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CHEBYSHEV COEFFICIENTS FOR L^1-PREDUALS AND FOR SPACES WITH THE EXTENSION PROPERTY

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Abstract

We apply the Chebyshev coefficients λ_f and λ_b, recently introduced by the authors, to obtain some results related to certain geometric properties of Banach spaces. We prove that a real normed space E is an L^1-predual if and only if $\lambda_f(E) = 1/2$, and that if a (real or complex) normed space E is a $\mathcal{P}_1$ space, then $\lambda_b(E)$ equals $\lambda_b(\mathbb{K})$, where $\mathbb{K}$ is the ground field of E.

In this note, $\mathbb{K}$ will be the real or complex field, and E a normed space over $\mathbb{K}$; when we want to state a result only for the real case or the complex case, we will indicate it specificaly. We will use the notations of [**2**], to which we refer for all concepts of the theory of normed spaces which may appear without defining them here.

If S is a non empty subset of E, the number

$$r(S) = \inf_{y \in E} \sup_{x \in S} \|x - y\|$$

is called the Chebyshev radius of S, and $\delta(S)$ denotes the diameter of S.

Definition. We will call the *finite Chebyshev coefficient* of E the real number

$$\lambda_f(E) = \sup\{r(S)/\delta(S) : S \subset E,\ S \text{ finite},\ \delta(S) > 0\},$$

and the *bounded Chebyshev coefficient* of E the real number

$$\lambda_b(E) = \sup\{r(S)/\delta(S) : S \subset E,\ 0 < \delta(S) < \infty\}.$$

It is easy to prove that, in general,

$$1/2 \leq \lambda_f(E) \leq \lambda_b(E) \leq 1.$$

Moreover, when E is finite dimensional, we have $\lambda_f(E) = \lambda_b(E)$.

Specifically, the Chebyshev coefficients associated to the scalar fields are

$$\lambda_f(\mathbb{R}) = \lambda_b(\mathbb{R}) = 1/2$$
$$\lambda_f(\mathbb{C}) = \lambda_b(\mathbb{C}) = 1/\sqrt{3}.$$

Let us recall that a $\mathcal{P}_\alpha(\mathbb{K})$ space, where α is a real number greater than or equal to 1, is a Banach space E for which any of following equivalent condition holds:

(i) Given two Banach spaces, F and G, a linear isometry into, $\phi : F \to G$, and a bounded linear operator, $L : F \to E$, there exists a bounded linear operator $\hat{L} : G \to E$, which extends L, in the sense of $\hat{L} \circ \phi = L$, and such that $\|\hat{L}\| \leq \alpha\|L\|$ (α-extension property).

(ii) Given a Banach space F, and a linear isometry, $\phi : E \to F$, there exists a projection, $P : F \to \phi(E)$, such that $\|P\| \leq \alpha$ (α-projection property).

It is said that a Banach space E is a N_α space, where α is a real number greater than or equal to 1, when there exists a collection $(E_\gamma)_{\gamma\in\Gamma}$ of finite dimensional subspaces of E, which is upwards directed, their union is dense in E and every one of them is a $\mathcal{P}_\alpha(\mathbb{K})$ space. Note that a Banach space is an L^1-predual space if and only if it is a N_α space for every $\alpha > 1$ ([**2**, theorem 2, pg. 232]).

Theorem 1. *If the Banach space E is an L^1-predual, then*

$$\lambda_f(E) = \lambda_f(\mathbb{K}).$$

Proof: We fix an $\alpha > 1$. Given that E is an L^1-predual, it is a N_α space, and, so, there exists a collection $(E_\gamma)_{\gamma\in\Gamma}$ of subspaces of E according to the definition above. We put $F = \bigcup_{\gamma\in\Gamma} E_\gamma$, which, because it is dense in E, satisfies $\lambda_f(E) = \lambda_f(F)$.

