings are left and right FTF rings. In the commutative case, Enochs showed [**3**] that a noetherian commutative ring R is FTF if and only if R_P is Gorestein for every minimal prime ideal P of R. Noncommutative examples of left FTF rings can be found in [**5**], [**6**].

The aim of this note is to construct new examples of left FTF rings. The tool will be the strongly graded ring theory.

The paper is organized as follows. In Section 1 we record the fundamental results on left FTF rings showed in [**5**] and [**6**] which we will use in the rest of the paper. We recall also the basic machinery from torsion theories and graded rings that we will use.

The main result, Theorem 2.5, appear in Section 2.

Finally, we give some applications of this result in Section 3.

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2. Preliminaries and general notation

Let $\mathbf{C}$ be a Grothendieck category. Recall that a torsion theory [**10**, Ch. VI] on $\mathbf{C}$ is a pair $\tau = (\mathcal{T}, \mathcal{F})$ of classes of objects of $\mathbf{C}$ complete with respect to the relation $\mathrm{Hom}(T, F) = 0$, for $T \in \mathcal{T}$ and $F \in \mathcal{F}$. The objects of $\mathcal{T}$ (resp. of $\mathcal{F}$) are called the τ-torsion (resp. the τ-torsionfree) objects. We will use also the notation $\mathcal{T}(\tau)$ and $\mathcal{F}(\tau)$ for the classes $\mathcal{T}$ and $\mathcal{F}$. Every object C of $\mathbf{C}$ contains a largest subobject $\tau(C)$ belonging to $\mathcal{T}$, called the τ-torsion of C. This gives an idempotent radical $\tau : \mathbf{C} \to \mathbf{C}$ that determines uniquely the torsion theory because $\mathcal{T} = \{C \in \mathbf{C} \mid \tau(C) = C\}$ and $\mathcal{F} = \{C \in \mathbf{C} \mid \tau(C) = 0\}$. A class $\mathcal{F}$ of objects of $\mathbf{C}$ is a torsionfree class for some torsion theory τ on $\mathbf{C}$ if and only if $\mathcal{F}$ is closed under subobjects, products and extensions [**10**, VI.2.2]. Dually, a class $\mathcal{T}$ of objects of $\mathbf{C}$ is a torsion class for s ome torsion theory τ on $\mathbf{C}$ if and only if $\mathcal{T}$ is closed under qu otients, coproducts and extensions [**10**, VI.2.1]. The torsion theory is uniquely determined by its torsion class or by its torsionfree class.

The torsion theory τ is said to be hereditary if $\mathcal{T}$ is closed under submodules or, equivalently, if $\mathcal{F}$ is closed under injective envelopes [**10**, VI.3.2]. A torsion theory on $\mathbf{C}$ is hereditary if and only if its associated idempotent radical is left exact [**10**, VI.3.5].

We recall some ideas from torsion theories on categories of (graded) modules. All rings considered are associative with identity element and the (left or right) R-modules are unital. By R-Mod (resp. Mod-R) we will denote the Grothendieck category of all the left (resp. right) R-modules. Let G be a multiplicative group with identity element e. A graded ring R is a ring with identity 1, together with a direct decomposition $R = \oplus_{g \in G} R_g$ as additive subgroups such that $R_g R_h \subseteq R_{gh}$ for all $g, h \in$ G. Thus R_e is a subring of $R, 1 \in R_e$ and for every $g \in G, R_g$ is an R_e-bimodule. A G-graded left R-module is a left R-module M endowed with an internal decomposition $M = \oplus_{g \in G} M_g$ where each M_g is a subgroup of the additive group of M such that $R_g M_h \subseteq M_{gh}$ for all $\in G$. Let M a nd N be graded left modules over the graded ring R. For every $g \in G$ we set

$$\mathrm{HOM}_R(M, N)_g = \{f \in Hom_R(M, N) | f(M_h) \subseteq M_{gh} \text{ for all } h \in G\}$$

$\mathrm{HOM}_R(M, N)_g$ is an additive subgroup of the group $\mathrm{Hom}_R(M, N)$ of all R-linear maps from M to N. Observe that

$$\mathrm{HOM}_R(M, N) = \bigoplus_{g \in G} \mathrm{HOM}_R(M, N)_g$$

is a subgroup of $\mathrm{Hom}_R(M,N)$ and it is a graded abelian group of type G. Clearly $\mathrm{HOM}_R(M,N)_e$ is just $\mathrm{Hom}_{R-gr}(M,N)$, i.e. the group of all morphisms from M to N in the category $R-gr$ of all graded left R-modules. Define for $g \in G$ the *g-suspension* $M(g)$ of a graded left R-module M as follows: $M(g)$ is the left R-module M graded by G by putting $M(g)_h = M_{hg}$ for all $h \in G$. Observe that

$$\mathrm{HOM}_R(M,N)_g = \mathrm{Hom}_{R-gr}(M,N(g)) = \mathrm{Hom}_{R-gr}(M(g^{-1}),N).$$

It is well known that $R-gr$ is a Grothendieck category (See [**9**]). Observe that if $G=\{e\}$ is the group of one element, then $R-gr = R$-Mod. Therefore, we can consider R-Mod as a particular case of the concept of category of graded modules.

