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# APPROXIMATION PROBLEMS IN MODULAR SPACES OF DOUBLE SEQUENCES

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*Abstract*

Let $X$ denote the space of all real, bounded double sequences, and let $\Phi, \varphi, \Gamma$ be $\varphi$-functions. Moreover, let $\Psi$ be an increasing, continuous function for $u \geq 0$ such that $\Psi(0) = 0$.

In this paper we consider some spaces of double sequences provided with two-modular structure given by generalized variations and the translation operator.

We apply the $\gamma(\tilde{v}_\Phi, \tilde{\rho}_\varphi)$-convergence in $\tilde{X}(\Phi, \Psi)$ in order to obtain an approximation theorem by means of the $(m,n)$-translation, i.e. a result of the form $(\tau_{mn}x - x) \to 0$ in an Orlicz sequence space $l^\Gamma$.

## 1. Notation

**1.1.** A function $\varphi$ defined in the interval $[0,\infty)$, continuous and nondecreasing for $u \geq 0$ and such that $\varphi(u) > 0$ for $u > 0, \varphi(u) \to \infty$ as $u \to \infty$ and $\varphi(0) = 0$, is called a $\varphi$-function. We will consider three $\varphi$-functions $\Phi, \varphi$ and $\Gamma$. Moreover, let $\Psi$ be a nonnegative, nondecreasing function of $u \geq 0$ such that $\Psi(u) \to 0$ as $u \to 0+$, (see [**3**]).

**1.2.** Let $X$ be the space of all real, bounded double sequences. Throughout this paper sequences belonging to $X$ will be denoted by $x = (t_{\mu\nu}) = ((x)_{\mu\nu})$ or $(t_{\mu\nu})_{\mu,\nu=0}^{\infty} = ((x)_{\mu\nu})_{\mu,\nu=0}^{\infty}$ and $|x| = (|t_{\mu\nu}|)$, $y = (s_{\mu\nu})$, $x^p(t^p_{\mu\nu})$ for $p = 1, 2, \ldots$. By a convergent sequence we shall mean a double sequence converging in the sense of Pringsheim. The symbols $X_d$ or $X_i$ denote subspaces of the space $X$ such that, for every fixed $\overline{\mu}$ and for every fixed $\overline{\nu}$ the sequences $(t_{\overline{\mu}\nu})$ and $(t_{\mu\overline{\nu}})$ are nonincreasing or nondecreasing, respectively.

**1.3.** Let $\rho_\varphi : X \to \langle 0, \infty)$ be a functional generated by the $\varphi$-function $\varphi$ such that for arbitrary $x, y \in X$ and $\alpha, \beta \geq 0$.

1' $\rho_\varphi(0) = 0$,

1" $\rho_\varphi(x) = 0$ implies $x = 0$,

2' $\rho_\varphi(-x) = \rho_\varphi(x)$,

3' $\rho_\varphi(\alpha x + \beta y) \leq \rho_\varphi(x) + \rho_\varphi(y)$, for $\alpha + \beta = 1$,

3" $\rho_\varphi(\alpha x+\beta y)\le \alpha\rho_\varphi(x)+\beta\rho_\varphi(y)$, for $\alpha+\beta=1$,
3"' $\rho_\varphi(\alpha x+\beta y)\le \alpha^{\overline{s}}\rho_\varphi(x)+\beta^{\overline{s}}\rho_\varphi(y)$, for $\alpha^{\overline{s}}+\beta^{\overline{s}}=1$, $0<\overline{s}\le 1$.

The functional $\rho_\varphi$ is called *pseudomodular* or *modular*, if satisfies the conditions 1', 2', 3' or 1', 1", 2', 3', respectively. If in place of 3' there holds 3", then $\rho_\varphi$ is called convex *psedomodular* or *convex modular*, respectively. In the case when in place of 3' there holds 3"' then we have $\overline{s}$*-convex pseudomodular* or $\overline{s}$*-convex modular*, respectively. Let us remark, that if $\rho_\varphi$ is pseudomodular or modular in $X$, then $X_{\rho_\varphi}=\{x\in X:\rho_\varphi(\lambda x)\to 0 \text{ as } \lambda\to 0+\}$ is called a *modular space*. Evidently, $X_{\rho_\varphi}$ is a vector subspace of $X$; if $\rho_\varphi(x)=\sum_{\mu,\nu=0}^{\infty}\varphi(|t_{\mu\nu}|)$ then $\rho_\varphi$ is modular and moreover the modular space $X_{\rho_\varphi}$ is the Orlicz sequence space $l^\varphi$.

