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UNITARY SUBGROUP OF INTEGRAL GROUP RINGS

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Abstract

Let A be a finite abelian group and $G = A \rtimes \langle b \rangle$, $b^2 = 1$, $a^b = a^{-1}$, $\forall a \in A$. We find generators up to finite index of the unitary subgroup of $\mathbb{Z}G$. In fact, the generators are the bicyclic units. For an arbitrary group G, let $B_2(\mathbb{Z}G)$ denote the group generated by the bicyclic units. We classify groups G such that $B_2(\mathbb{Z}G)$ is unitary.

Let $\mathbb{Z}G$ be the integral group ring of an arbitrary group G and let $f : G \to U(\mathbb{Z}) = \{\pm 1\}$ be an orientation homomorphism. For each $x = \sum_{g\in G} \alpha_g g$, we put $x^f = \sum \alpha_g f(g) g^{-1}$. In particular, if f is trivial, x^f coincides with the standard x^*. Let $U(\mathbb{Z}G)$ be the group of units of $\mathbb{Z}G$. Then $u \in U(\mathbb{Z}G)$ is called f-unitary if $u^{-1} = u^f$ or $u^{-1} = -u^f$. All f-unitary elements of $U(\mathbb{Z}G)$ form a subgroup $U_f(\mathbb{Z}G)$ containing $G \times U(\mathbb{Z})$. We refer to $U_f(\mathbb{Z}G)$ as the f-unitary subgroup of $U(\mathbb{Z}G)$. Interest in the group $U_f(\mathbb{Z}G)$ arose in algebraic topology and unitary K-theory [**4**].

We are interested in the constructive description of $U_f(\mathbb{Z}G)$. If G is finite cyclic, then Bovdi [**1**] gave a linearly independent set of generators for a torsion free subgroup of finite index in $U_f(\mathbb{Z}G)$. This was extended to finite abelian groups by Hoechsmann-Sehgal in [**3**]. We give generators up to finite index of $U_f(\mathbb{Z}G)$ if G is a finite dihedral group. In fact, the generators consist of the bicyclic units. The subgroup $B_2(\mathbb{Z}G)$ of $U(\mathbb{Z}G)$ generated by all the bicyclic units of $\mathbb{Z}G$ plays an important role in the study of $U(\mathbb{Z}G)$ (see [**5**], [**6**]). In Theorem 2, we characterize groups G for which $B_2(\mathbb{Z}G)$ is unitary.

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2. $U_f(\mathbb{Z}G)$ for dihedral groups

First, we recall some definitions. For an element $a \in G$ of finite order n write $\overline{a} = 1 + a + \cdots + a^{n-1}$. Denote by $t(G)$ the set of all torsion elements of G. If $a,\ b \in G,\ \ o(a) < \infty$, then

$$u_{a,b} = 1 + (1-a)b\overline{a}$$

has an inverse $u_{a,b}^{-1} = 1 - (1-a)b\overline{a}$. Moreover, $u_{a,b} = 1$ if and only if b normalizes $\langle a \rangle$. The elements $u_{a,b},\ \ a,\ b \in G$ are called *bicyclic units* of $\mathbb{Z}G$ and the group generated by them is denoted by $B_2(\mathbb{Z}G)$. We recall [**5**] that by $B_1(\mathbb{Z}G)$ is understood the group generated by the Bass cyclic units of $\mathbb{Z}G$. It is known [**5**] that if G is a finite dihedral group and Z is the centre of $U(\mathbb{Z}G)$ then $\langle Z, B_2(\mathbb{Z}G)\rangle$ (equivalently, $\langle B_1(\mathbb{Z}G), B_2(\mathbb{Z}G)\rangle$) is of finite index in $U(\mathbb{Z}G)$. We prove

Theorem 1. *Let G be the dihedral group*

$$D_{2n} = \langle a^n = 1 = b^2 \mid a^b = a^{-1}\rangle.$$

Suppose f is an orientation homomorphism of G with kernel $\langle a \rangle$. Then the index $\big(U_f(\mathbb{Z}G) : B_2(\mathbb{Z}G)\big)$ is finite.

