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THE FORMAL COMPLETION OF THE NÉRON MODEL OF $J_0(p)$

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Abstract

For any prime number $p > 3$ we compute the formal completion of the Néron model of $J_0(p)$ in terms of the action of the Hecke algebra on the $\mathbb{Z}$-module of all cusp forms (of weight 2 with respect to $\Gamma_0(p)$) with integral Fourier developement at infinity.

Let p be a prime number greater than three. Let $\mathcal{J}_{|\mathbb{Z}}$ be the Néron model of the jacobian $J_0(p)_{|\mathbb{Q}}$ of the modular curve $X_0(p)_{|\mathbb{Q}}$. In a joint work with Deninger we proved that the formal completion of $\mathcal{J}$ along the zero section is determined by the relative L-series of $J_0(p)$ with respect to $\mathbb{T} \otimes \mathbb{Q}$, where $\mathbb{T}$ is the Hecke algebra [**2**]. In fact, we explained how to construct a formal group law for $\mathcal{J}^\wedge$ from a formal Dirichlet series made up with the integral matrices reflecting the action of the Hecke operators on the Lie algebra of $\mathcal{J}$.

In this note we show that such a formal group law can also be constructed with the integral matrices reflecting the action of $\mathbb{T}$ on the $\mathbb{Z}$-module $S_2(\Gamma_0(p), \mathbb{Z})$ of all cusp forms (of weight 2, with respect to $\Gamma_0(p)$) with integral Fourier developement at infinity. We obtain in this way an effective result since, with the aid of a computer, it is possible to find explicit $\mathbb{Z}$-basis of $S_2(\Gamma_0(p), \mathbb{Z})$ and to compute the action of the Hecke algebra.

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Let g be the dimension of $J_0(p)$. Our aim is to prove the following theorem:

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Theorem. *Let $U_p \in M_g(\mathbb{Z})$ and $T_l \in M_g(\mathbb{Z})$, for all primes $l \neq p$, be the matrices of the Atkin-Lehner operator and the Hecke operators, with respect to any $\mathbb{Z}$-basis of $S_2(\Gamma_0(p), \mathbb{Z})$. Since these matrices commute, the formal Dirichlet series:*

$$\sum_{n=1}^{\infty} A_n \cdot n^{-s} = (I_g - U_p \cdot p^{-s})^{-1} \cdot \prod_l (I_g - T_l \cdot p^{-s} + I_g \cdot p^{1-2s})^{-1},$$

is well-defined and $A_n \in M_g(\mathbb{Z})$ for all n. Let $L(X,Y)$ be the g-dimensional formal group law with logarithm:

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n} A_n X^n \in \mathbb{Q}[X_1, \ldots, X_g]^g,$$

where X^n is the notation for $(X_1^n, \ldots, X_g^n)^t$. Then, $L(X,Y)$ is defined over $\mathbb{Z}$ and it is isomorphic to the formal completion of $\mathcal{J}$ along the zero section.

Honda [4] proved an analogous result for Shimura curves, but a finite (fairly big) set of primes had to be left aside. In fact, our proof follows the same pattern, but we have at our disposal deep results of Deligne-Rapoport [**1**], Deligne [**5**, thm. A.1] (which was implicitely used in [**2**]) and Mazur [**5**, II, sections 3 and 6], which allow us to deal with the bad primes.

After [**2**], in order to prove the theorem it is sufficient to show that Lie $(\mathcal{J})$ and $S_2(\Gamma_0(p), \mathbb{Z})$ are isomorphic as $\mathbb{T}$-modules. To this aim is devoted the rest of the paper. The proof consists on adding some details (checking of some compatibilities, essentially) to certain results of Mazur.

For any integer $N \geq 5$, let $M_0(N)$ be the curve over $\mathbb{Z}$ representing the fine moduli stack classifying generalized elliptic curves over $\mathbb{Z}[1/N]$ with a cyclic subgroup of order N. Let $X_0(N) \xrightarrow{i} M_0(N)$ be its minimal regular resolution. These two curves become isomorphic over $\mathbb{Z}[1/N]$.

The Atkin involution w=w$_N$ extends to an involution of $M_0(N)$ [**1**, IV, Prop. 3.19] and by minimality, to an involution of $X_0(N)$ commuting with i. Hence, w acts on $H^1(X_0(N), \mathcal{O})$ and on $H^1(M_0(N), \mathcal{O})$ in a compatible way. That is, we have a commutative diagram:

$$(1) \qquad \begin{array}{ccc} H^1(X_0(N), \mathcal{O}) & \xrightarrow{w^*} & H^1(X_0(N), \mathcal{O}) \\ {\scriptstyle i^*}\uparrow & & \uparrow{\scriptstyle i^*} \\ H^1(M_0(N), \mathcal{O}) & \xrightarrow{w^*} & H^1(M_0(N), \mathcal{O}). \end{array}$$

