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WORK OF PERE MENAL ON NORMAL SUBGROUPS

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Abstract

We describe subgroups of GL_2A which are normalized by elementary matrices for rings A satisfying the first stable range condition, Banach algebras A, von Neumann regular rings A, and other rings A.

Let A be an associative ring with 1 and, for any natural number n, let GL_nA be the group of all invertible n by n matrices with entries in A (thus GL_1A is the group of units of A). Let E_nA be the subgroup of GL_nA generated by all elementary matrices $a^{i,j}$, where $a \in A$ and $1 \leq i \neq j \leq n$.

For any subset X of A, let E_nX denote the subgroup of GL_nA generated by all elementary matrices with entries in X, and let $E_n(A,X)$ denote the normal subgroup of E_nA generated by E_nX. If $n \geq 3$ and B is an ideal of A, then $E_n(A,B)$ is generated by elements of the form $a^{i,j}b^{j,i}(-a)^{i,j}$, where $a \in A$, $b \in B$, and $1 \leq i \neq j \leq n$.

Denote by $G_n(A,B)$ the inverse image of the center of $GL_n(A/B)$ under the canonical homomorphism $GL_nA \to GL_n(A/B)$; if $n \geq 2$, then $G_n(A,B)$ consists of all matrices of GL_nA which, reduced modulo B, are scalar matrices xI with x in the center of A/B.

For various classes of rings A (see [**3**], [**36**]-[**38**]) including all von Neumann regular rings, all Banach algebras, all commutative rings, and all stable range 1 rings, one has a complete description of all subgroups H of GL_nA which are normalized by E_nA for $n \geq 3$. Namely, for any subgroup H of GL_nA:

(a) If there exists an ideal B of A such that $E_n(A,B) \subset H \subset G_n(A,B)$, then $E_n(A,B) = [E_nA, E_nB] = [H, E_nA] = [G_n(A,B), E_nA]$ and so H is normalized by E_nA.

(b) If H is normalized by E_nA, then there is a unique ideal B of A satisfying

$$E_n(A,B) \subset H \subset G_n(A,B).$$

Note that these cannot be extended to arbitrary rings A when $n \geq 3$ (see [**10**]), nor are they true when $n = 2$. In fact, the structure of E_2A is intractable even for the integers $\mathbf{Z}$ (see [**9**], [**18**], [**27**]-[**32**]). Since there are nonstandard normal subgroups of E_2A when A is a field of 2 or 3 elements (see [**8**]), the same is true of any ring A with a residue field of 2 or 3 elements.

A ring A is called *von Neumann regular* if for every $a \in A$ there is an $x \in A$ with $axa = a$. By using xax instead of x, if necessary, we have $axa = a$ and $xax = x$.

Some partial positive results about the structure of E_2A were known for von Neumann regular rings (see [**37**]), Banach algebras (see [**38**]), and rings with stable range 1 (see [**3**]), including all commutative rings A with stable range 1 (see [**5**], [**20**], [**34**], [**43**]).

In particular, it was proved in [**37**] that for $A/\mathrm{rad}\ A$ von Neumann regular and B an ideal of A, then $[E_2A, G_2(A,B)] \subset E_2(A,B)$. This inclusion implies that every subgroup H of GL_2A satisfying condition (a) with $n = 2$ for some ideal B is normalized by E_2A. However, these results excluded those normal subgroups of E_2A which do not satisfy any ladder condition $E_2(A,B) \subset H \subset G_2(A,B)$ for any ideal B, for example when A is the field of 2 or 3 elements and $H = [E_2A, E_2A]$. In [**24**], Menal and Vaserstein replaced $E_2(A,B)$ by $[E_2A, E_2(A,B)]$ to obtain the following complete description of normal subgroups of E_2A for any von Neumann regular ring A.

Theorem 1. *Let A be a von Neumann regular ring and H a subgroup of GL_2A:*

(a) *If there is an ideal B of A with $[E_2A, E_2B] \subset H \subset G_2(A,B)$, then*

$$[H, E_2A] \subset E_2B = E_2(A,B),$$

and

$$[G_2(A,B), [E_2A, E_2A]] = [E_2A, [E_2A, H]] = \\ = [H \cap E_2A, E_2A] = [E_2A, E_2B] = [E_2B, E_2B]$$

$\subset H$. *In particular, H is normalized by $[E_2A, E_2A]$ and both $H \cap E_2A$ and HE_2B are normalized by E_2A;*

(b) *If $[H, E_2A] \subset H$, then there is a unique ideal B of A such that*

$$[E_2A, E_2B] \subset H \subset G_2(A,B).$$

This gives the following classification result.

