THE WORK OF
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The aim of this paper is to review a set of articles ([6], [10], [11], [13], [16], [25]) of which José Luis Rubio de Francia was author or co-author written between 1985 and 1987.

I had the luck of being his graduate student around this time so that we collaborated in some of this work. It is hard to say in a few words how was José Luis Rubio but at least I would like to point out that he influenced my career in a decisive way and that he was one of the nicest persons I have ever met.

1. Singular integrals with rough kernels:
L^p theory ([16], [6], [13])

In all these papers a common approach in used to study the boundedness of several singular integrals, based on the following idea: decompose the operator T as a sum

$$T = \sum_{k=-\infty}^{\infty} \tilde{T}_k$$

in such a way that

(1) $$\|\tilde{T}_k f\|_2 \leq C 2^{-\alpha|k|} \|f\|_2 \text{ for some } \alpha > 0;$$

if now one of the following inequalities happens

(2) $$\|\tilde{T}_k f\|_1 \leq C \|f\|_{H^1};$$

(3) $$\|\tilde{T}_k f\|_{p_0} \leq C \|f\|_{p_0} \text{ for some } p_0 \neq 2;$$

(4) $$\|\tilde{T}_k f\|_{L^2(w)} \leq C \|f\|_{L^2(w)} \text{ for some weight } w,$$

interpolation with (1), summing a geometric series and duality give the boundedness of T in L^p, $1 < p < \infty$, in L^p, $p_0 < p < p_0'$ (or $p_0' < p < p_0$) and $L^2(w^\theta)$, $0 \le \theta < 1$, respectively. Polynomial growth in k can be allowed in inequalities (2), (3) and (4) with the same conclusion.

To get (1) we'll start with a natural descomposition of T as $\sum_{j=-\infty}^{\infty} T_j$ where each T_j is given by convolution with a measure σ_j.

If, for example, one can prove

$$(5) \qquad |\hat{\sigma}_j(\xi)| \le C \min(|2^j\xi|, |2^j\xi|^{-1})^\alpha \text{ for some } \alpha > 0,$$

then we can construct $\tilde{T}_k$ as follows: choose a function $\psi \in \mathcal{C}^\infty(\mathbf{R}^n)$, supported in $\frac{1}{2} < |\xi| < 2$ and such that

$$(6) \qquad \sum_{j=-\infty}^{\infty} \psi(2^j\xi) = 1 \qquad \forall \xi \ne 0,$$

define S_j as

$$(S_jf)\hat{}(\xi) = \psi(2^j\xi)\hat{f}(\xi)$$

and take

$$(7) \qquad \tilde{T}_k = \sum_{j=-\infty}^{\infty} T_j S_{j+k}.$$

Under these circunstances, (1) is easily verified using (5) and Plancherel's theorem.

a) The simplest application will consider the singular integral

$$(8) \qquad Tf(x) = \text{p.v.} \int_{\mathbf{R}^n} \frac{\Omega(y)}{|y|^n} f(x-y)\, dy$$

where Ω is homogeneous of degree zero, its restriction to the unit sphere has mean value zero and is in $L^q(S^{n-1})$ for some $q > 1$. It is well-known that T is bounded in $L^p(\mathbf{R}^n)$, $1 < p < \infty$, by using the method of rotations but the present method offers an alternative approach. $T_j f$ is the integral restricted to $2^j \le |y| < 2^{j+1}$ and σ_j is the integrable function given by $\Omega(y)|y|^{-n}\mathcal{X}_{\{2^j \le w|y| < 2^{j+1}\}}$ where $\mathcal{X}_A$ stands for the characteristic function of the set A. The estimation of an oscillatory integral shows that (5) happens for any $\alpha < 1/q'$.

Theorem 1. *Let $\{\sigma_j\}$ be a sequence of Borel measures supported in $\{x \in \mathbf{R}^n : |x| \leq 2^j\}$, with uniform total variation and integral zero such that*

$$|\hat{\sigma}_j(\xi)| \leq C|2^j\xi|^{-\alpha} \text{ for some } \alpha > 0.$$

Then, $Tf = \sum_j \sigma_j * f$ is bounded in L^p, $1 < p < \infty$.

Defining $\tilde{T}_k$ as above we only need to compute the Hörmander constant of its kernel to show that (2) is verified with constant $C(1 + |k|)$.