We need the

Proposition. *Let G be a group containing a subgroup A of index 2 and an element b such that $G = \langle A, b\rangle$ and $b^{-1}ab = a^{-1}$ for all $a \in A$. Suppose that $A^2 \neq 1$. If f is an orientation homomorphism of G with kernel A, then*

1) *the centre of $U_f(\mathbb{Z}G)$ coincides with $t_2(A) \times \langle -1 \rangle$, where*

$$t_2(A) = \{a \in t(A) : a^2 = 1\};$$

2) *the centre of $U(\mathbb{Z}G)$ is the direct product of $t_2(A) \times \langle -1\rangle$ and a torsion free abelian group T such that $U(\mathbb{Z}A) = \langle -1 \rangle \times A \times T$ and $x = x^*$ for all $x \in T$.*

Proof: Let $x = x_1 + x_2 b,\ \ x_i \in \mathbb{Z}A$ be a central unit in $\mathbb{Z}G$. Since G is a subgroup of $U(\mathbb{Z}G)$,

$$x = b^{-1}xb = x_1^* + x_2^* b \qquad \text{and} \qquad x = a^{-1}xa = x_1 + a^{-2}x_2 b$$

for all $x \in A$. Then $x_i = x_i^*$ and

$$x_2(1-a^2) = 0 \tag{1}$$

for all $a \in A$. We wish to prove that $x_2 = 0$.

Let us suppose that $x_2 \neq 0$. From (1) we obtain that A^2 is finite. Let $\widehat{A^2}$ denote the sum of all elements of A^2. If H is a normal subgroup of G, then denote by $\Delta(G, H)$ the ideal of $\mathbb{Z}G$ generated by elements of the form $h - 1$ with $h \in H$. Clearly,

$$\mathbb{Z}G/\Delta(G, H) \cong \mathbb{Z}(G/H).$$

If $\chi(y)$ is the sum of the coefficients of y, then the element

$$x + \Delta(G, A) = \chi(x_1) + \chi(x_2)b + \Delta(G, A)$$

is trivial, because $|G/A| = 2$ [**7**, p. 46]. This implies that one of the numbers $\chi(x_1)$ or $\chi(x_2)$ equals ± 1 and the other is zero. From (1) we obtain $x_2 = z\widehat{A^2}$, $z \in \mathbb{Z}A$, $\chi(x_2) = \chi(z)|A^2|$, and this is possible only in the case when $\chi(x_2) = 0$.

Suppose $A = A^2$. Then $x_2 = \gamma \sum_{a \in A} a$ for some $\gamma \in \mathbb{Z}$. From the equality $\chi(x_2) = \gamma|A| = 0$ we obtain $\gamma = 0$ and $x_2 = 0$, which leads to a contradiction. Thus $A \neq A^2$. Write $x_2 = \left(\sum_i \alpha_i c_i\right)\widehat{A^2}$ with $\alpha_i \in \mathbb{Z}$ where c_i's are a transversal of A^2 in A. Then

$$\begin{aligned} x_1 + x_2 b + \Delta(G, A^2) &= x_1 + \left(\sum_i \alpha_i c_i\right)\widehat{A^2} b + \Delta(G, A^2) \\ &= x_1 + \left(|A^2| \sum_i \alpha_i c_i\right) b + \Delta(G, A^2) \end{aligned}$$

is a unit in $\mathbb{Z}(G/A^2)$. Since G/A^2 is an abelian group of exponent two, by Higman's theorem [**7**, p. 57], all units of $\mathbb{Z}(G/A^2)$ are trivial. Obviously, $\sum_i \alpha_i = 0$ and if $\alpha_i \neq 0$ for some i, then $\alpha_i|A^2| \neq \pm 1$. Thus, $\alpha_i = 0$ for all i and the equality $x_2 = 0$ is contradictory. Hence, $x = x_1 \in U(\mathbb{Z}A)$ and $x^* = x = x_1^* = x_1$. Clearly, if $x \in U(\mathbb{Z}A)$ and $x^* = x$, then x is a central unit of $\mathbb{Z}G$.