We will denote by $\mathcal{H}$ the set of all homogeneous left ideals of R. In other words, $\mathcal{H}$ is the set of all subobjects in $R-gr$ of the graded left R-module R. The left ideals in $\mathcal{H}$ are also called *graded left ideals* of R. By $h(R)$ we denote the set of all homogeneous elements of R, that is, $h(R)=\bigcup\{R_g : g\in G\}$.

Following [**9**], an hereditary torsion theory τ on $R-gr$ is said to be *rigid* if for any τ-torsion graded left R-module M and for every g in G, $M(g)$ is τ-torsion. A rigid hereditary torsion theory is determined by certain subset of $\mathcal{H}$. Concretely, let $\mathcal{L}(\tau)=\{I\in\mathcal{H} \mid R/I \text{ is } \tau\text{-torsion}\}$. Then $\mathcal{L}(\tau)$ is a left *graded Gabriel topology* (or a graded filter of left ideals) on R, i.e., the following conditions are satisfied (see [**9**] or [**8**]).

$(G1)$ If $I\in\mathcal{L}(\tau)$ and $r\in h(R)$ then $(I:r)\in\mathcal{L}(\tau)$.

$(G2)$ If I and J are homogeneous left ideals of R, $J\in\mathcal{L}(\tau)$ and $(I:r)\in\mathcal{L}(\tau)$ for all $r\in J\cap h(R)$, then $I\in\mathcal{L}(\tau)$.

The τ-torsion graded submodule of a graded left R-module M can be computed from $\mathcal{L}(\tau)$ as

$$(1) \qquad \tau(M)=\{x\in M \mid Ix=0 \text{ for some } I\in\mathcal{L}(\tau)\}.$$

Observe that if R is any ring, it can be considered as a G-graded ring by $G=\{e\}$ and all the hereditary torsion theories on $R-gr = R$-Mod are rigid. However, if R is graded by any group G, there are some hereditary torsion theories on R-Mod that induce nicely rigid torsion theories on $R-gr$. These are the *graded torsion theories* and they are characterized as the torsion theories on R-Mod whose associatted (ungraded) Gabriel topology has a cofinal subset of homogeneous left ideals. For a graded torsion theory τ on R-Mod and M any graded left R-module, $\tau(M)$ is a graded subobject of M. This permits the induction of the rigid torsion theory on R-gr.

If τ is an hereditary torsion theory on R-Mod and M is a left R-module, then we can construct the abelian group

$$Q_\tau(M) = \varinjlim \{\mathrm{Hom}_R(I, M/\tau(M)) | I \in \mathcal{L}(\tau)\}.$$

It is well known [**10**, Ch. IX] that it is possible to give a canonical structure of ring on $Q_\tau(R)$ such that the canonical map $R \to Q_\tau(R)$ is a ring morphism and $Q_\tau(M)$ is a left $Q_\tau(M)$-module for every $M \in R$-Mod. Moreover, the canonical map $M \to Q_\tau(M)$ is an R-homomorphism with kernel and cokernel τ-torsion. Therefore, every τ- torsionfree left R-module M is isomorphic to an R-submodule of a left $Q_\tau(R)$-module, namely, $Q_\tau(M)$. The converse is not true in general and a torsion theory τ for which the class of τ-torsionfree left R-modules is precisely the class of all R-submodules of left $Q_\tau(R)$-modules is said to be *perfect* [**4**, Proposition 45.1]. The ring $Q_\tau(R)$ (together with the canonical ring morphism $R \to Q_\tau(R)$) is called the *quotient ring* of R with respect to τ.

The most useful quotient ring associated to an arbitrary ring R is the left maximal quotient ring of R, denoted by $Q^l_{\max}(R)$. This ring is the quotient ring $Q_\lambda(R)$ of R with respect to the torsion theory λ on R-Mod cogenerated [**10**, Ch. VI] by the injective hull $E({}_RR)$ of R in R-Mod. This torsion theory λ is called the *Lambek torsion theory* (on the left). Analogously, it is possible to define the right maximal quotient ring of R, denoted by $Q^r_{\max}(R)$. If the canonical ring monomorphism $R \to Q^l_{\max}(R)$ provides a right maximal quotient ring for R then we will say that $Q = Q^l_{\max}(R)$ is a *twosided maximal quotient ring* for R.

For more information on torsion theories the reader is referred to [**10**], [**4**] and [**9**].

Given left R- modules M and N we will say that M *embeds in* N whenever there is a monomorphism of left R-modules from M to N. Let $\mathcal{F}^R_0$ (or $\mathcal{F}_0$, if there is no risk of confusion) denote the class consisting of the left R-modules that embed in some flat left R-module. We say usually that $\mathcal{F}_0$ is *the class of submodules of flat left modules.* Our interest is centered in the study of the rings R for which $\mathcal{F}_0$ is a torsionfree class for some hereditary torsion theory τ_0 on R-Mod.

