

# ON FINITE GROUPS ACTING ON ACYCLIC COMPLEXES OF DIMENSION TWO

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## Abstract

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We conjecture that every finite group  $G$  acting on a contractible  $CW$ -complex  $X$  of dimension 2 has at least one fixed point. We prove this in the case where  $G$  is solvable, and under this additional hypothesis, the result holds for  $X$  acyclic.

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## 0. Introduction

Let  $G$  be a group and  $A$  an abelian group. Dicks and Dunwoody ([4, Chapter IV]) proved that for each element  $\zeta$  of  $H^1(G; AG)$  there exists a  $G$ -tree  $T$  with finite edge stabilizers, with the property that for each subgroup  $H$  of  $G$ , the restriction of  $\zeta$  to  $H$  is zero if and only if  $H$  fixes a point of  $T$ . It is natural to look for analogous geometric explanations of elements of higher cohomology groups; thus, for example, one can ask if for each element  $\zeta$  of  $H^2(G; AG)$  there exists a contractible 2-dimensional  $CW$ -complex  $X$  admitting an action of  $G$  with finite stabilizers for 2-cells, with the property that for each subgroup  $H$  of  $G$ , the restriction of  $\zeta$  to  $H$  is zero if and only if  $H$  acts trivially on  $X$  in some sense, perhaps leaving invariant a subtree of the 1-skeleton of  $X$ . The restriction of  $\zeta$  to any finite subgroup of  $G$  is zero, but if a finite group leaves a subtree invariant then it fixes a point. With this motivation, we optimistically conjecture that every finite group  $G$  acting on a contractible 2-dimensional  $CW$ -complex  $X$  has at least one fixed point.

In this note we prove this conjecture in the case where  $G$  is solvable. Our argument is based on a classical result of P.A. Smith ([8], [9]), stating that every action of a finite  $p$ -group on a finite dimensional  $\mathbf{Z}/p$ -acyclic  $CW$ -complex has a  $\mathbf{Z}/p$ -acyclic fixed-point set (see [2, Chapter III] and further developments e.g. in [1], [3], [7]).

In our context, the hypothesis that  $X$  has no cells above dimension 2 is essential. It is known that any finite nilpotent group whose order is not a prime power acts on some contractible 3-dimensional  $CW$ -complex without fixed points ([1]).

On the other hand, we shall prove that for a finite solvable group  $G$  acting on a 2-dimensional  $CW$ -complex  $X$ , in order to ensure the existence of a fixed point it suffices to assume that  $X$  is *acyclic*. For  $X$  acyclic, however, the condition that  $G$  be solvable cannot be removed, because the alternating group  $A_5$  acts on the 2-skeleton of the Poincaré sphere—which is acyclic—without fixed points ([6]). Recall that the 1-skeleton of the Poincaré sphere is the complete graph on 5 vertices, and the 2-skeleton is obtained by adding 6 pentagonal faces so as to extend the natural action of  $A_5$  on the set of vertices. The fundamental domain of the action is a triangle with angles  $\pi/2$ ,  $\pi/5$ ,  $3\pi/10$ , and the 60 copies of this fundamental domain triangulate the 2-skeleton, from which it follows that there are no fixed points. The fundamental group of this space is isomorphic to  $SL_2(\mathbf{F}_5)$ .

Since  $X$  being contractible is equivalent to  $X$  being simply-connected and acyclic, the question that remains open is: If we add the condition that  $X$  be simply-connected, can we delete the condition that  $G$  be solvable?

## 1. Statement and proof of the result

Let  $G$  be a finite group acting on a  $CW$ -complex  $X$  of dimension 2, and denote by  $X^G$  the set of fixed points under the action of  $G$ . We shall assume that the action is *cellular* ([5]); that is, each translation of an open cell is an open cell, and, if a cell is invariant, then it is pointwise fixed. Thus  $X^G$  is a subcomplex of  $X$ . For a subcomplex  $Y \subseteq X$ , we denote by  $C_n(X, Y)$  the group of relative cellular  $n$ -chains of the pair  $(X, Y)$ .