As an application, singular integrals of the type (8) are bounded in $L^p(\mathbf{R}^n)$, $1 < p < \infty$. If one introduces a bounded radial function in the kernel of (8), the method of rotations is not applicable but theorem 1 gives again the L^p-boundedness, $1 < p < \infty$.

Before modifiying some aspects of this theorem, let us state a new one related to maximal operators.

Theorem 2. *Let $\{\mu_j\}$ a sequence of positive Borel measures supported in $\{x \in \mathbf{R}^n : |x| \leq 2^j\}$, with uniform total variation such that*

$$|\hat{\mu}_j(\xi)| \leq C|2^j\xi|^{-\alpha} \text{ for some } \alpha > 0,$$

then, $\mathcal{M}f(x) = \sup_j |\mu_j * f(x)|$ defines a bounded operator in $L^p(\mathbf{R}^n)$, $1 < p \leq \infty$.

To prove this theorem, define $\sigma_j = \mu_j - \hat{\mu}_j(0)\varphi_j$ where $\varphi_j = 2^{-jn}\varphi(2^{-j}\cdot)$ and φ is a $\mathcal{C}^\infty(\mathbf{R}^n)$ function, supported in the unit ball and such that $\hat{\varphi}(0) = 1$. Then, the sequence $\{\sigma_j\}$ satisfies the hypotheses of theorem 1 and the same is true for $\{\varepsilon_j\sigma_j\}$, where $\varepsilon_j = \pm 1$ arbitrarily, with constants independent of the sequence of signs. It is enough to observe that

$$(9) \qquad \mathcal{M}f(x) \leq \left(\sum_j |\sigma_j * f(x)|^2\right)^{1/2} + cMf(x)$$

(where M stands for the Hardy-Littlewood maximal function) and apply theorem 1 to get the L^p-boundedness of the square function (via the uniform boundedness of $\sum_j \varepsilon_j\sigma_j * f$ and the usual argument with Rademacher functions).

A consequence of theorem 2 is the L^p-boundedness of the lacunary spherical maximal function (take $\mu_j = $ normalized Lebesgue measure over the sphere of radius 2^j). Obviously one can substitute the sphere by any other compact surface with enough curvature to ensure the required decay condition of the Fourier transform for the Lebesgue measure carried by it.

b) Given a matrix A whose eigenvalues have nonnegative real part, we can define the associated group of dilations $\{\delta_t\}$ by $\delta_t x = t^A x$ and a "norm" in $\mathbf{R}^n$

such that $\|\delta_t x\| = t\|x\|$, $t > 0$, (see [26]). If this norm is used in theorems 1 and 2 instead of the euclidean norm, they are still true. Apart from standard modifications of operators like (8), this provides other interesting results.

Given a curve $\Gamma : t \to \gamma(t)$ in $\mathbf{R}^n$, two operators are usually associated to it: the Hilbert transform along Γ

$$H_\Gamma f(x) = p.v. \int_{-\infty}^{\infty} f(x - \gamma(t))\frac{dt}{t}$$

and the maximal function along Γ

$$M_\Gamma f(x) = \sup_{h>0} \frac{1}{2h} |\int_{-h}^{h} f(x - \gamma(t))dt|.$$

$H_\Gamma f = \sum_j \sigma_j * f$ where σ_j is the measure of size $\frac{1}{t}$ over the portion of Γ where $2^j \le |t| < 2^{j+1}$ and M_Γ is equivalent to $\sup_j |\mu_j * f|$ where μ_j is the measure of size 2^{-j-1} over the same portion of Γ. A homogeneous curve is given by

$$\gamma(t) = t^A u, \quad t > 0, \qquad \gamma(t) = (-t)^A v, \quad t < 0.$$

where $u, v \in S^{n-1}$ and the positive and negative parts of γ generate the same subspace of $\mathbf{R}^n$. The boundedness of the Hilbert transform and the maximal function along a homogeneous curve are now a consequence of the estimates for $\hat{\sigma}_j$ and $\hat{\mu}_j$, which can be found in [26]. In that paper L^p-boundedness for $p \neq 2$ is proved via analytic interpolation which we avoid here. The same result holds for well-curved curves, see again [26] for the definition and the proof of the key estimate.