Let S be a finite subset of F with more than one point. There exists $\gamma \in \Gamma$ such that $S \subset E_\gamma$, and, if we indicate with subindices the Chebyshev radii in subspaces of E, we have

$$r(S) = r_F(S) \leq r_{E_\gamma}(S) \leq \delta(S).\lambda_f(E_\gamma) \leq \delta(S).\alpha.\lambda_f(\mathbb{K}),$$

where the last inequality is due to $E_\gamma \in \mathcal{P}_\alpha(\mathbb{K})$.

Therefore, $\lambda_f(E) = \lambda_f(F) \leq \alpha.\lambda_f(\mathbb{K})$, for every $\alpha > 1$, so $\lambda_f(E) \leq \lambda_f(\mathbb{K})$.

On the other hand, by the Hanh-Banach theorem, there exists a projection of norm 1 from E to $\mathbb{K}$, so $\lambda_f(\mathbb{K}) \leq \lambda_f(E)$. ∎

If ${}^*E$ is a non-standard enlargement of E, then, over the set $\text{fin}^*E = \{x \in {}^*E : \exists y \in E, \|x - y\|$ is a finite hyperreal number$\}$ of the finite elements of ${}^*E$, consider the equivalence relation "x is infinitely close to y", denoted by $x \cong y$, and defined by "$\|x - y\|$ is infinitesimal". In the quotient set, denoted $\hat{E}$, the norm $\|\hat{x}\| = st\|x\|$, $\hat{x} \in \hat{E}$, is defined, and the resulting normed space is called an infinitesimal hull of E.

Lemma 2. $\lambda_f(\hat{E}) = \lambda_f(E)$.

Proof: Let S be a finite subset of E with more than one point. Then, S is a finite subset of fin$^*E$ without infinitely close points.

It is obvious that $\delta(\hat{S}) = \delta(S)$ and $r(\hat{S}) \leq r(S)$. We suppose that $r(\hat{S}) < r(S)$, and take a real number t such that $r(\hat{S}) < t < r(S)$. Then, there exists $c \in$ fin$^*E$ such that $\hat{S} \subset B[\hat{c}, t]$, and so, $\|x - c\| < r(S)$ for every $x \in S$. Since S is finite, $^*S = S$, and we have a $c \in {}^*E$ such that $\|x - c\| < r(S)$ for every $x \in {}^*S$. Applying the Transfer Principle, there exists a standard element $c \in E$ such that $\|x - c\| < r(S)$ for every $x \in S$, and again because S is finite, this would imply $S \subset B[c, \rho]$, with $\rho = \max_{x \in S} \|x - c\| < r(S)$. Therefore, it is true $r(\hat{S}) = r(S)$ and we conclude $\lambda_f(E) \leq \lambda_f(\hat{E})$.

Let S be now a finite subset of fin$^*E$ with some points not infinitely close. Then, S is a *-finite subset of $^*E$ with some points not infinitely close and $\hat{S}$ is a finite subset of $\hat{E}$ such that $\delta(\hat{S}) \cong {}^*\delta(S)$.

Since the relation $r(T) \leq \delta(T).\lambda_f(E)$ is true for every finite subset T of E, by the Transfer Principle, we have $^*r(T) \leq {}^*\delta(T).\lambda_f(E)$ for every *-finite T, and, in particular, $^*r(S) \leq {}^*\delta(S).\lambda_f(E) \cong \delta(\hat{S}).\lambda_f(E)$.