Given a nonzero abelian group  $A$ , a space  $X$  is said to be  *$A$ -acyclic* if  $\tilde{H}_k(X; A) = 0$  for all  $k$ , where  $\tilde{H}$  denotes reduced homology. Recall that the condition  $\tilde{H}_{-1}(X; A) = 0$  is equivalent to the augmentation homomorphism  $C_0(X) \otimes A \rightarrow A$  being surjective, and hence equivalent to  $X$  being nonempty.

We prove

**Theorem 1.1.** *Let  $G$  be a finite solvable group acting on a  $CW$ -complex  $X$  of dimension 2. If  $\tilde{H}_*(X; \mathbf{Z})$  is finite, and the orders of  $G$ ,  $H_1(X; \mathbf{Z})$  are coprime, then the natural map  $\tilde{H}_*(X^G; \mathbf{Z}) \rightarrow \tilde{H}_*(X; \mathbf{Z})$  is injective.*

*Proof:* Under our assumptions, the graded group  $\tilde{H}_*(X; \mathbf{Z})$  is necessarily concentrated in degree 1, since it is free abelian in all other degrees. Moreover,  $H_1(X; \mathbf{Z}/p) = 0$  (and hence  $X$  is  $\mathbf{Z}/p$ -acyclic) for every prime  $p$  dividing the order of  $G$ .

Since  $G$  is solvable, we can find a series of subgroups

$$(1.1) \quad \{1\} = G_0 \subset G_1 \subset G_2 \subset \dots \subset G_k = G$$

in which each  $G_{i-1}$  is normal in  $G_i$  and  $G_i/G_{i-1} \cong \mathbf{Z}/p_i$ , where  $p_i$  is a prime. Then the action of  $G$  on  $X$  induces an action of  $G_i/G_{i-1}$  on  $X^{G_{i-1}}$  and

$$(1.2) \quad X^{G_i} = (X^{G_{i-1}})^{G_i/G_{i-1}}$$

We prove inductively that the map  $\tilde{H}_*(X^{G_i}; \mathbf{Z}) \rightarrow \tilde{H}_*(X; \mathbf{Z})$  is a monomorphism for all  $i = 0, \dots, k$ . This is trivial for  $G_0$ . Thus suppose that it has been established for  $G_{i-1}$ . Then  $X^{G_{i-1}}$  is  $\mathbf{Z}/p$ -acyclic for every prime  $p$  dividing the order of  $G$ . Since the order of  $G_i/G_{i-1}$  is a prime  $p_i$ , applying Smith's Theorem ([9]) to the action of  $G_i/G_{i-1}$  on  $X^{G_{i-1}}$  we obtain, by (1.2), that  $X^{G_i}$  is  $\mathbf{Z}/p_i$ -acyclic. This tells us in particular that  $X^{G_i}$  is nonempty and connected. Further, for every abelian group  $A$  we have an exact sequence

$$(1.3) \quad 0 \rightarrow H_2(X^{G_i}; A) \rightarrow H_2(X; A) \rightarrow H_2(X, X^{G_i}; A) \rightarrow \\ \rightarrow H_1(X^{G_i}; A) \rightarrow H_1(X; A) \rightarrow H_1(X, X^{G_i}; A) \rightarrow 0,$$

from which we infer that  $H_2(X, X^{G_i}; \mathbf{Z}/p_i) = 0$ . But, since  $X$  has no cells above dimension 2, the group  $H_2(X, X^{G_i}; \mathbf{Z})$  embeds in the free abelian group  $C_2(X, X^{G_i})$  and hence it is free abelian itself. This forces  $H_2(X, X^{G_i}; \mathbf{Z}) = 0$ , showing that  $H_1(X^{G_i}; \mathbf{Z})$  embeds in  $H_1(X; \mathbf{Z})$ . ■

**Corollary 1.2.** *Every action of a finite solvable group  $G$  on a  $\mathbf{Z}$ -acyclic CW-complex  $X$  of dimension 2 has at least one fixed point.*

*Proof:* It follows from Theorem 1.1 that the fixed-point set  $X^G$  is  $\mathbf{Z}$ -acyclic, so in particular it is nonempty. ■

**Note.** Robert Oliver has kindly informed us that an, as yet unpublished, paper by Yoav Segev contains a different proof of Corollary 1.2, with the additional assumption that  $X$  be finite.

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