It is well known (see [**2**]) that $U(\mathbb{Z}t(A)) = \pm t(A) \times T$ and $U(\mathbb{Z}A) = \pm A \times T$, where every element $u \in T$ satisfies the condition $u = u^*$. Therefore the centre of $U(\mathbb{Z}G)$ is the direct product of subgroups $\pm t_2(A)$ and T. This is 2) of the Proposition.

Suppose that $x = x_1 + x_2 b$ is a central unit in $U_f(\mathbb{Z}G)$. Since G is a subgroup of $U_f(\mathbb{Z}G)$, x is central in $U(\mathbb{Z}G)$. It follows that $x = x_1$ and $xx^f = x_1 x_1^* = x_1^2 = \pm 1$. Therefore, by Higman's theorem $x_1 = \pm a$ where $a \in t_2(A)$. This completes the proof of the Proposition. ∎

Proof of Theorem 1: Let G be the dihedral of order $2n$ given by $G = \langle a^n = 1 = b^2, a^b = a^{-1}\rangle$. If $n = 2$, then the theorem is trivial. So we may apply the last Proposition. Let Z be the centre of $U(\mathbb{Z}G)$. Then we know that $\big(U(\mathbb{Z}G) : \langle B_2(\mathbb{Z}G), Z\rangle\big) < \infty$. We have seen in the Proposition above that Z_1, the centre of $U_f(\mathbb{Z}G)$, is finite and $Z_1 < Z$. It suffices to prove, therefore, that $B_2(\mathbb{Z}G)$ is unitary. If $u_{x,y} \neq 0$, then $o(x) = 2$ and

$$u_{x,y} = 1 + (1-x)y(1+x).$$

Now, $y = a^i x^\varepsilon$, $\varepsilon = 0$ or 1. Since $x(1+x) = 1+x$ we have, in any case,

$$u_{x,y} = 1 + (1-x)\, a^i\, (1+x).$$

Then $u^f_{x,y} = 1 + (1+x)^f\, (a^i)^f\, (1-x)^f = 1 + (1-x)\, a^{-i}\, (1+x)$. Therefore, $u_{x,y}\, u^f_{x,y} = 1 + (1-x)(a^i + a^{-i})(1+x) = 1$ as $(a^i + a^{-i})$ is central. This completes the proof of the theorem. ∎

Remark. The last theorem holds for nonabelian groups $G = \langle A, b\rangle$ where A is finite abelian and $b^2 = 1$, $a^b = a^{-1}$ for all $a \in A$. If A is an elementary 2-group, then so is G and there is nothing to prove. Suppose $A^2 \neq 1$. The nonlinear irreducible representations ρ of G are induced from those of A and $\rho(\mathbb{Z}G) = \rho(D)$ for some dihedral subgroup D of A. The result follows.

3. Unitarity of the subgroup $B_2(\mathbb{Z}G)$

Theorem 2. *Let $G = \langle A, b\rangle$ where A is the kernel of the nontrivial orientation homomorphism $f : G \to U(\mathbb{Z})$. The subgroup $B_2(\mathbb{Z}G)$ is nontrivial and f-unitary if and only if G is non-Hamiltonian in which an element $b \neq 1$ of finite order can be chosen such that one of the following conditions is fulfilled:*

1) *A is an abelian group, the order of the element b divides 4 and $bab^{-1} = a^{-1}$ for all $a \in A$;*
2) *A is a Hamiltonian 2-group, G is the semidirect product of A and $\langle b \mid b^2 = 1\rangle$, and every subgroup of A is normal in G;*
3) *A is a Hamiltonian 2-group and G is the direct product of a Hamiltonian 2-subgroup of A and a cyclic group $\langle b\rangle$ of order 4;*
4) *$t(A)$ is an abelian group, every subgroup of $t(A)$ is normal in G and $bab^{-1} = a^{-1}b^{4i}$ for all $a \in A$, where the integer i depends on a.*

We need the following.