Definition. A ring R is said to be a left FTF ring (or, shortly, is left FTF) if the class $\mathcal{F}_0$ of submodules of flat left R-modules is the class of the τ_0- torsionfree left R-modules for some hereditary torsion theory τ_0 on R-Mod.

The class of left FTF rings includes the regular Von Neumann and QF rings, in fact every left IF ring (see [**1**]) is a left FTF ring with $\mathcal{F}_0 = R$-Mod. On the other hand, every semiprime left and right Goldie ring is left FTF.

We started the study of left FTF rings in [**5**], where the following basic properties were proved. By $r_R(X)$ we denote the right annihilator of a subset X of R.

Proposition 2.1. ([5]) *For a left FTF ring the following statements hold:*

(1) τ_0 *is of finite type, that is,* $\mathcal{L}(\tau_0)$ *contains a cofinal subset of finitely generated left ideals.*

(2) *For each finitely presented left* R *-module* P,

$\tau_0(P) = Ker\ f_1 \cap \ldots \cap Ker\ f_n$ *for some* $f_i \in Hom_R(P, R)$.

(3) $\mathcal{L}(\tau_0)$ *is the set of the left ideals* I *such that there are* $x_1, \ldots, x_n \in I$ *with* $r_R(\{x_1, \ldots, x_n\}) = 0$.

The following result is essentially proved in [**5**, Theorem 4.6], but with a slightly different presentation.

Theorem 2.2. ([5]) *Let* R *be a ring for which* λ *is of finite type. Then* R *is left FTF if and only if every direct product of copies of* $E({}_RR)$ *is a flat left* R*-module. In such a case,* $\tau_0 = \lambda$ *and* $Q_{\tau_0}(R) = Q^l_{\max}(R)$.

We will finish this section with some basic results on the behaviour of the class of the submodules of flat modules with respect to a ring monomorphism. Let $\rho : S \to T$ denote a ring monomorphism. We can consider the class consisting of the left S-modules that are isomorphic to an S-submodule of some left T-module. Denote this class by $\mathcal{F}(\rho)$. It is not hard to see that a left S-module M is in $\mathcal{F}(\rho)$ if and only if the canonical homomorphism of left S-modules $\theta_M : M \to T \otimes_S M$ given by $\theta_M(x) = 1 \otimes x$ for all $x \in M$ is injective. Using this observation, it is easy to prove the following relationships among the classes $\mathcal{F}(\rho), \mathcal{F}_0^T$ and $\mathcal{F}_0^S$. We advise that the class of left T-modules $\mathcal{F}_0^T$ will be considered also as a class of left S-modules.

Lemma 2.3. *(1)* $\mathcal{F}_0^S \subseteq \mathcal{F}(\rho)$.

(2) If $P \in S$*-Mod is* S*-flat, then* $T \otimes_S P$ *is* T*-flat.*

(3) Assume that T_S *is flat. If* $M \in \mathcal{F}_0^S$ *then* $T \otimes_S M \in \mathcal{F}_0^T$.

(4) Assume that ${}_ST$ *is flat. If* $M \in T$*-Mod is* T*-flat then* ${}_SM$ *is* S*-flat.*

(5) Assume that ${}_ST$ *is flat. Then* $\mathcal{F}_0^T \subseteq \mathcal{F}_0^S$.

(6) Assume that ${}_ST$ *and* T_S *are flat. A left* S*-module* M *is in* $\mathcal{F}_0^S$ *if and only if* M *is isomorphic to a left* S*-submodule of a flat left* T*-module.*

3. Strongly graded left FTF rings

Let $R = \oplus_{g \in G} R_g$ be a strongly graded ring by a group G with neutral element e. This means that $R_g R_h = R_{gh}$ for all $g, h \in G$. For a strongly graded ring, R_g is a finitely generated projective left and right R_e-module. Moreover, the functors $R \otimes_{R_e} -$ and $(-)_e$ establish an equivalence between the categories R_e-Mod and $R - gr$ (see [**2**, Theorem 2.8]).

The class of all left R-modules that embed in flat left R-modules is denoted throughout this section by $\mathcal{F}_0^R$ and we reserve the notation $\mathcal{F}_0$ for the class of the submodules of flat left R_e-modules. If the ring R is left FTF, we denote by τ_0^R the hereditary torsion theory for which $\mathcal{F}_0^R$ is the class of the τ_0^R-torsionfree left R-modules. When R_e is left FTF, the analogous notation will be τ_0.