c) Inequalities like (3) can be used instead of (2). In order to get them one modifies the choice of ψ above requiring

$$(10) \qquad \sum_{j=-\infty}^{\infty} |\psi(2^j\xi)|^2 = 1, \; \forall \xi \neq 0$$

instead of (6) so that $\sum_j S_j^2 = I$ and Littlewood-Paley type inequalities occur in both senses (see [17, chap. V]). The following chain of inequalities can be written

$$(11) \qquad \begin{aligned} \|\tilde{T}_k f\|_{p_0} = \|\sum_j T_j S_{j+k}^2 f\|_{p_0} &\le C\|(\sum_j |T_j S_{j+k} f|^2)^{1/2}\|_{p_0} \le \\ &\le C\|(\sum_j |S_{j+k} f|^2)^{1/2}\|_{p_0} \le C\|f\|_{p_0}, \end{aligned}$$

provided the $\{T_j\}$ satisfy the following vector-value inequality

$$\|(\sum_j |T_j f_j|^2)^{1/2}\|_{p_0} \leq C\|(\sum_j |f_j|^2)^{1/2}\|_{p_0}.$$

This inequality is easily obtained from an uniform weighted inequality

$$(12) \qquad \int |T_j f|^2 u \leq C \int |f|^2 A u$$

where A is bounded from L^q to L^q and $q = (\frac{p_0}{2})'$. For convolution operators T_j with kernel σ_j, (12) holds with $Aw = \sigma^*(w) = \sup_j ||\sigma_j| * w|$.

We can state the following theorem:

Theorem 3. *Let* $\{\sigma_j\}$ *be a sequence of Borel measures in* $\mathbf{R}^n$ *with uniform total variation, such that*

$$(5) \qquad |\hat{\sigma}_j(\xi)| \leq C \min(|2^j \xi|, |2^j \xi|^{-1})^\alpha \text{ for some } \alpha > 0.$$

If σ^* *is bounded in* $L^q(\mathbf{R}^n)$ *for some* $q \geq 1$, *then* $Tf = \sum_j \sigma_j * f$ *is bounded in* $L^p(\mathbf{R}^n)$, $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{2q}$.

We can avoid the compactness assumption for the support of σ_j but no new interesting results come from this generalization. Its main interest with respect to theorem 1 lies in the modifications to be given below.

Theorem 2 is also obtained from theorem 3 by using a bootstraping argument (again we don't need to assume that supp μ_j is compact but then we have to add $|\hat{\mu}_j(\xi) - \hat{\mu}_j(0)| \leq C|2^j \xi|^\alpha$). As before we define $\sigma_j = \mu_j - \hat{\mu}_j(0)\varphi_j$ and apart from (9) we also have

$$(13) \qquad \sigma^*(f) \leq \mathcal{M}f + CMf.$$

This inequality together with theorem 3 and (9) imply: *if* $\mathcal{M}$ *is bounded in* L^q, *it is also bounded in* L^p, $\frac{1}{p} - \frac{1}{2} < \frac{1}{2q}$, *(i.e.,* $p > \frac{2q}{q+1}$*).*

Starting with $q = 2$ where the result is given by the hypotheses on $\hat{\mu}_j(\xi)$, any $p > 1$ is reached after a finite number of steps.

d) Let $\xi = (\xi_1, \xi_2) \in \mathbf{R}^m \times \mathbf{R}^{n-m}$, theorem 3 can be modified in the following way:

Theorem 4. *If in theorem 3 we assume*

$$(14) \qquad |\hat{\sigma}_j(\xi)| \leq C \min(|2^j \xi_1|, |2^j \xi_1|^{-1})^\alpha \text{ for some } \alpha > 0,$$

instead of (5), *the same conclusion holds.*

The same proof works after taking operators S_j which act only on the variable ξ_1.

Condition (14) implies $\hat{\sigma}_j(0, \xi_2) = 0$, $\forall \xi_2 \in \mathbf{R}^{n-m}$ which is usually too strong. It seems better to assume

$$(15) \qquad \begin{aligned} |\hat{\sigma}_j(\xi_1, \xi_2) - \hat{\sigma}_j(0, \xi_2)| &\leq C|2^j \xi_1|^\alpha \\ |\hat{\sigma}_j(\xi)| &\leq C|2^j \xi_1|^{-\alpha} \end{aligned}$$

But then one has to make some hypothesis on $\hat{\sigma}_j(0, \xi_2)$, for example,

$$(16) \qquad |\hat{\sigma}_j(0, \xi_2)| \leq C \min(|2^j \xi_2|, |2^j \xi_2|^{-1})^\alpha.$$

Writting

$$\hat{\sigma}_j(\xi_1, \xi_2) = [\hat{\sigma}_j(\xi_1, \xi_2) - \hat{\sigma}_j(0, \xi_2)\varphi_j(\xi_1)] + \hat{\sigma}_j(0, \xi_2)\varphi_j(\xi_1)$$

theorem 4 is applied twice.