It is well know that if $\rho_\varphi$ is convex modular, then a sequence $x^p=(t^p_{\mu\nu})$, $p=1,2,\dots$ of elements of $X_{\rho_\varphi}$ is $\rho_\varphi$-bounded if, and only if, there exists a positive constant $k$ such that $\rho_\varphi(kx^p)\le 1$ for $p=1,2,\dots$ . Modular convergence $x^p\overset{\rho_\varphi}{\to} Tx$ in $X_{\rho_\varphi}$ means that $\rho_\varphi(k(x^p-x))\to 0$ as $p\to\infty$ for a $k>0$ depending on the sequence $(x^p)$, (compare [2], [3] and also [4]).

**1.4.** Now we shall consider a two-modular space $\langle X,\rho,\rho'\rangle$, where $\rho$ is convex modular and $\rho'$ is modular. A sequence $(x^p)$ of elements $X$ is called $\gamma$-*convergent* to $x\in X$, if $x^p\overset{\rho'}{\to} x$ as $p\to\infty$ and $(x^p)$ is $\rho$-bounded. We denote this by $x^p\overset{\gamma(\rho,\rho')}{\longrightarrow} x$ or shortly $x^p\overset{\gamma}{\to} x$. A sequence $(x^p)$ of elements of $X$ satisfies the $\rho'$-Cauchy condition, if there exists a constant $k>0$ with the property that for every $\varepsilon>0$ there is an $N$ such that $\rho'(k(x^p-x^q))<\varepsilon$ for $p,q>N$. The two-modular space $\langle X,\rho,\rho'\rangle$ will be called $\gamma$-*complete*, if for every fixed $\rho$-ball $K$ in $X_\rho(K=\{x\in X:\rho(k_0x)\le M_0,\ k_0 \text{ and } M_0 \text{ are some positive numbers }\})$, any sequence $(x^p)$ of elements of $K$ satisfying the $\rho'$-Cauchy condition, is $\gamma$-convergent to an element of $K$, (see for instance [4] and [6]).

## 2. Some subspaces of the space of double sequences

**2.1.** The $(m,n)$-translation of the sequence $x$ is defined as a sequence $((\tau_{mn}x)_{\mu\nu})$ where

$$(\tau_{mn}x)_{\mu\nu}=\begin{cases} t_{\mu,\nu} & \text{for } \mu<m \text{ and } \nu<n,\\ t_{\mu+m,\nu} & \text{for } \mu\ge m \text{ and } \nu<n,\\ t_{\mu,\nu+n} & \text{for } \mu<m \text{ and } \nu\ge n,\\ t_{\mu+m,\nu+n} & \text{for } \mu\ge m \text{ and } \nu\ge n.\end{cases}$$

The $\varphi$-modulus of the sequence $x$ is defined by the formula

$$\omega_\varphi(x;r,s)=\sup_{m\ge r}\sup_{n\ge s}\sup_{\mu,\nu}\varphi(|(x)_{\mu\nu}-(\tau_{m0}x)_{\mu\nu}-(\tau_{0n}x)_{\mu\nu}+(\tau_{mn}x)_{\mu\nu}|).$$

An easy computation shows that

$$\omega_\varphi(x;r,s) = \sup_{m\geq r}\sup_{n\geq s}\sup_{\mu\geq m}\sup_{\nu\geq n}\varphi(|t_{\mu,\nu} - t_{\mu+m,\nu} - t_{\mu,\nu+n} + t_{\mu+m,\nu+n}|).$$

Let the function $\Psi$ and the convex $\varphi$-function $\varphi$ be given. The functional

$$\rho_\varphi(x) = \sup_{r,s} rs\Psi(\omega_\varphi(x;r,s))$$

defined for every $x \in X$ is pseudomodular in $X$, and in consequence we may define the modular space $X_{\rho_\varphi}$ and the respective $F$-norm $\|x\|_{\rho_\varphi} = \inf\{\varepsilon > 0 : \rho_\varphi(\frac{x}{\varepsilon}) \leq \varepsilon\}$, (compare [**5**]).