Corollary. *Let A be a von Neumann regular ring. Then a subgroup H of E_2A is normal if and only if there is an ideal B of A such that $[E_2A, E_2B] \subset H \subset G_2(A, B)$.*

In the case of a commutative regular ring A this result was proved by Costa and Keller in [**5**]. Note that in this case A has stable range 1 (see [**11**, cor. 4.5]), so $E_2A = SL_2A$. In general, the stable range of A is not always 1 (see [**23**]), and it remains an open problem as to whether A is always a GE-ring, i.e., GL_nA is generated by elementary and diagonal matrices.

The uniqueness of the ideal B in the theorem is easy to see. This is a consequence of the following fact which is true for an arbitrary ring A: if H is a subgroup of GL_2A such that $[E_2A, E_2B] \subset H \subset G_2(A, B)$ for some ideal B of A then $B = \{x \in A : x^{1,2}x^{2,1} \in H\}$. To verify this fact, observe that $x^{1,2}x^{2,1} = [x^{1,2}, 1^{1,2}(-1)^{2,1}1^{1,2}] \in [E_2B, E_2A] \subset H$ for any $x \in B$. Finally, since both off-diagonal entries of the matrix are x, the inclusion $x^{1,2}x^{2,1} \in G_2(A, B)$ implies that $x \in B$.

In general, the intersection $H \cap E_2A$ in the theorem cannot be replaced by H because $[E_2B, E_2A] \neq [H, E_2A]$ when $H = G_2(A, B)$ and $A = B$ is the field of three elements. However, when E_2A is perfect (i.e., $E_2A = [E_2A, E_2A]$), the theorem implies that $[E_2A, E_2B] = [G_2(A, B), E_2A]$. In this case, a subgroup H of GL_2A is normalized by E_2A whenever $[E_2A, E_2B] \subset H \subset G_2(A, B)$ for an ideal B of A.

In the same paper, Menal and Vaserstein also applied these techniques to Banach algebras. However, it was shown in [**39**, section 4] that there are normal subgroups of E_2A corresponding to any quasi-ideal Y of A which do not satisfy any ladder condition $[E_2A, E_2B] \subset H \subset G_2(A, B)$ for an ideal B of A if Y is not itself an ideal, for example, $E_2(A, Y)$ (see [**33**, prop. 4.2]).

Recall that a quasi-ideal of A is an additive subgroup Y of A such that $aya \in Y$ and $yay \in Y$ for any $y \in Y$ and $a \in A$. Note that in many cases, every quasi-ideal is an ideal, for example, when $2A = A$ and A is commutative [**39**, lemma 4.1], but that this is not true in general. In fact, if $2A \neq A$, then quasi-ideals appear even in commutative local rings (see [**1**]).

The previous results did not use quasi-ideals because every quasi-ideal of a von Neumann regular ring is an ideal. This is also the case for every Banach algebra which is simple or commutative (see [**38**], [**39**]), but is not true for an arbitrary Banach algebra. However, the exterior (Grassman) algebra on a 2-dimensional real vector space is an example of a 4-dimensional algebra with a quasi-ideal that is not an ideal.

Let $C_2(A, Y)$ be the set of all $g \in GL_2A$ with $[g, E_2A] \subset E_2(A, Y)$.

Clearly $C_2(A,B) \subset G_2(A,B)$ for any ring A and any ideal B of A; the equality $C_2(A,B) = G_2(A,B)$ is equivalent to the inclusion $[E_2A, G_2(A,B)] \subset E_2(A,B)$. Menal and Vaserstein then proved the following description of normal subgroups.