For the maximal operator conditions like (15) are to be assumed on $\{\mu_j\}$ and an extra hypothesis on the boundedness of the maximal operator associated to $\hat{\mu}_j(0, \xi_2)$. All the technical details can be found in [**16**].

Now we can prove the boundedness of the Hilbert transform and the maximal function along a homogeneous curve with A diagonal by induction without using non-isotropic dilations. If the entries in A are integers, the estimations we need are also much easier. In addition we get a result for flat curves which is not given by theorem 1 and 2:

Corollary 5. *Let* $\Gamma = (t, \varphi(t))$ *be a curve in* $\mathbf{R}^2$ *such that* $\varphi(0) = \varphi'(0) = 0$, $\varphi''(t) > 0$ *and increasing for* $t > 0$, φ *odd or even, then* H_Γ *and* M_Γ *are bounded in* $L^p(\mathbf{R}^2)$, $1 < p < \infty$.

e) If in the hypotheses of corollary 5 we merely assume $\varphi''(t) > 0$ (i.e. not necessarily increasing) the conclusion can be false. In [**21**] the following result was proved: *let* $\Gamma = (t, \varphi(t))$ *be an even convex curve in* $\mathbf{R}^2$, *then* H_Γ *is bounded in* L^2 *if and only if* $\exists C > 1$ *such that*

$$(17) \qquad\qquad \varphi'(Ct) \geq 2\varphi'(t), \ \ \forall t > 0.$$

Assuming (17), inequality (14) fails in an angular sector which moves with j. This sequence of sectors is lacunary so that one can apply Littlewood-Paley inequalities associated to them as was proved by Nagel, Stein and Wainger [22]. Combining these inequalities with theorem 4, the following is proved in [6]:

Theorem 6. *Let $\Gamma = (t, \varphi(t))$ be a curve in $\mathbf{R}^2$, odd or even, $\varphi''(t) > 0$ for $t > 0$, satisfying (17). Then, M_Γ and H_Γ are bounded in $L^p(\mathbf{R}^2)$, $1 < p < \infty$.*

Together with the result in [21] this theorem implies that for φ even, (17) is necessary and sufficient for the L^p boundedness.

In [13], A. Córdoba and José Luis Rubio de Francia generalized the preceding theorem to the case where the curve is neither odd nor even. The proof works when some balance condition between the positive and negative parts of the curve is assumed. They also proved that the condition is necessary.

Theorem 7. *Let $\Gamma = (t, \varphi(t))$ be a curve in $\mathbf{R}^2$ such that $\varphi(0) = \varphi'(0) = 0$; $|\varphi'(t)|$ increasing if $t > 0$ and decreasing if $t < 0$; $\exists C > 1$ such that $|\varphi'(Ct)| \geq 2|\varphi'(t)|, \forall t \neq 0$ and $\exists k > 1$ s.t. $|\varphi(k^{-1}t)| \leq |\varphi(-t)| \leq |\varphi(kt)|$ for every $t > 0$ (balance condition). Then M_Γ and H_Γ are bounded in $L^p(\mathbf{R}^2)$, $1 < p < \infty$.*

f) Inequality (4) is also useful and gives weighted inequalities for singular integrals with rough kernels. In the chain of inequalities (11), one can use the $L^2(w)$ norm instead of L^{p_0} if $w \in A_2$ and the first and third inequalities still hold for the Littlewood-Paley theory (see [20]).

If T is given by (8) with $\Omega \in L^\infty$, $T_j f \leq CMf$ and the vector valued inequality also holds in $L^2(w)$ (see [17]). Then

Theorem 8. *Let T be as in (8) with $\Omega \in L^\infty(S^{n-1})$. Then, T is bounded in $L^p(w)$, $\forall w \in A_p$.*

To pass from $L^2(w)$ inequalities to all $L^p(w)$, $w \in A_p$, one uses the extrapolation theorem of Rubio de Francia ([24]).

g) For all the singular integrals studied above, including Hilbert transforms along curves, the maximal operator over the truncated integrals is shown to be bounded in the same spaces giving the a.e. convergence of the truncated integrals.