Lemma. *Suppose that G has a subgroup A of index 2 with $G = \langle A, b\rangle$ and $o(b) < \infty$. Suppose further that $A \neq N_A(\langle b\rangle)$ and*

1) *$t(A)$ is abelian and all subgroups of $t(A)$ are normal in A;*
2) *$bgb^{-1} = g^{-1}$ for all $g \in A \setminus N_A(\langle b\rangle)$.*

Then $bab^{-1} = a^{-1}$ for all $a \in A$ and $b^4 = 1$.

Proof: Let $c \in N_A(\langle b\rangle)$. Choose $a \in A \setminus N_A(\langle b\rangle)$. At first, suppose c has finite order. Then by (2) we have

$$a^{-1}bcb^{-1} = b(ac)b^{-1} = c^{-1}a^{-1}.$$

If $a \in t(A)$, then by (1) we have $bcb^{-1} = c^{-1}$. If a has infinite order, there exists an integer n such that $a^n c = ca^n$, since $\langle c\rangle$ is normal in A. By hypothesis, $a^n c \notin N(\langle b\rangle)$ and thus

$$a^{-n}bcb^{-1} = b(a^n c)b^{-1} = c^{-1}a^{-n}.$$

It follows that $bcb^{-1} = c^{-1}$ as desired. Now it is enough to prove that c cannot have infinite order. Suppose that $o(c) = \infty$ and $o(a) < \infty$. Then there is an n such that $c^n a = ac^n$. Clearly, $ac^n \notin N(\langle b\rangle)$. We have

$$a^{-1}bc^n b^{-1} = b(ac^n)b^{-1} = c^{-n}a^{-1}.$$

It follows that $bc^n b^{-1} = c^{-n}$. This is impossible because $c^n \in N(\langle b\rangle)$. Now let $o(c) = \infty$, $o(a) = \infty$. There exists an n such that $bc^n = c^n b$ and $a^{-1}c^n = bac^n b^{-1} = c^{-n}a^{-1}$. It follows that $[c^n, a^2] = 1$. Clearly, $a^2 c^n \notin N(\langle b\rangle)$ and we get

$$a^{-2}c^n = ba^2 c^n b^{-1} = a^{-2}c^{-n}$$

which implies $c^{2n} = 1$, a contradiction. Since $b^2 \in A$, $bb^2b^{-1} = b^{-2}$ and we have $b^4 = 1$, completing the proof of the lemma. ■

Proof of Theorem 2:

"*Necessity.*"

Suppose that $B_2(\mathbb{Z}G)$ is nontrivial and f-unitary. Let us first prove that every finite subgroup $\langle a\rangle$ of A is normal in G. Let n be the order of $\langle a\rangle$. If $g \notin N_G(\langle a\rangle)$, then $u_{a,g} = 1 + (1-a)g\overline{a} \neq 1$. Then from the equality $u_{a,g}^{-1} = u_{a,g}^{f}$ we have

$$\overline{a}g^{-1}f(g)(1-a^{-1}) = -(1-a)g\overline{a}.$$

Multiplying by $\overline{a}$ we obtain $n(1-a)g\overline{a} = 0$, which is impossible. Therefore, every subgroup of $t(A)$ is normal in G. Because $B_2(\mathbb{Z}G) \neq 1$, $G \setminus A$

contains an element c of finite order with $\langle c\rangle$ not normalized by A. Then $c^2 \in t(A)$ and $\overline{c^2}$ is central in $\mathbb{Z}G$. Clearly,

$$u_{c,g} = 1 + (1-c)g(1+c)\overline{c^2}$$

and $f(c) = -1$. Since $u_{c,g}$ is f-unitary, $u_{c,g}u^f_{c,g} = 1$ and it follows that

$$(g + g^{-1}f(g))(1+c)\overline{c^2} = c(g+g^{-1}f(g))(1+c)\overline{c^2}. \tag{1}$$