Theorem 2. *Suppose that $2A = A$ and that for any $a \in A$ there is a finite sequence $x_1, x_2, ..., x_n$ in A with $x_1 + x_2 + ... + x_n = 1$ and $1 - ax_i \in GL_1A$ for all i. Let H be a subgroup of GL_2A.*

(a) *If there is a quasi-ideal Y of A with $E_2(A,Y) \subset H \subset C_2(A,Y)$ then $E_2(A,Y) = [E_2A, E_2Y] = [E_2A, E_2(A,Y)] = [E_2A, H] = [E_2A, C_2(A,Y)] \subset H$ and so H is normalized by E_2A. Furthermore, when Y is an ideal, $C_2(A,Y) = G_2(A,Y)$;*

(b) *If H is normalized by E_2A, then there is a unique quasi-ideal Y of A such that $E_2(A,Y) \subset H \subset C_2(A,Y)$.*

This leads to the following classification result.

Corollary. *Suppose that $2A = A$ and that for any $a \in A$ there is a finite sequence $x_1, x_2, ..., x_n$ in A with $x_1 + x_2 + ... + x_n = 1$ and $1 - ax_i \in GL_1A$ for all i. A subgroup H of E_2A is normal if and only if there is a quasi-ideal Y of A such that $E_2(A,Y) \subset H \subset C_2(A,Y)$.*

Menal and Vaserstein proved some parts of the theorem under weaker hypotheses on the ring A. It should also be noted that under slightly different hypotheses on the ring A, the theorem was proved in [**39**] in the case when every quasi-ideal of A is an ideal.

The hypotheses of Theorem 2 are satisfied not only by Banach algebras but also by many other rings, for example, the ring of all bounded smooth functions on any smooth manifold. In fact, these hypotheses are satisfied by any connected topological ring A with GL_1A open in A.

Theorem 2 can also be extended to other classes of rings. We call A *semilocal* if $A/\text{rad } A$ is a (not necessarily finite) direct product of matrix rings over division rings — for example, any Artinian ring is semilocal. Menal and Vaserstein showed in [**24**] that the semilocal rings satisfying the hypotheses of Theorem 2 are precisely those rings A with $2A = A$ and which contain no factor ring isomorphic to a matrix ring $M_n(\mathbf{Z}/3\mathbf{Z})$, which extends Theorem 2 to any subgroup H of GL_2A in the following manner.

Corollary. *Let $2A = A$ be a semilocal ring without factor rings isomorphic to a matrix ring $M_n(\mathbf{Z}/3\mathbf{Z})$. Then the conclusions of Theorem 2 hold for any subgroup H of GL_2A.*

For a commutative local ring, the corollary was proved in ([1], [13], [15], [16], [19], [21], [33]); results for commutative semilocal rings can be found in ([2], [7]). Kolotilina and Vavilov showed in [14] that for any normal subgroup H of GL_2A there is a unique ideal B of A such that $E_2(A,B) \subset H \subset G_2(A,B)$, provided that A is a semilocal ring satisfying two conditions. First, the center of A must contain a unit ε such that $1 - \varepsilon$ is also a unit; second, that A has neither factor rings which are division algebras with centers of cardinality less than 5 nor factor rings which are isomorphic to $M_2(\mathbf{Z}/2\mathbf{Z})$.

The use of quasi-ideals opens up a wider range of rings. Many authors had some treatment of commutative local rings, but the results that Menal and Vaserstein obtained in [24] included noncommutative local rings A for which $2A = A$. In [26], they extended their results to local rings A such that $A/\text{rad } A$ contained at least 4 elements.

We call a ring A *local* if $A/\text{rad } A$ is a division ring, where rad A denotes the Jacobson radical of A. Note that in a local ring A, if Y is a quasi-ideal of A and $Y \neq A$, then $Y \subset \text{rad } A$, since $A = yAy \subset Y$ if Y contains a unit y.

Let A be the exterior (Grassman) algebra on a vector space V over a field F with $\dim_F V \geq 2$ and let Y be the F-subspace of A spanned by the monomials of odd degree. Then A is a local ring with $A/\text{rad } A = F$ and Y is a quasi-ideal which is not an ideal. If $2F = F$, then $2A = A$ and A is not commutative. If $2F = 0$, then $2A = 0$ and A is commutative.