M. Christ used in [8] and [9] methods similar to those developed here, independently. In fact, theorem 1 is a modification of [16] following his ideas. In [8], Hilbert transform and maximal functions along homogeneous curves in nilpotent groups are studied.

Extensions of the theory to the multiparameter setting with applications to multiple singular integrals and operators along hypersurfaces are in [14], to operators which are not necessarily of convolution type in [3] and [5]. Further results on curves are in [4] (see also Wainger's lecture in this Proceedings) and more weighted inequalities appear in [15] and [27].

2. Maximal functions with continuous parameter [25]

Maximal functions were studied in the preceding section only when they were controlled by their dyadic version but this is not the general case as the spherical maximal function shows. Moreover, this is also an example where the dyadic maximal function is bounded in a range which is larger than the one for the continuous maximal function. In [25], José L. Rubio de Francia gave a simple proof of the theorem of Stein on the boundedness of the spherical maximal function for $p > \dfrac{n}{n-1}$, $n \geq 3$, (see [26] for the original proof).

Theorem 9. *Let m be the Fourier transform of a compactly supported positive measure μ in $\mathbf{R}^n$ such that*

$$(18) \qquad\qquad |m\,(\xi)| \leq C|\xi|^{-a} \text{ for some } a > \frac{1}{2}.$$

Then, the maximal operator

$$T^* f(x) = \sup_{t>0} (m\,(t\cdot)\hat{f})^{\vee}(x)$$

is bounded in $L^p(\mathbf{R}^n)$, $p > \dfrac{2a+1}{2a}$.

($\vee$ stands for the inverse Fourier transform.)

Let us sketch the proof: take a cutting function $\psi \in C^\infty(\mathbf{R}^n)$ supported in $\dfrac{1}{2} < |\xi| < 2$ and also $\varphi \in C^\infty(\mathbf{R}^n)$ supported in $|\xi| < 1$, such that

$$\varphi(\xi) + \sum_{j=0}^{\infty} \psi(2^{-j}\xi) \equiv 1$$

and consider $m_j = m\,\psi(2^{-j}\cdot)$. Since the maximal function associated to $m\,\varphi$ is bounded by the Hardy-Littlewood maximal function, it is enough to prove that

$$T_j^* f(x) = \sup_{t>0} (m_j(t\cdot)\hat{f})^{\vee}(x)$$

satisfies
$$\|T_j^* f\|_p \leq C2^{-\varepsilon j}\|f\|_p$$
for the desired range of p's. This is achieved in a way similar to the one in the preceding paragraph, starting with an L^2−inequality. In fact, due to the size hypothesis on m one gets

$$\|T_j^* f\|_2 \leq C2^{-j(a-\frac{1}{2})}\|f\|_2.$$

For $p = 1$, looking at T_j^* as a vector valued singular integral, one can compute the Hörmander constant of the kernel to obtain

$$\|T_j^* f\|_1 \leq Cj2^j\|f\|_{H^1}.$$

Interpolating and summing in j gives the result for $p \leq 2$. But for $p = \infty$ the theorem holds trivially and the proof is ended.

When μ is the Lebesgue measure over the unit sphere, (18) is satisfied with $a = \dfrac{n-1}{2}$ and Stein's result is obtained. For other hypersurfaces a theorem of Greenleaf in [18] is obtained.

Since this method is based on a good L^2−estimate which is false in $n = 2$, it is not applicable to get Bourgain's result [1].

If m is not the Fourier transform of a measure as before, the L^∞ estimate can be false. In [25] there is also a theorem concerning this case.

Theorem 10. *Let s be an integer* $> \dfrac{n}{2}$, $m \in \mathcal{C}^{s+1}(\mathbf{R}^n)$ *such that*

$$|D^\alpha m(\xi)| \leq C|\xi|^{-a} \qquad \forall|\alpha| \leq s+1 \text{ with } a > \frac{1}{2}.$$

Then, T^* is bounded in $L^p(\mathbf{R}^n)$ for

$$\frac{2n}{n+2a-1} < p < \frac{2n-2}{n-2a}$$

(1 on the left if $a \geq \dfrac{n+1}{2}$, ∞ on the right if $a \geq \dfrac{n}{2}$.)