Now, let l be a prime different from p and consider the finite morphism [**5**, II, section 6]:

$$c : M_0(pl) \longrightarrow M_0(p), \; (E, (H_l, H_p)) \longrightarrow (E, H_p).$$

Here (H_l, H_p) denotes a cyclic subgroup of E of order pl (canonically) decomposed as a product of its p-primary and l-primary parts. By minimality c raises to a finite morphism between the regular resolutions fitting into a commutative diagram:

$$\begin{array}{ccc} X_0(pl) & \xrightarrow{c} & X_0(p) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\ M_0(pl) & \xrightarrow{c} & M_0(p). \end{array} \tag{2}$$

Let us denote $X = X_0(p)$, $X' = X_0(pl)$, $M = M_0(p)$, $M' = M_0(pl)$. The morphism $c : X' \longrightarrow X$ induces covariant and contravariant homomorphisms:

$$\mathrm{Pic}^0_{X'/\mathbb{Z}} \underset{c_*}{\overset{c^*}{\leftrightarrows}} \mathrm{Pic}^0_{X/\mathbb{Z}}.$$

At the level of invertible sheafs, c^* is the usual homomorphism and c_* is the norm-homomorphism defined by Grothendieck [**3**, 6.5]. Via the canonical identification of $H^1(X, \mathcal{O})$ with the tangent space of $\mathrm{Pic}^0_{X/\mathbb{Z}}$ at the zero-section, c_* and c^* induce homomorphisms:

$$H^1(X', \mathcal{O}) \underset{c_*}{\overset{c^*}{\leftrightarrows}} H^1(X, \mathcal{O}).$$

c^* is the natural homomorphism induced by $\mathcal{O}_X \longrightarrow c_*\mathcal{O}_{X'}$ and the homomorphism $H^1(X, c_*\mathcal{O}_{X'}) \longrightarrow H^1(X', \mathcal{O}_{X'})$ given by the Leray spectral sequence; whereas c_* is the trace-homomorphism defined in terms of Čech cocycles by:

$$c_*(f_{\alpha\beta}) = Tr_{X'|X}(f_{\alpha\beta}),$$

for any affine open covering: $X' = \cup_\alpha c^{-1}(U_\alpha)$, for U_α an affine open covering of X. This trace is well-defined since $\Gamma(c^{-1}U_\alpha, \mathcal{O}_{X'})$ is a finite $\Gamma(U_\alpha, \mathcal{O}_X)$-module.

In fact, the identification of $H^1(X, \mathcal{O})$ with the tangent space of $\mathrm{Pic}^0_{X/\mathbb{Z}}$ can be realized through the exact sequence:

$$0 \longrightarrow H^1(X, \mathcal{O}) \xrightarrow{\exp} H^1(X \otimes \mathbb{Z}[\varepsilon], \mathcal{O}^*) \longrightarrow H^1(X, \mathcal{O}^*),$$

where $\mathbb{Z}[\varepsilon]$ is the ring of dual numbers and $\exp(f) = 1 + f\varepsilon$. The above description of the action of c^* and c_* can be easily deduced from this sequence, working with Čech cocycles and having in mind that $1 + Tr_{X'/X}(f)\,\varepsilon$ is the norm of $1 + f\varepsilon$.

By Grothendieck duality we obtain homomorphisms:

$$H^0(X',\Omega_{X'}) \overset{c^*}{\underset{c_*}{\leftrightarrows}} H^0(X,\Omega_X),$$

where Ω_X is the dualizing sheaf, that is, the sheaf of regular differentials, which is defined as the only non-vanishing homology group (in degree -1) of the complex $R\pi^!\mathcal{O}_{\mathrm{Spec}\,\mathbb{Z}}$, where π is the structural morphism of X.

We need to check the compatibility of these homomorphisms c_*, c^* with the analogous homomorphisms defined by Mazur at the level of the curves $M_0(N)$ [**5**, page 88], which we denote by $(c^*)_M$, $(c_*)_M$. More precisely, we need the following diagrams to commute:

$$\begin{array}{ccc} H^1(X',\mathcal{O}) & \xleftarrow{\ c^*\ } & H^1(X,\mathcal{O}) \\ {\scriptstyle i^*}\uparrow & & \uparrow{\scriptstyle i^*} \\ H^1(M',\mathcal{O}) & \xleftarrow{(c^*)_M} & H^1(M,\mathcal{O}) \end{array} \tag{3}$$

$$\begin{array}{ccc} H^0(X',\Omega) & \xleftarrow{\ c^*\ } & H^0(X,\Omega) \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle i_*} \\ H^0(M',\Omega) & \xleftarrow{(c^*)_M} & H^0(M,\Omega), \end{array} \tag{4}$$

where i_* is defined from i^* by duality. Now, diagram (3) commutes since it is obtained from (2) by taking everywhere the natural homomorphisms induced by i and c. Since the $\mathbb{Z}$-modules involved are free [**5**, II, Lemma 3.3 and (3.2)] it is sufficient to check the commutativity of diagram (4) after tensoring with $\mathbb{Q}$. Then, the commutativity amounts to the fact that the natural homomorphism: $H^0(X'_{\mathbb{Q}},\Omega^1) \xleftarrow{c^*} H^0(X_{\mathbb{Q}},\Omega^1)$ is dual to the trace-homomorphism, $Tr_{X'/X} : H^1(X'_{\mathbb{Q}},\mathcal{O}) \longrightarrow H^1(X_{\mathbb{Q}},\mathcal{O})$, under Serre duality, and this is a consequence of the classical trace-formula [**7**, page 32].