For any quasi-ideal $Y \neq A$ of an arbitrary ring A denote by $T_2(A,Y)$ the set of all elements of the form $x^{2,1}\text{diag}(u,v)y^{1,2}$, where $x, y \in Y$, $u, v \in GL_1A$, and $v^{-1}au - a, uav^{-1} - a \in Y$ for all $a \in A$. Set $T_2(A,A) = GL_2A$. When A is local, it is clear that $T_2(A,B) = G_2(A,B)$ for every ideal B of A. Note that for any ring A, $G_2(A,0)$ is the center of $GL_2A = G_2(A,A)$ which consists of scalar matrices over the center of the ring A. Moreover, $G_2(A,0)$ is the centralizer of E_2A in GL_2A.

Then Menal and Vaserstein proved the following result.

Theorem 3. *Suppose A is a local ring such that $A/\text{rad } A$ has at least 4 elements. Then:*

(a) $E_2(A,Y) = [E_2A, E_2Y] = [E_2A, E_2(A,Y)] = [E_2A, T_2(A,Y)]$ *for any quasi-ideal Y of A;*

(b) *If H is a subgroup of GL_2A which is normalized by E_2A, then there exists a unique quasi-ideal Y of A such that $E_2(A,Y) \subset H \subset T_2(A,Y)$.*

This result uses the following lemma, which follows from $T_2(A,Y)$ being a subgroup of GL_2A invariant under conjugation by E_2A.

Lemma. *Let A be a local ring, Y a quasi-ideal of A. Then $T_2(A,Y)$ is a subgroup of GL_2A and $[E_2A, T_2(A,Y)] \subset E_2(A,Y) \subset T_2(A,Y)$. Therefore $[H, E_2A] \subset H$ for any subgroup H of GL_2A such that $E_2(A,Y) \subset H \subset T_2(A,Y)$ for some quasi-ideal Y of A.*

The unique quasi-ideal in part (b) of Theorem 3 is in fact the level of H, $L(H) = \{a \in A : a^{1,2} \in H\}$ which plays an important role in most classification theorems. The uniqueness of Y is easy to see, because $Y \subset L(H)$ whenever $E_2(A,Y) \subset H$, and $L(H) \subset Y$ whenever $H \subset T_2(A,Y)$. This is true for an arbitrary ring A and a subset Y; however in general it is not true that $E_2(A,Y) \subset T_2(A,Y)$.

If card $(A/\text{rad } A) < 4$, then E_2A maps onto the group $E_2F = SL_2F$ where $F = A/\text{rad } A$ is a field, and since E_2F is not perfect (see [**8**]), E_2A is not perfect.

Furthermore, in this case, $H = [E_2A, E_2A]$ is a normal subgroup of E_2A such that no quasi-ideal Y of A exists with $E_2(A,Y) \subset H \subset T_2(A,Y)$. In fact, if A is the field of 2 elements, then H is a non-central normal subgroup of GL_2A containing no nontrivial elementary matrices.

In the case of a commutative local ring A, the theorem is essentially due to Abe (see [**1**]). This case was also treated in ([**13**], [**15**], [**16**], [**19**], [**21**], [**33**]). When $2A = A$, the theorem is contained in Menal and Vaserstein (see [**24**]). The main difficulties in proving the theorem lie in the case when $A/\text{rad } A$ is a small field of characteristic 2.

It was shown in [**42**] that for any local ring A with $A/\text{rad } A$ having at least 3 elements and any normal subgroup H of GL_2A there is a unique ideal B of A such that $E_2(A,B) \subset H \subset G_2(A,B)$. Under the additional condition that the center of $A/\text{rad } A$ has at least 7 elements, this was proved previously by Kolotilina and Vavilov in [**14**]. However, it is not true in general that every subgroup H of GL_2A such that $E_2(A,B) \subset H \subset G_2(A,B)$ for an ideal B of A is normal (see [**42**]). By the lemma, such a subgroup H is always normalized by E_2A.

In [**25**], Menal and Vaserstein used these methods on stable range one rings. In this paper they generalized to noncommutative rings the description of E_2A-normalized subgroups given by Costa and Keller in [**5**]. The quasi-ideals which appear in the results of Menal and Vaserstein do not appear in many previous results as every quasi-ideal is an ideal in the commutative case with $2A = A$.

A ring A satisfies the first Bass stable range condition if for any $a, b \in A$ with $aA + bA = A$, there is an $r \in A$ such that $(a + br)A = A$ (see [**35**]). Equivalently, this means that there is an r such that $a + br$ is a unit. We denote this by $sr(A) \leq 1$. A result of Kaplansky [**40**] says that when $sr(A) \leq 1$ every one-sided unit in A is a unit.