For $p \leq 2$ the proof follows the same way as in Theorem 9 but now

$$\|T_j^* f\|_1 \leq C_\beta 2^{j(\frac{1}{2}+\beta-a)}\|f\|_{H^1} \qquad \forall\beta > \frac{n}{2}.$$

When $p > 2$, the theory of vector valued singular integrals is again applied to get an L^∞−BMO estimate:

$$\|T_j^* f\|_{BMO} \leq C_\beta 2^{j(\beta-a)}\|f\|_\infty \qquad \forall\beta > \frac{n}{2}.$$

This constant is smaller than the $H^1 - L^1$ constant so that the range in Theorem 10 is not symmetric.

Notice that for $p < 2$, Theorem 9 is better than theorem 10 if $a < \frac{n-1}{2}$ and conversely if $a > \frac{n-1}{2}$. Since this second theorem applies also in the first case one must combine both results to obtain the optimal situation when $\mathcal{M}$ comes as the Fourier transform of a measure.

3. Operators related to the method of rotations and the Radon transform [10]

a) Given an one-dimensional operator S bounded in $L^p(\mathbf{R})$, we can define a collection of n-dimensional analogues: for any $u \in S^{n-1}$,

$$S_u f(x) = S(f(x + \cdot u))(0).$$

All these operators are uniformly bounded in $L^p(\mathbf{R}^n)$. We look for inequalities of the type

$$(19) \qquad \left(\int_{\mathbf{R}^n} \left(\int_{S^{n-1}} |S_u f(x)|^q du \right)^{p/q} dx \right)^{1/p} \leq C \|f\|_p.$$

The left-hand side is called the $L^p(L^q)$ mixed norm of the family $\{S_u f\}$. Inequality (19) is trivial for $p = q$ (hence for $p > q$) because the order of integration can be reversed.

When S is the Hilbert transform, Hardy-Littlewood maximal function or maximal Hilbert transform, inequalities like (19) are used in the method of rotations for singular integral operators with variable kernel ([2]):

$$(20) \qquad Tf(x) = p.v. \int \frac{\Omega(x, y')}{|y|^n} f(x - y) \, dy$$

where

$$\sup_x \|\Omega(x, \cdot)\|_{L^r(S^{n-1})} < +\infty \text{ for some } r > 1.$$

Taking f = characteristic function of the unit ball, one proves that (19) only can be true when $\frac{1}{q} > \frac{n - p}{(n - 1)p}$ for the three operators listed above. In [10] the following is proved

Theorem 11.
(i) *When $S = M$, (19) holds for* $\frac{1}{q} > \frac{n - p}{(n - 1)p}$ *whenever* $p \leq \max(2, \frac{n+1}{2})$.
(ii) *When S is an one-dimensional operator bounded in $L^r(w)$ for every weight $w \in A_r(\mathbf{R})$, then (19) holds in the same range as (i).*

In both cases interpolation with the trivial case $p = q$ gives a result for $p > \max(2, \dfrac{n+1}{2})$ which is sharp if $n = 2$ but leaves an undecided region when $n \geq 3$.

The proof of (a) for $p = 2$ uses an estimate from L^2 to $L^2(L^2_\beta)$ where L^2_β is a Sobolev space. By the embedding theorems, for any $q < \dfrac{2(n-1)}{n-2}$, there exists $\beta < \dfrac{1}{2}$ such that $L^2_\beta(S^{n-1}) \subset L^q(S^{n-1})$. The estimate is obtained via the Fourier transform. For $p = \dfrac{n+1}{2}$, the X-ray transform and its L^p mapping properties are used. See [10] for details. (b) is then obtained from (a).

Applying theorem 11 to the method of rotations in [2] we get:

Corollary 12. *Let T be a singular integral operator like* (20) *and T^* the maximal operator of the truncated integrals. Then, T and T^* are bounded in $L^p(\mathbf{R}^n)$ if $1 < p \leq \max(2, \dfrac{n+1}{2})$ and $r > \dfrac{n-1}{n}p'$. If $n = 2$ also for $2 \leq p < \infty$ for any $r > 1$ and if $n \geq 3$, for $\dfrac{n+1}{2} \leq p < \infty$ if $r > \dfrac{2p}{2p-1}$.*

This result is sharp in $n = 2$ but there is probably a better result if $n \geq 3$.