**2.2.** The $\Phi$-variation $v_\Phi$ of the sequence $x$ is defined by

$$v_\Phi(x) = \sup_{(m_\mu),(n_\nu)} \sum_{\mu,\nu=1}^{\infty} \Phi(|t_{m_{\mu-1},n_{\nu-1}} - t_{m_{\mu-1},n_\nu} - t_{m_\mu,n_{\nu-1}} + t_{m_\mu,n_\nu}|),$$

where the supremum runs through all increasing subsequences $(m_\mu)$ and $(n_\nu)$ of indices. It is easily seen that $v_\Phi$ is pseudomodular, defined in $X$. The symbols $\|\cdot\|_{v_\Phi}$ and $X_{v_\Phi} \equiv X_\Phi$ denote respectively an $F$-norm and a modular space, (see for instance [**2**] and [**5**]).

**2.3.** In the following we shall define two vector subspaces of the space $X$:

$$X_\varphi(\Psi) = \{x \in X : rs\Psi(\omega_\varphi(\lambda x;r,s)) \to 0 \text{ as } r,s \to \infty \text{ for a } \lambda > 0\},$$
$$X(\Phi,\Psi) \equiv X_\varphi(\Phi,\Psi) = X_\varphi(\Psi) \cap X_{v_\Phi}.$$

We see at once that $X_\varphi(\Phi,\Psi) \subset X_\varphi(\Psi) \subset X_{\rho_\varphi}$.

**2.4.** In the sequel, $\overline{c}$ denotes the space of all double sequences $x = (t_{\mu\nu})_{\mu,\nu=0}^{\infty}$ such that $t_{\mu 0} = t_{0\nu} = a$ for all $\mu,\nu$, and $t_{\mu\nu} = b$ for $\mu \geq 1$ and $\nu \geq 1$, where $a$ and $b$ are two arbitrary numbers.

It is easy to check that:

(a) if $\rho_\varphi(x) = 0$ then $\rho_\varphi(2x) = 0$ for all $x \in X$,
(b) for $x, y \in X_{\rho_\varphi}$ such that $x, y \in \overline{c}$ we have the inequality $\rho_\varphi(x) \leq \rho_\varphi(2y)$, where $\varphi$ is convex,
(c) the condition $x \in \overline{c}$ implies that $x$ is convergent,
(d) $\overline{c} = \{x \in X : \rho_\varphi(x) = 0\} \subset X_\varphi(\Phi,\Psi)$,
(e) $x \in \overline{c}$ if and only if $\|x\|_{\rho_\varphi} = 0$.

Applying results of [**1**] we shall consider quotient spaces

$$\tilde{X}_{\rho_\varphi} = X_{\rho_\varphi}/\overline{c},\ \tilde{X}_\varphi(\Psi) = X_\varphi(\Psi/\overline{c} \text{ and } \tilde{X}(\Phi,\Psi) = X(\Phi,\Psi)/\overline{c}$$

Their elements will be denoted by $\tilde{x}$, $\tilde{y}$, etc. Moreover, applying the properties of $\overline{c}$, $\rho_\varphi$, $v_\Phi$, $\|\cdot\|_{\rho_\varphi}$, we may define the modular functionals $\tilde{\rho}_\varphi(\tilde{x}) = \inf\{\rho_\varphi(y) : y \in \tilde{x}\}$, $\tilde{v}_\Phi(\tilde{x}) = \inf\{v_\Phi(y) : y \in \tilde{x}\}$ and the norm $\|\tilde{x}\|_{\rho_\varphi} = \|x\|_{\rho_\varphi}$; we have also the formula $\tilde{X}_{\rho_\varphi} = (X_{\rho_\varphi}/\overline{c})_{\tilde{\rho}_\varphi}$, (compare [**5**]).

## 3. Completeness of a two-modular space

**3.1.** We are now going to investigate the completeness of two-modular space $\langle \tilde{X}(\Phi,\Psi), \tilde{v}_\Phi, \tilde{\rho}_\varphi \rangle$. The theorems on completeness of the spaces $\tilde{X}_{\rho_\varphi}$ and $\tilde{X}_\varphi(\Psi)$ with respect to the $F$-norm $\|\cdot\|_{\rho_\varphi}$ or the modular functional $\tilde{\rho}_\varphi$ have been obtained in [**7**] (compare also [**5**]). Let us remark that the space $\tilde{X}(\Phi,\Psi)$ is not complete with respect to $\|\cdot\|_{\rho_\varphi}$ and $\tilde{\rho}_\varphi$, respectively. Indeed, consider the following example.

