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UNIQUE CONTINUATION FOR SCHRODINGER OPERATORS WITH POTENTIAL IN MORREY SPACES

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0. Introduction

Let us consider in a domain Ω of $\mathbf{R}^n$ solutions of the differential inequality

$$|\Delta u(x)| \leq V(x)|u(x)|, \ x \in \Omega, \tag{1}$$

where V is a non smooth, positive potential.

We are interested in global unique continuation properties. That means that u must be identically zero on Ω if it vanishes on an open subset of Ω.

There is an extensive literature on the matter, mainly to relax the local integrability condition required to the potential V. When L^p_{loc} classes are considered, $p \geq n/2$ is a necessary and sufficient condition for the strong unique continuation property [**JK**] (see [**K**] for references). In this paper we shall consider some spaces introduced by Morrey [**M**], which have been recently used by C. Fefferman and D.H. Phong [**FP**] in studing the eigenvalues of Schrodinger operators; these spaces contain $L^{n/2}_{loc}$.

We say that $V \in F^p, L_{p,\lambda}$ with $\lambda = 2p - n$ in classical notation [**P**], if

$$\|V\|_{F_p} = \sup_Q |Q|^{2/n-p}(\int_Q |V|^p)^{1/p} < \infty$$

where the *sup* is taken over all cubes in $\mathbf{R}^n$ and $|Q|$ = Volume of Q. Notice $F^p \subset F^q$ if $p \geq q$.

In this paper we prove that any solution of (1) has the global unique continuation property if $V \in F^p_{loc}$ and $p > (n-2)/2$.

Very recently T. Wolf has obtained the same result with a different approach.

We would like to thank C. Kenig for telling us about T. Wolf's result.

This improves the previously known results where $p > \frac{(n-1)}{2}$ (see [**CS**] and [**ChR**]).

The point to obtain this improvement is that in the above works the Carleman estimate is seen as a consequence of a uniform Sobolev inequality (see [**KRS**]).

$$\|u\|_{L^2(V)} \leq C\|V\|_{F^p}\|(\Delta + a_j\partial/\partial x_j + b)u\|_{L^2(V^{-1})}, \tag{2}$$

where C is independent of the linear perturbation of the Laplacian. Nevertheless, we prove directly the Carleman estimate

(3) $$\|e^{\tau x_n} u\|_{L^2(V)} \leq C\|V\|_{F^p}\|e^{\tau x_n}\Delta u\|_{L^2(V^{-1})},$$

where C is independent of τ for τ in (τ_0, ∞).

As we shall see while (2) is based on the restriction theorem for the Fourier Transform on the $(n-1)-$dimensional sphere, together with classical theory of weights, our proof follows from a detailed analysis of the multiplier associated to (3) which just involves the restriction theorem in dimension n-2. Therefore the assumption in p comes from the restriction operator in the sphere. We think that this is just a technical obstruction and the restriction theorem should be true for $p \geq 1$. Notice that we are close in the case $n = 4$. We also remark that F^1_{loc} contains the so called Kato-Stummel class which B. Simon has conjectured is enough to assure unique continuation (see [**S**]).

In the sequel we denote by $H^2_{loc}(\Omega)$ the classical Sobolev space, and

$$Av_Q f = (1/|Q|)\int_Q f.$$

We define the local Morrey class as the functions W such that

$$|||W||| = \sup_{y\in\Omega}\limsup_{r\to 0}\|\chi_{B(y,r)}(.)W(.)\|_{F^p} < \infty.$$

The main theorem is:

Theorem 1. *Let $u \in H^2_{loc}(\Omega), n \geq 3$, be a solution of (1), then there exists an $\varepsilon > 0$, only depending on p and n, such that if $V \in F^p_{loc}, \|V\|_{F^p} < \varepsilon, p > (n-2)/2$. and u vanishes in an open subdomain of Ω, then u must be zero everywhere in Ω.*