A maximal operator related to the Bochner-Riesz multipliers is given by

$$M_\delta f(x) = \sup_R \frac{1}{|R|} \int_R |f(x - y)| \, dy$$

where for a fixed $\delta > 0$, the supremum is taken over all the parallelopipeds containing the origin and having one side of length r and $(n-1)$ sides of length δr, $\forall r > 0$. The conjecture is

(21) $$\|M_\delta f\|_n \leq C(\log \delta)^a \|f\|_n$$

which was proved by Córdoba [12] for $n = 2$, the only case where it is known.

If (21) was true, interpolating with the trivial estimate with constant $C\delta^{1-n}$ for p close to 1 would give

(22) $$\|M_\delta f\|_p \leq C(\log \delta)^a \delta^{1-n/p} \|f\|_p, \quad 1 < p \leq n$$

As a consequence of theorem 10 (a) we have:

Corollary 13. (22) *holds if $1 < p \leq \max(2, \dfrac{n+1}{2})$.*

For $n = 2$ this gives a new proof of Córdoba's result.

b) A maximal operator associated to the Radon transform is

$$Rf(x,u) = \sup_{r>0} \frac{1}{r^{n-1}} \int_{\substack{|y| \le r \\ <y,u>=0}} |f(x-y)| d\lambda(y), \quad x \in \mathbf{R}^n, \ u \in S^{n-1}$$

where λ is the Lebesgue measure on the hyperplane $<y,u>=0$. Remember that for $u \in S^{n-1}$ and $t \in \mathbf{R}$, the Radon transform of f in (u,t) is obtained by integrating f over the hyperplane $<x,u>=t$ (see [23]). Again one can consider mixed norm estimates

$$(23) \qquad \left(\int_{\mathbf{R}^n} \left(\int_{S^{n-1}} |Rf(x,u)|^q du \right)^{p/q} dx \right)^{1/p} \le C \|f\|_p$$

and the theorem proved in [10] is:

Theorem 14. *Inequality* (23) *holds whenever*

$$1 < p \le \frac{n+1}{n} \qquad and \qquad \frac{1}{q} > \frac{n}{p} - (n-1)$$

or

$$\frac{n+1}{n} \le p \le 2 \qquad and \qquad \frac{1}{q} > (\frac{2}{p}-1)\frac{1}{n-1}$$

or

$$p \ge 2 \qquad and \qquad q < \infty.$$

It is enough to prove the theorem for $p=2$, $q < \infty$ and $q = n+1$, $p > \dfrac{n+1}{n}$ and interpolate with the trivial case $p=q$. The L^2-theory is handled with the Fourier transform and the mapping properties of the Radon transform are used in the remainder.

c) Let us include two more results from [10]:

Theorem 15. *Let* Γ *be a rectifiable curve in* $\mathbf{R}^n$ *which crosses at most* M *times (*$M>0$ *given) every hyperplane in* $\mathbf{R}^n$ *and* $u_1,...,u_N$, N *points over* Γ. *If* $H_{u_1},...,H_{u_N}$ *are the Hilbert transforms in these directions, then*

$$\left\| \left(\frac{1}{N} \sum_{j=1}^N |H_{u_j} f|^q \right)^{1/q} \right\|_2 \le Cq \|f\|_2$$

and

$$\| \sup_{1 \le j \le N} |H_{u_j} f| \|_2 \le C \log N \|f\|_2.$$

Notice that S^1 has the finite crossing property of the theorem so that any set of N points in $\mathbf{R}^2$ satisfies those inequalities.

As a consequence we have

Corollary 16. *Let $u_1,...,u_N$ as in the preceding theorem, if $\dfrac{4}{3} \le p \le 4$,*

$$\left\| \left(\sum_{j=1}^{N} |H_{u_j} f_j|^2 \right)^{1/2} \right\|_p \le C(\log N)^{4|\frac{1}{p}-\frac{1}{2}|} \left\| \left(\sum_{j=1}^{N} |f_j|^2 \right)^{1/2} \right\|_p$$

with C independent of N.