Let $\Phi(u) = |u|$, $\varphi(u) = |u|$, $\Psi(u) = u^2$ and $x = (t_{\mu\nu})_{\mu,\nu=0}^\infty$, $x^p = (t^p_{\mu\nu})_{\mu,\nu=0}^\infty$, $p = 1, 2, \dots$, where

$$t_{\mu\nu} = \begin{cases} \frac{1}{(\mu+1)(\nu+1)} & \text{for } \mu = \nu, \\ 0 & \text{elsewhere}, \end{cases} \qquad t^p_{\mu\nu} = \begin{cases} t_{\mu\nu} & \text{for } \mu \le p \text{ and } \nu \le p, \\ 0 & \text{elsewhere}. \end{cases}$$

Since

$$\omega_\varphi(x^p; r, s) \le \sup_{m\ge r}\sup_{n\ge s}\sup_{p\ge\mu\ge m}\sup_{p\ge\nu\ge n} \frac{2}{(\mu+1)(\nu+1)} \le \frac{2}{(r+1)(s+1)},$$

$$rs\Psi\left(\omega_\varphi(x^p; r, s)\right) \le \frac{4}{(r+1)(s+1)} \to 0 \text{ as } r, s \to \infty$$

and

$$v_\Phi(x^p) = \sum_{1\le\mu,\nu\le p} (t_{\mu,\nu} + t_{\mu-1,\nu-1}) = 1 + \frac{1}{(p+1)^2} + 2\sum_{\mu=1}^{p-1} \frac{1}{(\mu+1)^2} < \infty,$$

then $x^p \in X(\Phi,\Psi)$. Further, if $r < p$ and $s < p$, we have

$$\omega_\varphi(x^p - x; r, s) \le \frac{2}{(p+1)^2},\ rs\Psi(\omega_\varphi(x^p - x; r, s)) \le \frac{4}{(p+1)^2},$$

if $r \ge p$ and $s \ge p$, we have

$$\omega_\varphi(x^p - x; r, s) \le \frac{2}{(r+1)(s+1)},\ rs\Psi(\omega_\varphi(x^p - x; r, s)) \le$$

$$\le \frac{4}{(r+1)(s+1)} \le \frac{4}{(p+1)^2}$$

and in consequence we obtain

$$\rho_\varphi(x^p - x) = \sup_{r,s} rs\Psi(\omega_\varphi(x^p - x; r, s)) \le \frac{4}{(p+1)^2} \to 0 \text{ as } p \to \infty.$$

This shows that $x^p \to x$ in the $F$-norm of $X_\varphi(\Psi)$. Moreover, we have

$$rs\Psi(\omega_\varphi(x; r, s)) \le \frac{4}{(r+1)(s+1)} \to 0 \text{ as } r, s \to \infty,$$

and so $x \in X_\varphi(\Psi)$. However

$$v_\Phi(x) = \sum_{\mu,\nu=1}^\infty |t_{\mu,\nu} + t_{\mu-1,\nu-1}| \ge 2\sum_{\mu,\nu=1}^\infty \frac{1}{(\mu+1)(\nu+1)} = \infty,$$

whence $x \notin X_\Phi$. Finally $x^p \in X(\Phi,\Psi)$, $\rho_\varphi(x^p - x) \to 0$ as $p \to \infty$, but $x \notin X(\Phi,\Psi)$.

**3.2.** In the sequel, for a given sequence $x \in X$ we define a new sequence $\overline{x} = (\overline{t}_{\mu\nu})_{\mu\nu=0}^{\infty}$ by the formulas

$$\overline{t}_{\mu\nu} = \begin{cases} t_{\mu 0} + a, & \text{for } \mu = 0,1,2,\dots \text{ and } \nu = 0, \\ t_{0\nu} + a, & \text{for } \mu = 0 \text{ and } \nu = 1,2,\dots, \\ t_{\mu\nu} + b, & \text{for } \mu \geq 1 \text{ and } \nu \geq 1, \end{cases}$$

where the constants $a$ and $b$ can be of the form $a = t_{\mu\nu} - t_{\mu 0}$, $b = t_{0\nu} - t_{00}$ ($\mu, \nu > 0$ are arbitrary indices). In the following we shall consider the sequence $\overline{x}$ defined by the constants $a = t_{11} - t_{10}$ and $b = t_{01} - t_{00}$.