In [**12**], Goodearl and Menal proved the stable range one condition for many classes of rings and algebras. They used a strong form of stable range one, unit 1-stable range (replace the r above by a unit u to obtain the definition), which followed from the following condition on a ring A : given any $x, y \in A$, there is a unit $u \in A$ such that $x - u$ and $y - u^{-1}$ are both units. Verification of this condition yields stable range one in the following cases: (1) any algebra over an uncountable field, in which all elements are zero-divisors or units and there are no uncountable direct sums of nonzero one-sided ideals; (2) any algebra over an uncountable field, in which there are only countably many primitive factor rings, all of which are Artinian; (3) the endomorphism ring of any noetherian module over an algebra as in (2); (4) any algebraic algebra over an infinite field; (5) any integral algebra over a commutative ring which modulo its (Jacobson) radical is algebraic over an infinite field; (6) any von Neumann regular algebra over an uncountable field, which has a rank function. They also use other techniques to prove stable range 1 for other rings, including finite Rickart C^*-algebras and certain strongly π-regular rings. For more examples of stable range one rings, see [**3**], [**6**], [**35**], [**40**], [**41**].

When $sr(A) \leq 1$, more is known about the structure of GL_nA. In [**3**], Bass showed

$$[E_nA, E_nB] = [E_nA, G_n(A, B)] = [GL_nA, GL_nB] = E_n(A, B)$$

for any ideal B of A and any $n \geq 3$, where the principal congruence subgroup GL_nB is defined as the kernel of the homomorphism $GL_nA \to E_n(A/B)$. Generally, $[GL_nA, G_n(A, B)] \neq E_n(A, B)$ even for local rings A. Moreover, $G_n(A, B)/GL_nB = G_2(A, B)/GL_2B$ for $n \geq 2$, hence this group does not depend on n, and it is an abelian group isomorphic to the group of units of the center of the ring A/B.

The group $GL_nB/E_n(A, B) = K_1(A, B) = GL_1B/W(A, B)$ is also an abelian group which does not depend on n for $n \geq 2$, where $W(A, B)$ is the subgroup of GL_1A generated by the elements of the form $(a + c + abc)(a + c + cba)^{-1}$ with $a \in B, b \in A, c \in 1 + B$ (see [**17**], [**22**]). So the group $G_n(A, B)/E_n(A, B)$, which classifies all subgroups H of GL_nA normalized by E_nA and with the same "level" B, is two-step nilpotent. When A is commutative, $E_n(A, B) = SL_nB$ for all ideals B of A, $G_n(A, B)/GL_nB = G1(A, B)/GL_1B = GL_1(A/B)$, and $GL_nB/E_n(A, B) = K_1(A, B) = GL_1B$.

The next theorem [**25**] gives more information about the structure of $E_2(A, Y)$ and $C_2(A, Y)$ for rings A with $sr(A) \leq 1$ and quasi-ideals Y of A. Part (b) generalizes to quasi-ideals a result of [**15**] describing the

kernel of the Whitehead determinant $GL_1B \to K_1(A,B)$ for ideals B (the case $B = A$ had been done in [**18**]).

Theorem 4. *Let A be an associative ring with $sr(A) \leq 1$. Then for any quasi-ideal Y of A:*

(a) *the group $E_2(A,Y)$ is generated by elements of the forms $a^{1,2}y^{2,1}(-a)^{1,2}$ and $a^{2,1}y^{1,2}(-a)^{2,1}$ with $a \in A$ and $y \in Y$;*

(b) *the group $E_2(A,Y)$ consists of all elements of the form $x^{1,2}y^{2,1}z^{1,2}d$, where $x,y,z \in Y$, and d is a diagonal matrix which is a product of diagonal matrices of the form $diag(a+c+abc,(a+c+cba)^{-1})$ with $a,b,c \in A, a+c+cba \in GL_1A$ and either $a-1, c \in Y$ or $a, c-1 \in Y$;*

(c) *when Y is an ideal of A, $C_2(A,Y) = G_2(A,Y)$ and this group consists of all elements of the form $x^{1,2}y^{2,1}z^{1,2}d$ with $x,y,z \in Y$ and a diagonal matrix $d = diag(u,v)$ with $u,v \in GL_1A$ such that $v^{-1}au - a, uav^{-1} - a \in Y$ for all $a \in A$.*