The proof is related to a restriction theorem for the Fourier Transform, obtained in [**CS**] and [**ChR**], for which we are going to give an easy proof. Let us define, for this purpose, the Morrey classes; we say that V is in $F^{\alpha,p}$ if

$$|||V|||_{\alpha,p} = \sup_{r,x} r^\alpha (Av_{B(x,r)} V^p)^{1/p} < \infty,$$

where the sup is taken on all the balls contained in Ω. This notation corresponds to $\mathcal{E}^{-\alpha,p}$ in [**P**], $1 \leq \alpha \leq n/p$. Also $F^{2,p} = F^p$.

Theorem 2. *Let $d\sigma$ be the uniform measure on the unit sphere S^{n-1} in $\mathbf{R}^n$, and $(d\sigma)\wedge$ its Fourier transform, let $V \in F^{\alpha,p}, p > (n-1)/2(\alpha-1)$, and consider the operator*

$$Tf(x) = (d\sigma)^\wedge * f(x).$$

Then there exists a constant C such that

$$\|Tf\|_{L^2(V)} \leq C|||V|||_{\alpha,p}\|f\|_{L^2(V^{-1})}$$

for any f in C_0^∞.

It would be interesting to understand how this theorem is related to the one in [**V**] for mixed norm introduced by Rubio de Francia in the study of Bochner-Riesz operators [**R**].

1. The Carleman estimate

It is standard to obtain Theorem 1 as a consequence of the following Carleman estimate. This reduction can be seen in the case of L^2 weighted estimates in [**CS**] or [**ChR**].

Theorem (1.1). *There exists a constant $C > 0$ such that for V in F^p, $p > (n-2)/2$, the inequality*

$$\|e^{\tau x_n}u\|_{L^2(V)} \leq C\|V\|_{F^p}\|e^{\tau x_n}\Delta u\|_{L^2(V^{-1})}, \tag{1.1}$$

holds for every u in C_0^∞ and τ in $\mathbf{R}$.

Proof: We can reduce to the case $\tau = 1$ in the following way:
Take $f(x) = e^{\tau x_n}u(x)$, then (1.1) reduces to

$$\|f\|_{L^2(V)} \leq C\|V\|_{F^p}\|P_\tau(D)f\|_{L^2(V^{-1})}, \tag{1.2}$$

where $P_\tau(D)$ has symbol $P_\tau(\xi) = |\xi|^2 - \tau^2 + i\tau\xi_n$.
The change of variable $f(\tau^{-1}x) = g(x)$ reduces (1.2) to

$$\|g\|_{L^2(V_\tau)} \leq C\|V_\tau\|_{F^p}\|P_1(D)g\|_{L^2(V_\tau^{-1})},$$

where $V_\tau(x) = V\left(\frac{x}{\tau}\right)$, since $\|V_\tau\|_{F^p} \leq \tau^2\|V\|_{F^p}$.
Consider the inverse operator given by the Fourier multiplier

$$(Tg)^\wedge(\xi) = \frac{1}{P_1(\xi)}g^\wedge(\xi).$$

Our theorem reduces to prove that $T : L^2(V^{-1}) \to L^2(V)$ for V in $F^p, p > (n-2)/2$.

We are going to use a decomposition of T in the phase space. Consider first

$$P_1(\xi)^{-1} = (\varphi_1(\xi) + \varphi_2(\xi) + \varphi_3(\xi))P_1(\xi)^{-1} = \sum_{i=1}^{3} m_i(\xi),$$

where φ_i is in C_0^∞, $i = 1, 2$; supp $\varphi_1 \subset \{|\xi| < 1/2\}, \varphi_1 \equiv 1$ in $\{|\xi| < 1/4\}$; supp $\varphi_3 \subset \{|\xi| > 2\}, \varphi_3 \equiv 1$ in $\{|\xi| > 3\}$.