Proposition 3.1. *If R is a left FTF ring, then R_e is a left FTF ring.*

Proof: We will proof first that $\mathcal{F}_0$ is the torsionfree class for some torsion theory τ_0 on R_e-Mod and then we will show that τ_0 is necessarily hereditary. Note that $\mathcal{F}_0$ is closed by submodules. We will prove that $\mathcal{F}_0$ is stable under extensions and direct products. Consider

$$0 \longrightarrow N \longrightarrow M \longrightarrow L \longrightarrow 0$$

an exact sequence of left R_e-modules with $N, L \in \mathcal{F}_0$. Since R_{R_e} is flat, the following sequence of left R-modules is exact

$$(2) \qquad 0 \longrightarrow R \otimes_{R_e} N \longrightarrow R \otimes_{R_e} M \longrightarrow R \otimes_{R_e} L \longrightarrow 0.$$

Since $N, L \in \mathcal{F}_0$; $R \otimes_{R_e} N$ and $R \otimes_{R_e} L$ are in $\mathcal{F}_0^R$ (Lemma 2.3.(3)). Therefore, (2) is an exact sequence of left R-modules with τ_0^R-torsionfree extreme points and this implies that $R \otimes_{R_e} M$ is τ_0^R- torsionfree, that is, $R \otimes_{R_e} M \in \mathcal{F}_0^R$. By Lemma 2.3.(6), $R \otimes_{R_e} M \in \mathcal{F}_0$. The map

$$\theta_M : M \longrightarrow R \otimes_{R_e} M$$

defined by $\theta_M(m) = 1 \otimes m$ for each $m \in M$ is a monomorphism of left R_e-modules. Hence, ${}_{R_e}M \in \mathcal{F}_0$ and this proves that $\mathcal{F}_0$ is closed under extensions.

Next, we prove that $\mathcal{F}_0$ is stable by direct products. Let $\{M_i : i \in I\}$ be a family of left R_e-modules in $\mathcal{F}_0$ and put

$$M = \prod \{M_i : i \in I\}.$$

According to Lemma 2.3, $R \otimes_{R_e} M_i \in \mathcal{F}_0^R$ for every $i \in$ I. Since R is left FTF, $\mathcal{F}_0^R$ is stable under direct products, and

$$\prod\{R \otimes_{R_e} M_i : i \in I\} \in \mathcal{F}_0^R.$$

Again Lemma 2.3 assures that

$$\prod\{R \otimes_{R_e} M_i : i \in I\} \in \mathcal{F}_0.$$

But there is an obvious monomorphism of left R_e-modules

$$\prod\{M_i : i \in I\} \longrightarrow \prod\{R \otimes_{R_e} M_i : i \in I\}$$

that shows that $\prod\{M_i : i \in I\} \in \mathcal{F}_0$. Therefore $\mathcal{F}_0$ is closed by direct products and it is the torsionfree class for some torsion theory τ_0 (at this moment, possibly not hereditary) on R_e-Mod.

We will finish by proving that τ_0 is in fact hereditary, that is, the class of the τ_0-torsion left R_e-modules is stable by submodules. For, consider M a τ_0-torsion left R_e-module and N a submodule of M. We claim that $R \otimes_{R_e} M$ is a τ_0^R- torsion left R-module. To check this assertion, it suffices to show that

$$\mathrm{Hom}_R(R \otimes_{R_e} M, P) = 0$$

for every flat left R-module P. But

$$\mathrm{Hom}_R(R \otimes_{R_e} M, P) \cong \mathrm{Hom}_{R_e}(M, {}_{R_e}P) = 0,$$

since ${}_{R_e}P$ is flat (Lemma 2.3.(6)) and, thus, τ_0-torsionfree. Now we are ready to show that $N \subseteq M$ is a τ_0-torsion left R_e- module. We check that $\mathrm{Hom}_{R_e}(N, F) = 0$ for every flat left R_e-module F. Since R is strongly graded, one has an isomorphim of abelian groups

$$\mathrm{Hom}_{R_e}(N, F) \cong \mathrm{Hom}_{R-gr}(R \otimes_{R_e} N, R \otimes_{R_e} F).$$

On the other hand, since R_{R_e} is flat, we have a monomorphism of left R-modules

$$R \otimes_{R_e} N \longrightarrow R \otimes_{R_e} M.$$

Hence, $R \otimes_{R_e} N$ is τ_0^R-torsion. According to Lemma 2.3.(1), $R \otimes_{R_e} F$ is a flat left R- module. Therefore

$$\mathrm{Hom}_R(R \otimes_{R_e} N, R \otimes_{R_e} F) = 0.$$

Since

$$\mathrm{Hom}_{R-gr}(R\otimes_{R_e} N, R\otimes_{R_e} F) \subseteq \mathrm{Hom}_R(R\otimes_{R_e} N, R\otimes_{R_e} F)$$

we obtain $\mathrm{Hom}_{R_e}(N,F)=0$. This concludes the proof. ■

In that follows, we will try an aproximation to the converse of Proposition 3.1. If R_e is a left FTF ring, then the inclusion $R_e \to R$ allows the construction of a torsion theory $\bar{\tau}_0$ on R-Mod with torsion class $\mathcal{T}(\bar{\tau}_0)$ consisting of those left R-modules that are τ_0-torsion considered as left R_e-modules. Every graded left R-module decomposes, when it is considered as left R_e-module, as a direct sum $\underline{M} = \oplus_{g\in G} M_g$ of R_e-submodules and every morphism $f : M \to N$ in $R-gr$ is, after forgeting the R-linear structure, a morphism of left R_e-modules $\underline{f} : \underline{M} \to \underline{N}$ that maps the g-th component M_g of $\underline{M}$ to the g-th component N_g of $\underline{N}$. This construction defines an exact functor

$$\underline{(\_)} : R-gr \longrightarrow R_e - \mathrm{Mod}.$$

This permits us to induce a rigid torsion theory τ_0^g on $R-gr$ from τ_0 by putting as torsion class

$$\mathcal{T}(\tau_0^g) = \{X \in R-gr \mid \underline{X} \text{ is } \tau_0 - \text{torsion}\}.$$

The following result gives some information about $\bar{\tau}_0$.