Let t be a hiperreal number such that $t > {}^*r(S)$, $t \cong \delta(\hat{S}), \lambda_f(E)$. There exists a $c \in {}^*E$ such that $S \subset B[c, t]$, and then,

$$\|\hat{x} - \hat{c}\| = st\|x - c\| \cong \|x - c\| \leq t \cong \delta(\hat{S}).\lambda_f(E), \; \forall \hat{x} \in \hat{S}.$$

Since the first and last members are standard, we have $\|\hat{x} - \hat{c}\| \leq \delta(\hat{S}).\lambda_f(E)$, for every $\hat{x} \in \hat{S}$, so that $r(\hat{S}) \leq \delta(\hat{S}).\lambda_f(E)$, and we conclude $\lambda_f(\hat{E}) \leq \lambda_f(E)$. ∎

Theorem 3. *If E is a real Banach space such that $\lambda_f(E) = 1/2$, then E is an L^1-predual.*

Proof: We consider an infinitesimal hull $\hat{E}$ of E. Then, $\hat{E}$ has the radial intersection property (2,4), that is, given four closed balls in $\hat{E}$ with the same radius, which intersect in pairs, the total intersection is non empty.

Indeed, let ρ be a positive real number, and let $\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4 \in \hat{E}$, such that $\|\hat{x}_i - \hat{x}_j\| \leq 2\rho$, for $i, j = 1, 2, 3, 4$. We take $S = \{\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4\}$, a finite subset of $\hat{E}$ with $\delta(S) \leq 2\rho$, and, so, $r(S) \leq \rho$. For every natural number p, there exists $\hat{c}_p \in \hat{E}$ such that $\|\hat{x}_i - \hat{c}_p\| < \rho + (1/2p)$, for $i = 1, 2, 3, 4$, and, consequently, there exists $c_p \in$ fin$^*E$ such that $\|x_i - c_p\| < \rho + (1/p)$, for $i = 1, 2, 3, 4$. We consider now the sequence $(c_p)_{p \in \mathbb{N}}$ in $^*E$, which can be enlarged to an internal sequence $(c_p)_{p \in {}^*\mathbb{N}}$ in $^*E$. The set of index $p \in {}^*\mathbb{N}$ such that $\|x_i - c_p\| < \rho + (1/p)$, for $i = 1, 2, 3, 4$ is an internal subset of $^*\mathbb{N}$ containing all standard natural numbers, and, so, if we work in a suitably saturated model (cf. [**3**]), it contains an infinite index, $\omega \in {}^*\mathbb{N}$. Then, the element $c_\omega \in {}^*E$ is finite, because $\|x_i - c_\omega\| \le \rho$, so that we can take $\hat{c}_\omega \in \hat{E}$ thus verifying that $\|\hat{x}_i - \hat{c}_\omega\| \le \rho$, for $i = 1, 2, 3, 4$.

Therefore, $\hat{E}$ is an L^1-predual ([**2**, theorem 6, pg. 212]), that is, $\hat{E}'$ is an L^1 space. Projecting $\hat{E}'$ over E' by means of the function $T \in \hat{E}' \to T|_E \in E'$, we have that E' is also an L^1 space ([**2**, theorem 3, pg. 162]), and then E is an L^1-predual. ■

Lemma 4. *$\lambda_b(E)$ is the infimum of the positive real numbers r such that for every $\gamma > 0$, whenever $(x_\alpha)_{\alpha \in I} \subset E$ is a γ-Cauchy net (that is, given $\varepsilon > 0$, there exists $\alpha_0 \in I$ such that $\|x_\alpha - x_\beta\| \le \gamma + \varepsilon$, for every pair of subindices $\alpha, \beta \in I$ greater than or equal to α_0), $(x_\alpha)_{\alpha \in I}$ has some $r\gamma$-limit x in E (that is, given $\varepsilon > 0$, there exists $\alpha_0 \in I$ such that $\|x_\alpha - x\| \le r\gamma + \varepsilon$, for every subindex $\alpha \in I$ greater than or equal to α_0).*

Proof: Let S be a bounded subset of E, with more than one point. For every natural number n, we consider $S^{(n)} = S \times \{n\}$ and the bijection

$$\begin{aligned} S &\longrightarrow S^{(n)} \\ x &\longmapsto x^{(n)} = (x, n). \end{aligned}$$