The Fourier multiplier corresponding to m_1 has a kernel rapidly decreasing and hence satisfies the inequality. For m_3 just observe that it behaves like $|\xi|^{-2}$ and by known results, see [**FeP**], satisfies the inequality for V in F^p with $p > 1$.

We may decompose m_2 as a finite sum of operators the worst of which is given by the multiplier

$$\tilde{m}(\xi) = p_1(\xi)^{-1}\psi_1(|\xi'|^2 - 1)\psi_2(\xi_n),$$

with $\xi' = (\xi_1, ..., \xi_{n-1})$, supp $\psi_2 \subset [-1, 1]$, supp $\psi_1 \subset [-1/4, 1/4], \psi_1 \in C_0^\infty$.

Now we may write

$$\tilde{m}(\xi) = \sum_{j=1}^{\infty} \tilde{m}_j(\xi),$$

for $\tilde{m}_j(\xi) \equiv m_\delta(\xi) = a_j(\xi)\psi_1\left(\frac{|\xi'|-1}{\delta}\right)\psi_2\left(\frac{\xi_n}{\delta}\right), \delta = 2^{-j}$, with appropiate a_j with $\delta^{-1} < |a_j| < 2\delta^{-1}$. ■

Hence we may reduce our inequality to the study of the operator K_δ given by a Fourier multiplier which has L^∞ norm as δ^{-1} and is supported in the "torus" $|\xi'| - 1 < 2\delta, |\xi_n| < \delta$. It is enough to prove:

Lemma. *For* $0 < \delta < 1/2$ *and* T_δ *defined by*

$$(T_\delta)^\wedge(\xi) = m(\xi)f^\wedge(\xi),$$

where

$$m(\xi) = \varphi\left(\frac{1 - |\xi'|}{\delta}\right)\varphi\left(\frac{\xi_n}{\delta}\right), \text{supp } \varphi \subset [-1, 1],\ \varphi \in C_0^\infty,$$

the following inequalities hold:

(i) $$\left(\int |T_\delta f|^2 V\right)^{1/2} \le C\delta|\log\delta|\|V\|_{F^{p_0}}\left(\int |f|^2 V^{-1}\right)^{1/2}, \ p_0 = (n-2)/2.$$

(ii)
$$\left(\int |T_\delta f|^2 V\right)^{1/2} \le C\delta^{1+\varepsilon}\|V\|_{F^p}\left(\int |f|^2 V^{-1}\right)^{1/2}, \ \textit{with } 0 < \varepsilon < 1-(n-2)/2p.$$

Proof: Let us call $K(x) = m^\wedge(x)$ and consider $\{\psi_j\}$ a smooth partition of unity

$$1 = \sum_{j=0}^{\infty} \psi_j, \text{ supp } \psi_j \subset (2^{j-1}, 2^{j+1}) j = 1, 2, \dots .$$

Define $T_j f = K_j * f$, where $K_j(x) = \psi_j(|x'|)K(x)$ and $x = (x', x_n) \in \mathbf{R}^{n-1} \times \mathbf{R}$. We shall obtain a good estimate for K_j which will allow us to sum in j.

On one hand observe that a straightforward calculation gives $|m_j(\xi)| = |(K_j)^\wedge(\xi)| \leq C \min\{2^j\delta, 1\}$ and, as a consequence,

$$\left(\int |T_j f|^2\right)^{1/2} \leq Cmin\{2^j\delta, 1\} \left(\int |f|^2\right)^{1/2}. \tag{1.3}$$

On the other hand for any natural number m there exists a constant C_m such that

$$|K_j(x)| \leq C_m \delta^2 2^{-j(n-2)/2}(1+\delta|x_n|)^{-m}(1+\delta 2^j)^{-m}. \tag{1.4}$$

Consider first the case $0 \leq j \leq 1 + [\log 1/\delta]$. For $k \in \mathbf{Z}$ we define

$$K_{jk}(x) = K_j(x) \cdot \chi_{[k\delta^{-1},(k+1)\delta^{-1}]}(x_n).$$

Then

$$|K_{jk}(x)| \leq C_m \delta^2 2^{-j(n-2)/2}(1+|k|)^{-m}.$$

Finally we can make in $\mathbf{R}^n$ a grid with paralellepipeds $\{Q_\nu\}$ such that the dimension of Q_ν are $2^j \times \ldots \times 2^j \times \delta^{-1}$.