Proposition 3.2. *Assume that R_e is a left FTF ring and let $\bar{\tau}_0$ be the torsion theory induced on R -Mod by τ_0. The following conditions are satisfied.*

(1) *τ_0 is G-stable, that is, for every τ_0-torsion left R_e- module T, the left R_e-module $R\otimes_{R_e} T$ is τ_0-torsion.*
(2) *$\mathcal{T}(\bar{\tau}_0)$ is the smallest torsion class on R-Mod containing the underlying left R-modules of the objects in $\mathcal{T}(\tau_0^g)$. Therefore, $\bar{\tau}_0$ is a graded torsion theory.*
(3) *The Gabriel topology $\mathcal{L}(\bar{\tau}_0)$ associated with $\bar{\tau}_0$ is*

$$\mathcal{L}(\bar{\tau}_0) = \{I \leq_R R \mid \exists x_1, \ldots, x_n \in I \cap R_e, \text{ with } l_{R_e}(\{x_1,\ldots,x_n\}) = 0\}.$$

(4) *A left R-module M is $\bar{\tau}_0$- torsionfree if and only if ${}_{R_e}M \in \mathcal{F}_0$.*

Proof: (1) It suffices to prove that for every left ideal I in the filter $\mathcal{L}(\tau_0)$, and for each $g \in G$, the left R_e-module $R_g \otimes_{R_e} R_e/I$ is τ_0-torsion. Since τ_0 is of finite type (Proposition 2.1) I contains a finitely generated left ideal I_0 such that I_0 is τ_0-dense in R_e. We will prove that $R_g \otimes_{R_e} R_e/I_0$ is τ_0-torsion and therefore $R_g \otimes_{R_e} R_e/I$ is τ_0- torsion since it is an epimorphic image of $R_g \otimes_{R_e} R_e/I_0$. Observe that

$$R_g \otimes_{R_e} R_e/I_0 \cong R_g/R_gI_0.$$

Since R_g is projective and finitely generated as left R_e-module, and I_0 is finitely generated, it follows that R_g/R_gI_0 is a finitely presented left R_e-module. According to Proposition 2.1.(2), $R_g \otimes_{R_e} R_e/I_0$ is τ_0- torsion if and only if

$$\mathrm{Hom}_{R_e}(R_g \otimes_{R_e} R_e/I_0, R_e) = 0.$$

But

$$\mathrm{Hom}_{R_e}(R_g \otimes_{R_e} R_e/I_0, R_e) \cong \mathrm{Hom}_{R_e}(R_e/I_0,\ \mathrm{Hom}_{R_e}(R_g, R_e)) = 0,$$

since $\mathrm{Hom}_{R_e}(R_g, R_e)$ is a flat (in fact, projective) left R_e- module.

(2) Let $\mathcal{T}$ be any torsion class containing the underlying left R-modules of the objects in $\mathcal{T}(\tau_0^g)$ and take M a $\bar{\tau}_0$-torsion left R-module. Then ${}_{R_e}M$ is τ_0-torsion. By G-stability, $R \otimes_{R_e} M$ is τ_0-torsion. Observe that $R \otimes_{R_e} M$ is canonically G-graded and, so, it is τ_0^g-torsion. This implies that $R \otimes_{R_e} M$ is in $\mathcal{T}$. Note that there exists a canonical epimorphism of left R-modules from $R \otimes_{R_e} M$ onto M. This shows that M is in $\mathcal{T}$. Therefore, $\mathcal{T}(\tau_0^g) \subseteq \mathcal{T}$.

The fact that $\bar{\tau}_0$ is a graded torsion theory on R-Mod follows from [**8**, Proposition 1.1].

(3) By G-stability, it is clear that

$$\mathcal{T}(\tau_0^g) = \{X \in R - gr \mid X_e \text{ is } \tau_0 - \text{torsion}\}.$$

Taking into account that $\bar{\tau}_0$ is graded, it is not hard to see that

$$\mathcal{L}(\tau_0^g) = \{I \leq_R R \mid I \cap R_e \in \mathcal{L}(\tau_0)\}.$$

Now, apply Proposition 2.1.

(4) This is a consequence of the foregoing facts together with [**8**, Proposition 2.1]. ■

We know from Proposition 3.1 that if R is a left FTF ring, then R_e is a left FTF ring too. The following result analyzes the relationships between the torsion theories $\bar{\tau}_0$ and τ_0^R on R-Mod. By $\mathcal{H}$ we denote the set of all homogeneous left ideals of R.

Proposition 3.3. *If R is a left FTF ring then the following statements hold:*

(1) $\mathcal{L}(\bar{\tau}_0) \subseteq \mathcal{L}(\tau_0^R)$

(2) $\mathcal{L}(\tau_0^g) = \mathcal{L}(\tau_0^R) \cap \mathcal{H}$.