We take now $I = \bigcup_{n \in \mathbb{N}} S^{(n)}$ with the order relation

$$\alpha \le \beta \iff \alpha = \beta \vee (\alpha = x^{(n)}, \beta = y^{(m)}, n < m),$$

which makes I a directed set. Over it, we build the net $(x_\alpha)_{\alpha \in I}$ defined by $x_\alpha = x$ if $\alpha \in I$ is such that $\alpha = x^{(n)}$, for some $n \in \mathbb{N}$. Thus, $(x_\alpha)_{\alpha \in I}$ is a $\delta(S)$-Cauchy net in E, with range S, and such that for every $y \in S$ and every $\alpha \in I$, there exists a $\beta \in I$, $\beta \ge \alpha$, which satisfies $x_\beta = y$, that is, for every $y \in S$ there exists and infinite index β which satisfies $x_\beta = y$.

We call $\lambda'(E)$ the infimum of the positive real numbers r such that for every $\gamma > 0$, every γ-Cauchy net has an $r\gamma$-limit in E, and let t be greater than $\lambda'(E)$. The previously builded net $(x_\alpha)_{\alpha \in I}$ has a $t\delta(S)$-limit $x \in E$, and, so, $\|x_\alpha - x\| \lesssim t\delta(S)$ for every infinite index α. Hence, we have $\|y - x\| \lesssim t\delta(S)$ for every $y \in S$, and, because both members are standard, $\|x - y\| \le t\delta(S)$ for every $y \in S$, that is, $S \subset B[s, t\delta(S)]$. Thus, $r(S) \le t\delta(S)$ for every $t > \lambda'(E)$, and $\lambda_b(E) \le \lambda'(E)$.

Conversely, if $(x_\alpha)_{\alpha \in I}$ is a γ-Cauchy net in E for some $\gamma > 0$, we put $S_\alpha = \{x_\beta : \beta \in I, \beta \ge \alpha\}$, for every $\alpha \in I$. Thus, every set S_α is bounded and we can suppose that it has more than one point (otherwise the proof is trivial); therefore, given $\varepsilon > 0$, there exists $\alpha \in I$ such that $\delta(S_\alpha) < \gamma + \varepsilon/\lambda_b(E)$. Then,

$$r(S_\alpha) \le \delta(S_\alpha).\lambda_b(E) < (\gamma + \varepsilon/\lambda_b(E)).\lambda_b(E) = \gamma.\lambda_b(E) + \varepsilon,$$

and, so, there exists, $c_\varepsilon \in E$ such that $\|x_\beta - c_\varepsilon\| \le \gamma.\lambda_b(E) + \varepsilon$ for every $\beta \ge \alpha$. Hence, c_ε is a $(\gamma.\lambda_b(E) + \varepsilon)$-limit of $(x_\alpha)_{\alpha \in I'}$ and, since this is true for every $\varepsilon > 0$ and for every γ-Cauchy net in E, it follows that $\lambda'(E) \le \lambda_b(E)$. ■

Theorem 5. *If E is a $\mathcal{P}_1(\mathbb{K})$ space, then $\lambda_b(E) = \lambda_b(\mathbb{K})$.*

Proof: If we embed $\mathbb{K}$ into E by means of a linear isometry, identifying it with a onedimensional subspace of E, the Hahn-Banach theorem assures the existence of a projection of norm 1, $P : E \to \mathbb{K}$, whence we deduce $\lambda_b(\mathbb{K}) \leq \lambda_b(E)$.

We will prove the reciprocal inequality in several stages:

(I) In the first place, we observe that if $E \in \mathcal{P}_1(\mathbb{K})$ and $\hat{E}$ is an infinitesimal hull of E, then $\lambda_b(E) \leq \lambda_b(\hat{E})$, because, considering the canonical linear isometry $E \to \hat{E}$, there exists a contractive projection $\hat{E} \to E$.