Call $f_\nu = f \cdot \chi_{Q_\nu}$. Then

$$\begin{aligned}
\int |K_{jk} * f|^2 w &= \int |K_{jk} * \sum_\nu f_\nu|^2 w \\
&\leq C \sum_\nu \int |K_{jk} * f_\nu|^2 w \\
&\leq C \left(\sup_\nu \int_{Q^*\nu} w\right) \sum_\nu \|K_{jk} * f_\nu\|^2_{L^\infty(Q^*\nu)},
\end{aligned}$$

where Q^*_ν is a paralellepiped with the same center as Q_ν and side ten times bigger than the sides of Q_ν. By (1.4) and Young's inequality

$$\begin{aligned}
&\leq C_m \delta^4 2^{-j(n-2)}(1+|k|)^{-2m} \left(\sup_\nu \int_{Q^*\nu} w\right) \sum_\nu \left(\int |f_\nu|\right)^2 \\
&\leq C_m \delta^4 2^{-j(n-2)}(1+|k|)^{-2m} \left(\sup_\nu \int_{Q^*\nu} w\right)^2 \int |f|^2 w^{-1}
\end{aligned}$$

Now observe that if $w = V^{p_0}$ and $V \in F^{p_0}$, then

$$\sup_\nu \int_{Q^*\nu} w \leq C(2^j\delta)^{-1} 2^{2j} \|V\|^{p_0}_{F^{p_0}}.$$

Thus,

$$\left(\int |K_j * f|^2 V^{p_0}\right)^{1/2} \leq C\delta 2^{-j(n-4)/2} \|V\|_{F^{p_0}}^{p_0} \left(\int |f|^2 V^{-p_0}\right)^{1/2}.$$

Interpolation with (1.3) gives

$$\left(\int |K_j * f|^2 V\right)^{1/2} \leq C\delta \|V\|_{F^{p_0}} \left(\int |f|^2 V^{-1}\right)^{1/2}, \text{ if } 0 \leq j \leq 1 + [\log 1/\delta].$$

In the case $j \geq 1+[\log 1/\delta]$, let us define K_{jk} as $K_j(x)\chi_{[k2^j,(k+1)2^j]}(x_n)$, with $k \in Z$. Now for j fixed we consider in $\mathbf{R}^n$ a grid of cubes of side 2^j. Repeating the above process we obtain

$$\left(\int |K_j * f|^2 V^{p_0}\right)^{1/2}$$

$$\leq C\delta^{2(1-m)} 2^{-j((n-2)/2+2m-2)} \|V\|_{F^{p_0}}^{p_0} \left(\int |f|^2 V^{-p_0}\right)^{1/2}.$$

Again interpolation with (1.3) gives for $j \geq 1 + [\log 1/\delta]$

$$\left(\int |K_j * f|^2 V\right)^{1/2} \leq C 2^{-j} \|V\|_{F^{p_0}} \left(\int |f|^2 V^{-1}\right)^{1/2}.$$

Adding up in j we prove (i).

In order to prove (ii) we proceed as follows:

Define $K_j(x) = \psi_j(\delta|x|)K(x)$, with ψ_j as above $j = 0, 1, \dots$ and the support of $K_j \subset B(0, 2^{j+1}\delta^{-1})$. Then fix j and construct a grid of cubes $\{Q_\nu\}$ os side $2^j\delta^{-1}$. Then it is enough to prove the estimate for $f_\nu = f \cdot \chi_{Q_\nu}$.