Therefore, $\tau_0^R = \bar{\tau}_0$ if and only if τ_0^R is graded.

Proof: (1) Let $I \in \mathcal{L}(\bar{\tau}_0)$. To prove that $I \in \mathcal{L}(\tau_0^R)$ it suffices to find a finitely generated left ideal I_0 contained in I such that $\mathrm{Hom}_R(R/I_0, R) = 0$ (Proposition 2.1). Since $I \in \mathcal{L}(\bar{\tau}_0)$, there exists (Proposition 3.2.(3)) a finitely generated left ideal J of R_e such that $J \subseteq I \cap R_e$ and $\mathrm{Hom}_{R_e}(R_e/J, R_e) = 0$. Let $I_0 = RJ$. It is clear that I_0 is a finitely generated left ideal of R contained in I. Observe that I_0 is an homogeneous left ideal of R. Consider $f \in \mathrm{Hom}_R(R/I_0, R)$. Since R/I is graded and finitely generated, $\mathrm{Hom}_R(R/I, R) = HOM_R(R/I, R)$ [**2**], that is, f can be expressed as a sum of graded R-linear maps. Therefore, we can assume without loss of generality that f is graded. Equivalently, f can be considered as a morphism in the category $R - gr$ from R/I to $R(g)$ for some $g \in$ G. If we denote by $f_|$ the restriction of f to the part of degree e of R/I, it is clear the $f_|$ is an R_e-homomorphism from R_e/J to R_g. Since R_g is projective as left R_e-module and R_e/J is τ_0-torsion, it follows that $f_| = 0$. This assures that $Imf \cap R_g = 0$. But R is strongly graded and, so, Imf must be zero.

(2) Observe that, by Proposition 3.2.(2) and part (1) in this proposition, $\mathcal{L}(\tau_0^g) = \mathcal{L}(\bar{\tau}_0) \cap \mathcal{H} \subseteq \mathcal{L}(\tau_0^R) \cap \mathcal{H}$. It remains to prove that $\mathcal{L}(\tau_0^R) \cap \mathcal{H} \subseteq \mathcal{L}(\tau_0^g)$. Given $I \in \mathcal{L}(\tau_0^R) \cap \mathcal{H}$, let $I_0 \subseteq I$ be a finitely generated left ideal satisfying $\mathrm{Hom}_R(R/I_0, R) = 0$. Consider $a_1, \dots, a_n$ a set of generators of I_0. For each $i = 1, \dots, n$, there is a decomposition $a_i = \sum_{g \in G} a_{ig}$, where a_{ig} is the g-th homogeneous component of a_i. Define H as the homogeneous left ideal generated by the set of homogeneous elements $\{a_{ig} : i = 1, \dots n, g \in G\}$. Because $I_0 \subseteq H$, it is clear that $\mathrm{Hom}_R(R/H, R) = 0$ and, in particular, $\mathrm{Hom}_{R-gr}(R/H, R) = 0$. Hence, $\mathrm{Hom}_{R_e}(R_e/H \cap R_e, R_e) = 0$. Equivalently, $r_{R_e}(H \cap R_e) = 0$. In view of Proposition 3.2.(3), if we prove that $H \cap R_e$ is a finitely generated left ideal of R_e, then H is in $\mathcal{L}(\tau_0^g)$ and, so, $I \in \mathcal{L}(\tau_0^g)$. Taking into account that H is a finitely generated homogeneous left ideal, there is a graded free left R-module F and an epimorphism of graded left R-modules from F onto H. Taking in this epimorphism components of degree e, an epimorphism of left R_e-modules from F_e onto $H \cap R_e$ is obtained. But $F_e \cong R_{g_1} \oplus \dots \oplus R_{g_n}$ as left R_e-modules, for some $g_1, \dots, g_n \in$ G. In this way, F_e is a finitely generated projective left R_e-module. This clearly implies that $H \cap R_e$ is finitely generated as left R_e- modules. Therefore we have obtained that $I \in \mathcal{L}(\bar{\tau}_0)$. ■

Our next objective is to prove the converse of Proposition 3.1 for strongly graded rings by locally finite groups. By a locally finite group we understand a group G satisfying that all its finitely generated subgroups are finite.