**Remark.** The following identity holds $\tilde{v}_\Phi(\tilde{x}) = v_\Phi(\overline{x})$.

*Proof:* Since $\overline{x} \in \tilde{x}$, then by definition of $\tilde{v}_\Phi(\tilde{x})$ we have

$$(+) \qquad \tilde{v}_\Phi(\tilde{x}) \leq v_\Phi(\overline{x}).$$

Now, let $y = (s_{\mu\nu})_{\mu,\nu=0}^{\infty} \in \tilde{x}$, then $s_{\mu 0} = t_{\mu 0} + A$, $s_{0\nu} = t_{0\nu} + A$ for $\mu = 0,1,2,\dots$, $\nu = 1,2,\dots$ and $s_{\mu\nu} = t_{\mu\nu} + B$ for $\mu \geq 1$ and $\nu \geq 1$, where $A$ and $B$ are two arbitrary numbers. In the following we may define the sequence $\overline{y} = (\overline{s}_{\mu\nu})_{\mu,\nu=0}^{\infty}$, where $\overline{s}_{\mu 0} = t_{\mu 0} + A + a$, for $\mu = 0,1,2,\dots$, $\overline{s}_{0\nu} = t_{0\nu} + A + a$, for $\nu = 1,2,\dots$, and $\overline{s}_{\mu\nu} = t_{\mu\nu} + B + b$ for $\mu \geq 1$ and $\nu \geq 1$, with $a = t_{11} + B - t_{10} - A$ and $b = t_{01} = t_{00}$. Obviously, $v_\Phi(y) \geq v_\Phi(\overline{y})$ and $v_\Phi(\overline{y}) = v_\Phi(\overline{x})$. Hence, $v_\Phi(y) \geq v_\Phi(\overline{x})$ for every $y \in \tilde{x}$. In consequence

$$(++) \qquad \tilde{v}_\Phi(\tilde{x}) \geq v_\Phi(\overline{x}).$$

Finally, by (+) and (++) we obtain $\tilde{v}_\Phi(\tilde{x}) = v_\Phi(\overline{x})$. ■

**3.3. Theorem.** *Let $\Phi, \varphi$ be $\varphi$-functions and let $\Psi$ be the function defined as in 1.1., which satisifies the condition:*

> *there exists a $u_0 > 0$ such that for every $\delta > 0$ there is an $\eta > 0$ satisfying the inequality $\Psi(\eta u) \leq \delta \Psi(u)$ for all $0 \leq u \leq u_0$.*

*Then, the two-modular space $\langle \tilde{X}(\Phi, \Psi), \tilde{v}_\Phi, \tilde{\rho}_\varphi \rangle$ is $\gamma$-complete.*

*Proof:* Let us suppose that $\tilde{K}$ is a $\tilde{v}_\Phi$-ball in $\tilde{X}(\Phi, \Psi)$ and let $\tilde{x}^p \in \tilde{K}$ for $p = 1,2,\dots,(\tilde{x}^p)$ be a $\tilde{\rho}_\varphi$-Cauchy sequence. It is easily seen that the sequence $(\tilde{x}^p)$ is $\tilde{\rho}_\varphi$-convergent to an element $\tilde{x} \in \tilde{X}_\varphi(\Psi)$, (see [7] or compare [5]). In consequence $\tilde{x}^p \xrightarrow{\gamma} x$, where $\gamma = \gamma(\tilde{v}_\Phi, \tilde{\rho}_\varphi)$. Next, we show that $\tilde{x} \in \tilde{K}$. Taking the sequence $(x^p)$, such that $x^p \in \tilde{x}^p$, $x^p \in X_\Phi$ we may define the sequence $(\overline{x}^p)$. Of course, we have

$$v_\Phi(k_0 \overline{x}^p) \leq M_0$$

for some positive numbers $k_0$ and $M_0$. If $\overline{x}^p = (\overline{t}^p_{\mu\nu})$, then