Take $V \in F^p$ and $(n-2)/2 = p_0 < p < \infty$, let us call $w = V^{p/p_0}$, then

$$\left(\int |T_j f_\nu|^2 w\right)^{1/2} \leq \left(\int_{Q*_\nu} |T_j f_\nu|^2 w\right)^{1/2} = \left(\int |T_j f_\nu|^2 w_\nu\right)^{1/2}, \text{ where}$$

$w_\nu = w\chi_{Q*_\nu}$; then $w_\nu \in F^{p_0}$ and

$$\|w_\nu\|_{F^{p_0}} \leq C\|V\|_{F^p}^{p/p_0} (2^j\delta^{-1})^{2(1-p/p_0)} \text{ and then by (i)}$$

$$\left(\int |T_j f_\nu|^2 w\right)^{1/2} \leq C\delta |\log\delta| (2^j\delta^{-1})^{2(1-p/p_0)} \|V\|_{F^p}^{p/p_0} \left(\int |f_\nu|^2 w^{-1}\right)^{1/2}.$$

But also

$$\left(\int |T_j f_\nu|^2\right)^{1/2} \leq C\left(\int |f_\nu|^2\right)^{1/2}, \text{ and by interpolation } \left(\int |T_j f_\nu|^2 V\right)^{1/2}$$

$$\leq C\delta^{2-p/p_0} |\log\delta|^{p_0/p} 2^{-2j(1-p_0/p)} \|V\|_{F^p} \left(\int |f_\nu|^2 V^{-1}\right)^{1/2},$$

and (ii) is proved. ■

2. The Restriction theorem

We give the proof of theorem 2. Let us remak again that this theorem is contained in [**CS**] and [**ChR**], but the simplicity of our proof justifies to write it here.

Proof of theorem 2: It is known that

$$K(x) = (d\sigma)^\wedge(x) = |x|^{-(n/2-1)} J_{n/2-1}(|x|),$$

where J_λ designs the Bessel function of order λ. Then decompose

$$K(x) = \sum_{j=0}^{\infty} K_j(x) \text{ with}$$
$$K_j(x) = (d\sigma)^\wedge(x)\psi_j(|x|), \quad j = 1, 2, ..., \text{ supp } \psi_j \subset [2^{j-1}, 2^{j+1}];$$
$$K_0(x) = (d\sigma)^\wedge(x)\psi(|x|), \text{ supp } \psi \subset [-1, 1].$$

The classical P. Tomas, estimate for the Fourier Transform of $K_j(x)$ gives us the boundedness of $T_j = K_j*$ from L^2 to L^2 with norm 2^j.

We can repeat the argument in the proof of theorem 1 and obtain, for $w = V^p$,

$$T_j : L^2(w^{-1}) \to L^2(w) \text{ with norm bounded by } 2^{-j(n-1)/2} \left(\sup_{Q_\nu} \int_{Q_\nu} w \right)$$

where Q_ν is a cube in the grid in $\mathbf{R}^n$ of side 2^j. Since $V \in F^{\alpha,p}$, we obtain

$$\|T_j\|_{L^2(w^{-1}) \to L^2(w)} \le C 2^{j(n-\alpha p-(n-1)/2)} \|V\|^p_{\alpha,p}.$$

Interpolation gives

$$\|T_j\|_{L^2(V^{-1}) \to L^2(V)} \le C(2^j)^{\frac{n-1+2p(1-\alpha)}{2p}} \|V\|_{\alpha,p},$$

the sum is convergent if $p > \frac{n-1}{2(\alpha-1)}$.

It is an open question if the above operator send $L^2(V^{-1})$ to $L^2(V)$ for V in $F^{\alpha,p}, p < (n-1)/2$. The answer to this question would be the corner stone to extend unique continuation properties to potential in F^p for $p \le (n-2)/2$.

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