Lemma 3.4. *Let R be a ring and consider $\{R_i : i \in I\}$ a directed family of subrings of R such that $R = \bigcup\{R_i : i \in I\}$. Let M be a left R-module such that ${}_{R_i}M$ is flat as left R_i-module for every $i \in I$. Then ${}_RM$ is a flat left R-module.*

Proof: We can apply [**10**, Proposition 10.7] to deduce that ${}_RM$ is flat considering that, since $R = \bigcup\{R_i : i \in I\}$, it follows that for every finite set X of elements of R, there exists $i \in I$ such that R_i contains X. ■

Theorem 3.5. *Assume that R is a strongly graded ring by a locally finite group G. Then R is a left FTF ring if and only if R_e is a left FTF ring. Moreover, in such a case, $\tau_0^R = \bar{\tau}_0$.*

Proof: According to Proposition 3.1 we only need to prove that if R_e is left FTF then R is a left FTF ring. Assume that R_e is a left FTF ring, that is, $\mathcal{F}_0$ is the class of the τ_0- torsionfree left R-modules for an hereditary torsion theory τ_0 on R_e-Mod. This torsion theory induces canonically an hereditary torsion theory $\bar{\tau}_0$ on R- Mod. By Proposition 3.2, $\mathcal{F}(\bar{\tau}_0)$ consists precisely of the left R-modules M such that ${}_{R_e}M \in \mathcal{F}_0$. This fact, together with Lemma 2.3, gives $\mathcal{F}_0^R \subseteq \mathcal{F}(\bar{\tau}_0)$ without hypothesis on the group G. We will prove the equality in the case that G is locally finite. In a first step, the group G is assumed to be finite. Given $M \in \mathcal{F}(\bar{\tau}_0)$, Proposition 3.3 says that ${}_{R_e}M \in \mathcal{F}_0$. Thus, there exist a flat left R-module P and a monomorphism of left R_e-modules $M \to P$. Tensorizing by the flat right R_e-module R_{R_e}, we obtain a monomorphism of left R-modules $R \otimes_{R_e} M \to R \otimes_{R_e} P$. It is clear that $R \otimes_{R_e} P$ is a flat left R-module. To conclude that ${}_RM \in \mathcal{F}_0^R$ we will exhibit a monomorphism of left R-modules from M to $R \otimes_{R_e} M$. Since R is strongly graded, $R_g R_{g^{-1}} = R_e$ for every $g \in G$. Then there exists, for each $g \in G$, a decomposition $1 = \sum_{i=1}^{n(g)} r(g)_i s(g^{-1})_i$, where $r(g)_i \in R_g$ and $s(g^{-1})_i \in R_{g^{-1}}$ for each $i = 1, \ldots, n(g)$. Define $\phi : M \to R \otimes_{R_e} M$ by $\phi(m) = \sum_{g \in G} \sum_{i=1}^{n(g)} r(g)_i \otimes s(g^{-1})_i m$. In an analogous way as in [**7**, Lemma 2.1] it can be proved that ϕ is R-linear. It is easy to see that ϕ is injective. Therefore ϕ is a monomorphism of left R-modules from M to $R \otimes_{R_e} M$ and this implies that $M \in \mathcal{F}_0^R$. Therefore, in the case that G is finite, $\mathcal{F}_0^R = \mathcal{F}(\bar{\tau}_0)$, that is, the class $\mathcal{F}_0^R$ is the class of all the $\bar{\tau}_0$-torsionfree left R-modules.

To work in the case that G is locally finite, we introduce some new notation. Let H be a finite subgroup of G and put $R_H = \oplus_{h \in H} R_h$. It is clear that R_H is a subring of R and that R_H is a strongly H-graded ring. We denote by τ_0^H to the hereditary torsion theory induced by τ_0 in R_H-Mod with torsion class consisting of those left R_H-modules that are τ_0-torsion as left R_e-modules. Since H is finite, the foregoing argument assures that the class of the τ_0^H-torsionfree left R_H-modules is precisely the class $\mathcal{F}_0^H$ of the submodules of flat left R_H-modules. On the other hand, Proposition 3.2.(4) says that a left R_H-module M is τ_0^H-torsionfree if and only if ${}_{R_e}M \in \mathcal{F}_0$. After all these observations, we are ready to finish the proof. As in the finite case, we only need check that $\mathcal{F}(\bar{\tau}_0) \subseteq \mathcal{F}_0^R$. For $M \in \mathcal{F}(\bar{\tau}_0)$, let $E = E({}_RM)$ be its injective hull in R-Mod. It is immediate that $E \in \mathcal{F}(\bar{\tau}_0)$. By Proposition 3.2.(4), ${}_{R_e}E \in \mathcal{F}_0$. But this implies that ${}_{R_H}E \in \mathcal{F}_0^H$ for every finite subgroup H of G. Since R is a projective right R_H-module, it follows that ${}_{R_H}E$ is injective as left R_H-module and, therefore, ${}_{R_H}E$ is a flat left R_H-module. Since G is locally finite, we have that

$$R = \bigcup \{R_H : H \text{ is a finite subgroup of } G\}.$$

Thus, Lemma 3.4 applies and ${}_RE$ is a flat left R- module. We have proved that M embeds in a flat left R- module, namely, its injective hull in R-Mod. This gives the equality $\mathcal{F}_0^R = \mathcal{F}(\bar{\tau}_0)$, that finishes the proof. ■

4. Applications

Recall [**1**] that a ring R is said to be left IF if every injective left R-module is flat or, in our notation, if $\mathcal{F}_0^R = R$-Mod. Colby [**1**, Theorem 3] proved that a group ring AG is a left IF ring if and only if A is left IF and G is locally finite. As a consequence of Theorem 3.5 we extend this result to general strongly graded rings.