$$\sum_{\mu,\nu=1}^{\infty} \Phi\left(k_0 \left|\overline{t}^p_{m_{\mu-1},n_{\nu-1}} - \overline{t}^p_{m_{\mu-1},n_\nu} - \overline{t}^p_{m_\nu,n_{\nu-1}} + \overline{t}^p_{m_\mu,n_\nu}\right|\right) \leq M_0$$

for all increasing sequences $(m_\mu)$ and $(n_\nu)$ of positive integers and for $p = 1, 2, \ldots$ . Since $\bar{t}^p_{\mu\nu} \to \bar{t}_{\mu\nu}$ as $p \to \infty$ for every $\mu$ and $\nu$, where $(\bar{t}_{\mu\nu}) = \bar{x}$, then we easily obtain

$$\sum_{\mu,\nu=1}^{\infty} \Phi\left(k_0 \left|\bar{t}_{m_{\mu-1},n_{\nu-1}} - \bar{t}_{m_{\mu-1},n_\nu} - \bar{t}_{m_\mu,n_{\nu-1}} + \bar{t}_{m_\mu,n_\nu}\right|\right) \leq M_0$$

for $(m_\mu), (n_\nu)$, $p$ as previously. Therefore $v_\Phi(k_0\bar{x}) \leq M_0$. Applying the above remark, we obtain $\tilde{v}_\Phi(k_0\tilde{x}) \leq M_0$, and consequently $\tilde{x} \in \tilde{K}$. ■

## 4. A theorem of approximation type

**4.1.** Let $\Phi, \varphi, \Psi, \Gamma$ be the functions defined as in part 1.1. We shall consider an Orlicz sequence space $l^\Gamma$ and the space $\tilde{X}(\Phi, \Psi)$, and we shall apply the $\gamma$-convergence in $\tilde{X}(\Phi, \Psi)$ in order to formulate a theorem of the form $\tau_{mn}x - x \to 0$ in the space $l^\Gamma$.

Let us denote $T(x, m, n, \mu, \nu) = |(\tau_{mn}x)_{\mu\nu} - (x)_{\mu\nu}|$ and $M(x, m, n, \mu, \nu) = |t_{\mu+m,\nu+n} - t_{\mu+m,\nu} - t_{\mu,\nu+n} + t_{\mu,\nu}|$, for all $m, n, \mu, \nu$.

**Lemma.**

(a) *If $x \in X_d$, then $T(x, m, n, \mu, \nu) \leq M(x, m, n, \mu, \nu)$ for all $m, n, \mu$ and $\nu$.*
(b) *If $x \in X_i$, then $T(x, m, n, \mu, \nu) \leq M(x, m, n, \mu, \nu)$ for all $m, n, \mu$ and $\nu$.*

*Proof (a):* For $\mu < m$ and $\nu < n$ we have $T(x, m, n, \mu, \nu) = 0$.

If $\mu \geq m$ and $\nu < n$, then $T(x, m, n, \mu, \nu) = |t_{\mu+m,\nu} - t_{\mu,\nu}| \leq |(t_{\mu,\nu+n} - t_{\mu+m,\nu+n}) + (t_{\mu+m,\nu} - t_{\mu,\nu})| = M(x, m, n, \mu, \nu)$.

If $\mu < m$ and $\nu \geq n$, then $T(x, m, n, \mu, \nu) = |t_{\mu,\nu+n} - t_{\mu,\nu}| \leq |(t_{\mu+m,\nu} - t_{\mu+m,\nu+n}) + (t_{\mu,\nu+n} - t_{\mu,\nu})| = M(x, m, n, \mu, \nu)$.

For $\mu \geq m$ and $\nu \geq n$ we have $T(x, m, n, \mu, \nu) = |t_{\mu+m,\nu+n} - t_{\mu,\nu}| \leq |(t_{\mu+m,\nu+n} - t_{\mu,\nu}) + (t_{\mu,\nu} - t_{\mu+m,\nu}) + (t_{\mu,\nu} - t_{\mu,\nu+n}) = M(x, m, n, \mu, \nu)$.

Finally $T(x, m, n, \mu, \nu) \leq M(x, m, n, \mu, \nu)$ for all $m, n, \mu$ and $\nu$. ■

*Proof (b):* For $\mu < m$ and $\nu < n$, $(\tau_{mn}x)_{\mu\nu} = t_{\mu\nu}$, then $T(x, m, n, \mu, \nu) = 0$.