We are ready to analize the action of the Hecke algebra. The Hecke algebra $\mathbb{T}$ is the subalgebra of $\mathrm{End}_{\mathbb{Q}}(J_0(p))$ generated by all the operators T_l and the Atkin involution w. The Hecke operator T_l is, by definition, the endomorphism of $J_0(p)$ induced by correspondence on $X_0(p)_{|\mathbb{Q}}$ determined by the morphisms:

$$\begin{array}{ccc} X'_{\mathbb{Q}} & \xrightarrow{\ c_{\mathbb{Q}}\ } & X_{\mathbb{Q}} \\ {\scriptstyle (c\mathrm{w}_l)_{\mathbb{Q}}}\downarrow & & \\ X_{\mathbb{Q}} & & \end{array}$$

To be more precise, T_l is the composition of the two homomorphisms:

$$T_l : J_0(p) \xrightarrow{(cw_l)^*_{\mathbb{Q}}} J_0(pl) \xrightarrow{(c_{\mathbb{Q}})_*} J_0(p),$$

induced by $c_{\mathbb{Q}}$ and $(cw_l)_{\mathbb{Q}}$ on $\mathrm{Pic}^0_{X_0(N)/\mathbb{Q}} = J_0(N)$, for $N = p, pl$. By the universal property of the Néron model, T_l operates on $\mathcal{J}$ and on its connected component as:

$$T_l : \mathcal{J}^0 \xrightarrow{(cw_l)^*_{\mathbb{Z}}} (\mathcal{J}')^0 \xrightarrow{(c_*)_{\mathbb{Z}}} \mathcal{J}^0,$$

where $(\mathcal{J}')^0$ is the connected component of the Néron model of $J_0(pl)$. By a theorem of Raynaud [**6**, 8.1.4], the connected component of the Néron model of $J_0(N)$ represents the functor $\mathrm{Pic}^0_{X_0(N)/\mathbb{Z}}$. Hence the homomorphisms:

$$\mathrm{Pic}^0_{X'/\mathbb{Z}} \overset{(cw_l)^*}{\underset{c_*}{\leftrightarrow}} \mathrm{Pic}^0_{X/\mathbb{Z}},$$

induced by the finite morphisms $X' \xrightarrow{c, cw_l} X$, coincide with $(cw_l)^*_{\mathbb{Z}}$, $(c_*)_{\mathbb{Z}}$, since they induce the same homomorphism on the generic fiber. Thus, T_l acts on $H^1(X, \mathcal{O})$ and (by duality) on $H^0(X, \Omega)$. We have a commutative diagram:

$$\begin{array}{ccccc} T_l : H^1(X,\mathcal{O}) & \xrightarrow{(cw_l)^*} & H^1(X',\mathcal{O}) & \xrightarrow{c_*} & H^1(X,\mathcal{O}) \\ {\scriptstyle i^*}\uparrow & & {\scriptstyle i^*}\uparrow & & \uparrow{\scriptstyle i^*} \\ H^1(M,\mathcal{O}) & \xrightarrow{(cw_l)^*_M} & H^1(M',\mathcal{O}) & \xrightarrow{(c_*)_M} & H^1(M,\mathcal{O}). \end{array} \tag{5}$$

The left-hand square is diagram (3) for cw_l and the right-hand square is the dual of diagram (4). Mazur shows that $i^* : H^1(M, \mathcal{O}) \longrightarrow H^1(X, \mathcal{O})$ is an isomorphism [**5**, II, Prop. 3.4]; hence, through this isomorphism we obtain (by (1) and (5)) the same structure of $\mathbb{T}$-module on $H^1(M, \mathcal{O})$ as the one taken by definition by Mazur. That is, we have isomorphisms as $\mathbb{T}$-modules:

$$H^1(X, \mathcal{O}) \cong H^1(M, \mathcal{O}), \qquad H^0(X, \Omega) \cong H^0(M, \Omega).$$

Therefore we have $\mathbb{T}$-isomorphisms:

$$\mathrm{Lie}(J^0) \cong T_0(J^0)^\wedge \cong H^1(X, \mathcal{O})^\wedge \cong H^0(X, \Omega) \cong H^0(M, \Omega),$$

and this last group is isomorphic to $S_2(\Gamma_0(p), \mathbb{Z})$ as a $\mathbb{T}$-module, as shown by Mazur [**5**, II, (4.6) and (6.2)].

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