Theorem 4.1. *Let R be a G-strongly graded ring, where G is a locally finite group. Then R is a left IF ring if and only if R_e is a left IF ring.*

In [**1**, Proposition 5] it is showed that a left IF ring with finite global weak dimension is regular. By combinning this result with Theorem 4.1 we obtain the following corollary.

Corollary 4.2. *Let R be a G-strongly graded ring, where G is a locally finite group. Then R is regular if and only if R_e is regular and R has finite global weak dimension.*

C. Năstăsescu proved that if R is a strongly graded ring by a finite group G, then R_e has a QF maximal left quotient ring if and only if R has a QF maximal left quotient ring [**7**, Theorem 5.1 and Corollary 2.10]. Here we will obtain an analogous result for QF twosided maximal quotient rings. This result is deduced from Theorem 3.5 and some facts on left FTF rings with finiteness conditions investigated in [**6**]. In [**6**, Theorem 11] we obtained that the rings R that have a QF twosided maximal quotient ring are exactly the τ_0^R-artinian left FTF rings with τ_0^R perfect.

Theorem 4.3. *Let R be a G-strongly graded ring by a finite group G. R has a QF twosided maximal quotient ring if and only if R_e has a QF twosided maximal quotient ring.*

Proof: By [**6**, Theorem 11] and Theorem 3.5. we can assume that both R_e and R are left FTF rings and that $\tau_0^R = \bar{\tau}_0$. By [**7**, Proposition 2.2] and Proposition 3.2 R is τ_0-artinian if and only if $\bar{\tau}_0$-artinian. Again by [**6**, Theorem 11], it remains only to prove that τ_0 is perfect if and only if $\bar{\tau}_0$ is perfect. Let $Q_e = Q_{\tau_0}(R)$ and $Q = Q_{\bar{\tau}_0}(R)$. Since $\mathcal{T}(\tau_0) \subseteq \mathcal{T}(\lambda), R_e$ is λ-artinian. Therefore, λ is of finite type and, by Theorem 2.2., $\tau_0 = \lambda$. An analogous argument can be constructed for τ_0^R. This gives that Q_e is the left maximal quotient ring of R_e and Q is the left maximal quotient ring of R. By [**7**, Theorem 5.1], there is a ring monomorphism $Q_e \to Q$, such that the following square of ring morphisms commutes

$$\begin{array}{ccc} R_e & \longrightarrow & R \\ \downarrow & & \downarrow \\ Q_e & \longrightarrow & Q \end{array}$$

Assume that τ_0 is perfect and let M be a left Q-module. Then M is a left Q_e-module and, by [**4**, Proposition 45.1], it is τ_0-torsionfree. Proposition 3.2.(4) gives that M is $\bar{\tau}_0$-torsionfree and, again by [**4**, Proposition 45.1], $\bar{\tau}_0$ is perfect.

Conversely, assume that $\bar{\tau}_0$ is perfect and consider M a left Q_e-module. Then $Q \otimes_{Q_e} M$ is a left Q-module and it follows from [**4**, Proposition 45.1] that it is $\bar{\tau}_0$-torsionfree. Proposition 3.2.(4) assures that $Q \otimes_{Q_e} M$ is τ_0-torsionfree. By [**7**, Theorem 5.1] Q_e is a direct summad of Q as right Q_e-module. Therefore there is a canonical Q_e-monomorphism $\theta : M \to Q \otimes_{Q_e} M$ given by $\theta(x) = 1 \otimes x$ for all $x \in$ M. Hence, M is τ_0-torsionfree and by [**4**, Proposition 45.1] τ_0 is perfect. ■

As a consequence of Theorem 4.3 and Theorem 4.1, it is possible to obtain the following known result ([**7**, Corollary 2.10]).

Corollary 4.4. *Assume that R is strongly graded by a finite group G. R is QF if and only if R_e is QF.*

Some others results of this kind can be deduced from Theorem 3.5 and characterizations of special types of left FTF rings. May be the most interesting are the following. For the definition of $QF-3$ ring an the basic properties of these rings, we refer to [**11**].

Theorem 4.5. *Let R be a strongly graded ring by a finite group G. Then R has a semiprimary $QF-3$ twosided maximal quotient ring if and only if R_e has.*

Proof: By [**6**, Proposition 8 and Remark 9.(A)] we have that a ring R is left FTF and τ_0^R-artinian if and only if it has a semiprimary $QF-3$ twosided maximal quotient ring. This, together with Theorem 3.5 and [**8**, Proposition 2.2], prove the result. ■

Theorem 4.6. *Assume that R is a strongly graded ring by a finite group. R is left artinian $QF-3$ if and only if R_e is left artinian $QF-3$.*

Proof: By [**6**, Remark 9.(C)], if R is a left artinian ri ng, then R is $QF-3$ if and only if R is left FTF. Theorem 3.5 and [**7**, Theorem 1.2] complet the proof. ■

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José Gómez:
Departamento de Matemática Aplicada
Universidad de Granada
Campus de Almería
04071 Almería
SPAIN

Blas Torrecillas:
Departamento de Algebra
Universidad de Granada
Campus de Almería
04071 Almería
SPAIN

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