Choose $b \in G \setminus A$ such that b is a 2-element of least order and let $g \in A$. In (1) taking $c = b$, $g = bg^{-1}b^{1+2i}$ whenever $g \notin N_A(\langle b\rangle)$. We obtain $bgb^{-1} = g^{-1}b^{2i'}$ for all $g \in A \setminus N_A(\langle b\rangle)$ and

$$(bg)^2 = (g^{-1}b^2g)^{i'+1}.$$

Clearly, bg is a 2-element in $G \setminus A$ and i' is even, otherwise the order of bg is less than the order of b, which is impossible. Therefore,

$$bgb^{-1} = g^{-1}b^{4j} \tag{2}$$

for all $g \in A \setminus N_A(\langle b\rangle)$.

a) *Suppose that the order of* b *divides* 4.

Then from (2) $bgb^{-1} = g^{-1}$ for all $g \in A \setminus N_A(\langle b\rangle)$.

If $t(A)$ is abelian, then, by the Lemma, A is abelian and $bab^{-1} = a^{-1}$ for all $a \in A$. This is case 1) of the theorem.

If $t(A)$ is nonabelian, then $t(A)$ is a Hamiltonian group and

$$t(A) = Q \times E \times T$$

where Q is the quaternion group of order 8, $E^2 = 1$ and all elements of T are of odd order.

We wish to prove that $A = t(A)$. Suppose that g is an element of infinite order of $A\setminus N(\langle b\rangle)$. Then $g^2 \in C_A(Q)$ and there exists an element w of order 4 of Q such that $[b,w] = 1$, because every subgroup of Q is normal in G. Clearly, $g^2w \notin N(\langle b\rangle)$ and by (2)

$$wg^{-2} = bwg^2b^{-1} = bg^2wb^{-1} = w^{-1}g^{-2},$$

which is impossible. Therefore, all elements of $A \setminus N(\langle b\rangle)$ have finite orders.

Let g be an element of infinite order from $N_A(\langle b\rangle)$ and let an $a \in A\setminus N_A(\langle b\rangle)$. Clearly there exists n such that $[g^n, a] = 1$, because the finite cyclic subgroup $\langle a\rangle$ is normal in G. Then $g^n a \in A \setminus N_A(\langle b\rangle)$ and $g^n a$ is of infinite order, which leads to a contradiction. Therefore, $t(A) = A$.

We claim that $T = 1$. Let v be an element of odd order from $A\setminus N(\langle b\rangle)$. Obviously, there exists an element w of order 4 in Q such that $[b, w] = 1$, as every subgroup of Q is normal in G. Thus $vw \notin N(\langle b\rangle)$ and by (2)

$$v^{-1}w = bvwb^{-1} = w^{-1}v^{-1},$$

which is impossible. Next, let v be an element of odd order from $N_A(\langle b\rangle)$. Because $\langle v\rangle \lhd G$, $[v, b] = 1$. Clearly there is an element w of order 4 in Q such that $b^{-1}wb = w^{-1}$ and $vw \notin N_A(\langle b\rangle)$. Then

$$w^{-1}v = bwvb^{-1} = v^{-1}w^{-1},$$

which is impossible. Hence, the structure of G is described in case 2) or 3) of the theorem.

b) *Suppose that the order of b is 2^k $(k \geq 3)$.*

Then by (2) b^2 belongs to the centre of $t(A)$, because $t(A)$ is abelian or Hamiltonian. Hence, $t(A)$ is abelian and every subgroup of $t(A)$ is normal in G. Then from (2) $bab^{-1} = a^{-1}b^{4j}$ for all $a \in A \setminus N_A(\langle b\rangle)$. Denote by $\langle b^{4r}\rangle$ the subgroup generated by $b^{4j} = abab^{-1}$, as a runs over $A \setminus N_A(\langle b\rangle)$.