(II) Let Γ be a non empty set. We denote by $l^\infty(\Gamma, \mathbb{K})$ the set of all bounded functions from Γ to $\mathbb{K}$, with the uniform norm. Giving to Γ the discrete topology, we know that $l^\infty(\Gamma, \mathbb{K})$ is linearly isometric to the space $C(\beta\Gamma, \mathbb{K})$ of continuous functions with values in $\mathbb{K}$ defined over the Stone-Čech compactification of Γ; so, $l^\infty(\Gamma, \mathbb{K}) \in \mathcal{P}_1(\mathbb{K})$ ([**2**]), and $\lambda_b(l^\infty(\Gamma, \mathbb{K})) \leq \lambda_b(\hat{l}^\infty(\Gamma, \mathbb{K}))$, by (I).

(III) We suppose that Γ is a finite set. We will prove that, in this case, $\lambda_b(l^\infty(\Gamma, \mathbb{K})) \leq \lambda_b(\mathbb{K})$.

Because $l^\infty(\Gamma, \mathbb{K})$ is a finite dimensional, we know that its bounded and finite Chebyshev coefficients are equal, as are those of $\mathbb{K}$.

Let ρ be a real number, $\rho > \lambda_f(\mathbb{K})$, and let $S = \{x_1, \dots, x_n\}$ be a finite subset of $l^\infty(\Gamma, \mathbb{K})$. Fixed $\gamma \in \Gamma$, we consider the finite subset of $\mathbb{K}$ $S_\gamma = \{x_1(\gamma), \dots, x_n(\gamma)\}$; then, $r(S_\gamma)/\delta(S_\gamma) \leq \lambda_f(\mathbb{K}) < \rho$, and

$$r(S_\gamma) < \rho.\delta(S_\gamma) = \rho. \max_{1 \leq i,\, j \leq n} |x_i(\gamma) - x_j(\gamma)| \leq \rho. \max_{1 \leq i,\, j \leq n} \|x_i - x_j\| = \rho.\delta(S).$$

Then, there exists a centre $c_\gamma \in \mathbb{K}$ such that $S_\gamma \subset B[c_\gamma, \rho.\delta(S)] \subset \mathbb{K}$. We define thus a function from Γ in $\mathbb{K}$ which associates c_γ to every γ, and which satisfies

$$\|x_i - c\| = \sup_{\gamma \in \Gamma} |x_i(\gamma) - c| \leq \rho.\delta(S),\ i = 1, \dots, n,$$

so that $S \subset B[c, \rho.\delta(S)] \subset l^\infty(\Gamma, \mathbb{K})$. Therefore, $r(S) \leq \rho.\delta(S)$, and we conclude $\lambda_f(l^\infty(\Gamma, \mathbb{K})) \leq \lambda_f(\mathbb{K})$.

(IV) The inequality $\lambda_b(l^\infty(\Gamma, \mathbb{K})) \leq \lambda_b(\mathbb{K})$ is valid also when Γ is an infinite set.

Indeed, let $(x_\alpha)_{\alpha \in I}$ be a γ-Cauchy net in $l^\infty(\Gamma, \mathbb{K})$, for $\gamma > 0$. We take $X_0 = \mathbb{K} \cup \Gamma \cup I$ in order to build a superstructure $\mathcal{X}$ with base X_0 and over it a polysaturated nonstandard model satisfying the $\aleph_0$-isomorphism property (cf. [**3**, sec. 0.4.]). In this case, we can identify $\hat{l}^\infty(\Gamma, \mathbb{K})$ to $\hat{l}^\infty(\omega, \mathbb{K})$, for every infinite natural number ω, since $\hat{l}^\infty(\omega, \mathbb{K})$ is isometrically isomorphic to $\hat{l}^\infty(\mathbb{N}, \mathbb{K})$, and this is so to $\hat{l}^\infty(\Gamma, \mathbb{K})$ ([**4**, theorem 2.11]).