4. Singular integrals with rough kernel: weak (1,1) estimates [11]

The singular integral (8) and the related maximal operator

$$(24) \qquad M_\Omega f(x) = \sup_{r>0} \frac{1}{r^n} \int_{|y|<r} |\Omega(y')f(x-y)| \, dy$$

are easily seen to be bounded in $L^p(\mathbf{R}^n)$, $1 < p < \infty$, by using the method of rotations. But this method does not apply to obtain a weak (1,1) estimate because the weak L^1–space is not a normed space. The question remained open for a long time until M. Christ gave in [7] the first proof of the weak (1,1) estimate for M_Ω in the two-dimensional case. Subsequently, in a joint paper with José L. Rubio de Francia [11], they were able to extend the result to all dimensions for M_Ω and to prove it for $n = 2$ for the singular integral (they claim that in this case the proof can be extended to $n \le 5$). S. Hofmann proved independently the result for the singular integral in [19].

Theorem 17. *The maximal operator M_Ω given by (24) is of weak type (1,1) when $n \ge 2$ and $\Omega \in L \log L(S^{n-1})$. The singular integral operator T given by (8) is of weak type (1,1) when $n = 2$, $\Omega \in L \log L(S^1)$ and $\int \Omega = 0$.*

The proof follows the usual Calderón-Zygmund argument with just one modification: after decomposing $f = g+b$ where b is a sum of functions b_i supported in disjoint dyadic cubes Q_i and with mean value zero, one takes away an exceptional set E formed by dilations of the cubes Q_i and usually proves that $\|Tb\|_{L^1(\mathbf{R}^n \setminus E)} \le C\|b\|_1$. Instead of this inequality, what is used in the above papers is

$$\|Tb\|_{L^2(\mathbf{R}^n \setminus E)}^2 \le C\lambda \|b\|_1$$

where $\lambda > 0$ is the height at which the Calderón-Zygmund decomposition has been made. The idea of using this inequality goes back to a paper of C. Fefferman.

In practice, one takes $K_j(x) = 2^{-jn}\eta(2^{-j}x)\Omega(x)$ where η is a radial C^∞ function, nonnegative, supported in $\dfrac{1}{2} \le |x| \le 4$ and identically one on $1 \le$

$|x| \leq 2$. Then $M_\Omega f \leq C \sup_j f * K_j$ if $f \geq 0$ and it is enough to prove the weak (1,1) estimate for $\sup_j f * K_j$.

If b is the "bad function" in the Calderón-Zygmund decomposition of f and the exceptional set E is constructed by taking the union of the cubes with same centers as Q_i and five times their sides, for a fixed j, $b_i * K_j(x)$ is different from zero in some point $x \notin E$ only if the side of the cube Q_i where b_i is supported is less than 2^j. For each $s \in \mathbf{Z}$, denote by B_s the sum of the b_i for which the sidelength of Q_i is 2^s; then, the key estimate is the following: assuming $\Omega \in L^\infty(S^{n-1})$ and $s > 0$,

$$\| \sup_j |K_j * B_{j-s}| \|^2_{L^2(\mathbf{R}^n)} \leq C 2^{-\varepsilon s} \|\Omega\|_\infty^2 \lambda \|b\|_1$$

for some $\varepsilon > 0$. By dilation invariance it is enough to prove it for $j = 0$. If $\tilde{K}_0(x) = \overline{K_0(-x)}$ we have

$$\|K_0 * B_{-s}\|_2^2 = < \tilde{K}_0 * K_0 * B_{-s}, B_{-s} >$$

and it is enough to prove

$$\|\tilde{K}_0 * K_0 * B_{-s}\|_\infty \leq C\|\Omega\|_\infty^2 2^{-\varepsilon s} \lambda.$$

The convolution $\tilde{K}_0 * K_0$ has better properties than K_0 alone and this makes possible the above estimate to hold. When $n = 2$, $\tilde{K}_0 * K_0$ is Hölder continuous outside the origin and this is enough (see [7]) but for $n \geq 3$ this Hölder property does not hold and one has to go into harder geometric considerations (see [11]).

For the singular integral there is an additional complication coming from the fact that the key estimate must be now proved for a sum instead of a supremum

$$\| \sum_j K_j * B_{j-s} \|^2_{L^2(\mathbf{R}^n)} \leq C 2^{-\varepsilon s} \lambda \|b\|_1 \|\Omega\|_\infty^2.$$

The square of the sum presents cross terms which are hard to handle. This is the reason why the proof works only in low dimensions.

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