Under the additional condition $2A = A$, Menal and Vaserstein proved the second conclusion of Theorem 4(c) for any quasi-ideal Y of A:

Theorem 5. *Let A be an associative ring with 1 such that $sr(A) \leq 1$ and $2A = A$. Then for any quasi-ideal Y of A, the group $C_2(A,Y)$ consists of all elements of the form $x^{1,2}y^{2,1}z^{1,2}d$ with $x,y,z \in Y$ and a diagonal matrix $d = diag(u,v)$ with $u,v \in GL_1A$ such that $v^{-1}au - a, uav^{-1} - a \in Y$ for all $a \in A$.*

In this case, Menal and Vaserstein obtained the following result.

Theorem 6. *Let A be an associative ring with 1 such that $2A = A$ and $sr(A) \leq 1$. Then a subgroup H of E_2A is normal if and only if $[E_2A, E_2(A,Y)] \subset H \subset C_2(A,Y)$ for some quasi-ideal Y of A.*

This classification result follows from:

Theorem 7. *Let A be an associative ring with 1 such that $2A = A$ and $sr(A) \leq 1$.*

(a) *If H is a subgroup of GL_2A normalized by E_2A, then there is a unique quasi-ideal Y of A such that $[E_2A, E_2(A,Y)] \subset H \subset C_2(A,Y)$;*

(b) *If Y is a quasi-ideal of A, then $[E_2A, E_2Y] = [E_2A, C_2(A,Y) \cap E_2A]$.*

For a stable range 1 ring A and $n \geq 3$, $(GL_nA)_{ab}$, the abelianization of the linear group GL_nA is well-known. Specifically, it is K_1A which is isomorphic to $GL_1A/W(A)$, which is described above. In [**4**], Berrick and Menal computed the abelianization of GL_2A, showing that K_1A is a direct summand of $(GL_2A)_{ab}$ in the following result.

Theorem 8. *If I is an ideal of A with stable range 1, then every element $g \in GL_2I$ can be written in the form $g = diag(u,v)a^{2,1}b^{1,2}c^{2,1}$, where $u,v \in GL_2I$ and $a,b,c \in I$. Moreover, there is a group isomorphism of $(GL_2I)_{ab}$ with $(I/I^2)^3 \oplus I^2/L \oplus GL_1I/W_1(I,I)$, where L is the ideal of I generated by $(x^2-x)I$ and $I(x^2-x)$ for all $x \in I$.*

In [**42**], the problem of developing a "sandwich theorem" for normal subgroups of GL_2A similar to the one for GL_nA with $n \geq 3$ was explored. In this paper, results previously discussed were generalized to rings A which satisfy one of three conditions; either A is a stable range 1 ring, or $A/\text{rad } A$ is von Neumann regular, or for every $a \in A$ there is a finite sequence $x_1, x_2, ..., x_n$ in A such that $x_1 + x_2 + ... + x_n = 1$ and $1 - ax_i \in GL_1A$ for all i. In this case, the following theorem was proved.

Theorem 9. *Suppose A is a ring satisfying one of the above three conditions such that no proper one-sided ideal of A contains all $u-1$, where $u \in GL_1A$, and every element of A is a sum of units. Then for every subgroup H of GL_2A which is normalized by GE_2A there is a unique ideal B of A such that $[E_2A, E_2B] \subset H \subset G_2(A,B)$.*

Here GE_2A is the subgroup of GL_2A generated by all its diagonal and elementary matrices. If A is a stable range 1 ring, then $GE_2A = GL_2A$. When no proper one-sided ideal of A contains all $u-1$, $E_2(A,B) \subset [E_2B, GE_2A]$ for every ideal B of A.

While the converse to this theorem is not always true, the same paper contains the following modification.

Theorem 10. *Suppose A satisfies one of the above three conditions. Then*

$$[G_2(A,B), [E_2A, E_2A]] \subset E_2(A,B)$$

for any ideal B of A. Therefore, if H is a subgroup of GL_2A such that $E_2(A,B) \subset H \subset G_2(A,B)$ for an ideal B of A, then $[H, [E_2A, E_2A]] \subset E_2(A,B)$, hence H is normalized by $[E_2A, E_2A]$.

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