Let p be a natural number. There exists an index $\alpha_p \in I$ such that $\|x_\alpha - x_\beta\| < \gamma + 1/(2p)$, when $\alpha, \beta \in {}^*I$, $\alpha, \beta \geq \alpha_p$. We consider the set

$S = \{x_\alpha : \alpha \in {}^*I, \alpha \geq \alpha_p\}$, an internal *-bounded subset of $l^\infty(\omega, \mathbb{K})$, where we can apply (III), and so

$$^*r(S)/^*\delta(S) \leq \lambda_b(\mathbb{K})$$
$$^*r(S) \leq \lambda_b(\mathbb{K})^*\delta(S) \leq \lambda_b(\mathbb{K})(\gamma + 1/(2p)) < t,$$

where $t = \lambda_b(\mathbb{K})(\gamma + 1/p)$. Since $^*r(S) < t$, there exists $c_p \in l^\infty(\omega, \mathbb{K})$ such that $S \subset B[c_p, t]$ and $\hat{c}_p \in \hat{l}^\infty(\omega, \mathbb{K}) = \hat{l}^\infty(\Gamma, \mathbb{K})$ satisfies $\|\hat{x}_\alpha - \hat{c}_p\| < t$, for $\alpha \in I$, $\alpha \geq \alpha_p$.

Bearing in mind that $l^\infty(\Gamma, \mathbb{K}) \in \mathcal{P}_1(\mathbb{K})$, consider the natural embedding $l^\infty(\Gamma, \mathbb{K}) \to \hat{l}^\infty(\Gamma, \mathbb{K})$; then, there exists a projection of norm 1, $P : \hat{l}^\infty(\Gamma, \mathbb{K}) \to l^\infty(\Gamma, \mathbb{K})$, and $x = P(\hat{c}_p)$ is an element of $l^\infty(\Gamma, \mathbb{K})$ which satisfies

$$\|x_\alpha - x\| = \|P(\hat{x}_\alpha) - P(\hat{c}_p)\| \leq \|\hat{x}_\alpha - \hat{c}_p\| \leq t, \ \alpha \in I, \ \alpha \geq \alpha_p.$$

Thus, for every $\rho > \lambda_b(\mathbb{K})$, taking a $p \in \mathbb{N}$ greater than the real number $\lambda_b(\mathbb{K})/\gamma(\rho - \lambda_b(\mathbb{K}))$, we have

$$\|x_\alpha - x\| \leq \lambda_b(\mathbb{K})(\gamma + \frac{1}{p}) < \rho\gamma, \ \alpha \in I, \ \alpha \geq \alpha_p,$$

that is, x is a $\rho\gamma$-limit of $(x_n)_n$ in $l^\infty(\Gamma, \mathbb{K})$. Since this is valid for every γ-Cauchy net in $l^\infty(\Gamma, \mathbb{K})$ and for any $\gamma > 0$ we conclude by Lemma 4 that $\lambda_b(l^\infty(\Gamma, \mathbb{K})) \leq \lambda_b(\mathbb{K})$.

(V) Now, we can embed E linearly and isometrically into $l^\infty(E, \mathbb{K})$ by means of the application

$$\begin{aligned} \phi : E &\longrightarrow l^\infty(E, \mathbb{K}) \\ x &\longrightarrow \phi(x) : E \longrightarrow \mathbb{K} \\ &\qquad\qquad y \longrightarrow \phi(x)(y) = f_y(x) \end{aligned}$$

where f_y is a continuous linear functional from E to $\mathbb{K}$ which satisfies $\|f_y\| = 1$ and $f_y(y) = \|y\|$, the existence of which is guaranteed by the Hahn-Banach theorem. So, there exists a projection of norm 1, $l^\infty(E, \mathbb{K}) \to E$, which permits us to deduce the inequality $\lambda_b(E) \leq \lambda_b(l^\infty(E, \mathbb{K}))$, and, from the result in the preceding paragraph, $\lambda_b(E) \leq \lambda_b(\mathbb{K})$. ■

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