Put $\widetilde{G} = G/\langle b^{4r}\rangle$, $\widetilde{A} = A/\langle b^{4r}\rangle$ and $\widetilde{b} = b\langle b^{4r}\rangle$. Then $\widetilde{G}$ satisfies the conditions of our Lemma and it follows that $r = 1$ and $\widetilde{b}a\widetilde{b}^{-1} = a^{-1}$ for all $a \in \widetilde{A}$. This is case 4) of the theorem.

"*Sufficiency.*"

Let G satisfy one of the conditions 1)–4) of the theorem. Clearly, if a finite subgroup $\langle c\rangle$ is not normal in G, then $c \in bA$, $\langle c^2\rangle = \langle b^2\rangle$ and $\overline{c^2}$ belongs to the centre of $\mathbb{Z}G$. Therefore,

$$u_{c,g} = 1 + (1-c)g(1+c)\overline{c^2}$$

and

$$u_{c,g}\, u^f_{c,g} = 1 + (1-c)(g + g^{-1}f(g))(1+c)\overline{c^2}.$$

Suppose that $g \in A$. Then $f(g) = 1$ and $(g + g^{-1})\overline{c^2}$ is a central element. This is obvious in cases 1), 2) and 3). Suppose that G satisfies the condition 4) of the theorem. Then $\langle c^2\rangle = \langle b^2\rangle$, and $bgb^{-1} = g^{-1}b^{4i}$ and $G/\langle b^4\rangle$ is abelian. Thus

$$b(g + g^{-1})\overline{c^2}b^{-1} = (g + g^{-1})\overline{c^2} = a^{-1}(g + g^{-1})\overline{c^2}a$$

and $(g + g^{-1})\overline{c^2}$ is central in $\mathbb{Z}G$.

If $g \in bA$, then $g = ba$, $f(g) = -1$ and

$$g^{-1} = a^{-1}b^{-1} = b^{-1}ab^{4i}.$$

Clearly, $g^{-1}\overline{c^2} = g\overline{c^2}$ and $(g + f(g)g^{-1})\overline{c^2} = 0$. Therefore, $u_{c,g}\, u_{c,g}^{f} = 1$ and the bicyclic units are f-unitary. Thus $B_2(\mathbb{Z}G)$ is an f-unitary subgroup, proving the theorem ■

References

1. A.A. BOVDI, Unitary subgroup of the multiplicative group of integral group ring of a cyclic group, *Math. Zametki* **41** (4) (1987), 467–474.
2. A.A. BOVDI, The multiplicative group of an integral group ring, Uzhgorod, 1987.
3. K. HOECHSMANN AND S.K. SEHGAL, On a theorem of Bovdi, to appear.
4. S.P. NOVIKOV, Algebraic construction and properties of Hermitian analogues of K-theory over rings with involution from the viewpoint of Hamiltonian formalism, Applications to differential topology and the theory of characteristic classes, II, *Izv. Akad. Nauk SSSR Ser. Mat.* **34** (1970), 475–500; *English transl. in Math. USSR Izv.* **4** (1970).
5. J. RITTER AND S.K. SEHGAL, Generators of subgroups of $U(\mathbb{Z}G)^*$, *Contemporary Math.* **93** (1989), 331–347.
6. J. RITTER AND S.K. SEHGAL, Construction of units in integral group rings, *Trans. A.M.S.* **324** (1991), 603–621.
7. S.K. SEHGAL, "*Topics in group rings*," M. Dekker, New York, 1978.

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