If $\mu \geq m$ and $\nu < n$, then $T(x, m, n, \mu, \nu) = |t_{\mu+m,\nu} - t_{\mu,\nu}| \leq |(t_{\mu,\nu} - t_{\mu+m,\nu}) + (t_{\mu+m,\nu+n} - t_{\mu,\nu+n})| = M(x, m, n, \mu, \nu)$.

If $\mu < m$ and $\nu \geq n$, then $T(x, m, n, \mu, \nu) = |t_{\mu,\nu+n} - t_{\mu,\nu}| \leq |(t_{\mu,\nu} - t_{\mu,\nu+n}) + (t_{\mu+m,\nu+n} - t_{\mu+m,\nu})| = M(x, m, n, \mu, \nu)$.

For $\mu \geq m$ and $\nu \geq n$ we have $T(x, m, n, \mu, \nu) = |t_{\mu+m,\nu+n} - t_{\mu,\nu}| \leq |(t_{\mu,\nu} - t_{\mu+m,\nu+n}) + (t_{\mu,\nu+n} - t_{\mu,\nu}) + (t_{\mu+m,\nu} - t_{\mu,\nu})| = M(x, m, n, \mu, \nu)$.

Thus $T(x, m, n, \mu, \nu) \leq M(x, m, n, \mu, \nu)$ for all $m, n, \mu$ and $\nu$. ■

**4.2.** Let us suppose that the functions $\Phi, \varphi, \Gamma$ and $\Psi$ satisfy the following condition:

(i) There exist positive constants $a$, $b$, $u_0$ such that

$$\Gamma(au) \leq b\Phi(u)\Psi(\varphi(u)) \text{ for } 0 \leq u \leq u_0.$$

First let us remark that the condition (i) is equivalent to the following one:

(ii) For every $u_1 \geq 0$ there exists a constant $c > 0$ such that

$$\Gamma(cu) \leq b\Phi(u)\Psi(\varphi(u)) \text{ for } 0 \leq u \leq u_1, \text{ (for a proof see [\textbf{5}]).}$$

**4.3.** Let the functions $\Phi, \varphi, \Psi, \Gamma$ satisfy the assumptions 1.1. and 4.2., and let $v_\Phi(\lambda x) < \infty$ for a $\lambda > 0$.

**Theorem 1.** *If $x \in X_d$ or $x \in X_i$, then*

$$(*) \qquad \sum_{\mu,\nu}^{\infty} \Gamma\left(c\lambda \left|(\tau_{rs}x)_{\mu\nu} - (x)_{\mu\nu}\right|\right) \leq brs\Psi(\omega_\varphi(\lambda x; r, s))v_\Phi(\lambda x)$$

*for all nonnegative integers $r$ and $s$, where $c$ and $b$ are some positive constants.*

*Proof:* We limit ourselves to the case when $x \in X_d$. By Lemma we have $|(\tau_{mn}x)_{\mu\nu} - (x)_{\mu\nu}| \leq |t_{\mu,\nu} - t_{\mu+m,\nu} - t_{\mu,\nu+n} + t_{\mu+m,\nu+n}|$ for arbitrary $m, n, \mu$ and $\nu$. Let a positive constant $\lambda$ and integers $r$ and $s$ be given. Since $x$ is a bounded sequence, taking $u_1 = 4\lambda \sup_{\mu,\nu} |t_{\mu,\nu}|$, and choosing $m \geq r$, $n \geq s$ arbitrary, by (i) we obtain

$$\Gamma(c\lambda M(x,m,n,\mu,\nu)) \leq b\Phi(\lambda M(x,m,n,\mu,\nu))\Psi(\varphi(\lambda M(x,m,n,\mu,\nu)))$$

for all $m, n, \mu, \nu$ such that $\lambda M(x,m,n,\mu,\nu) \leq u_1$. We have

$$\sum_{\mu,\nu=0}^{\infty} \Gamma(c\lambda|(\tau_{mn}x)_{\mu\nu} - (x)_{\mu\nu}|) \leq$$

$$\leq b\Psi(\sup_{m\geq r}\sup_{n\geq s}\sup_{\mu\geq m}\sup_{\nu\geq n}\varphi(\lambda M(x,m,n,\mu,\nu))) \sum_{\mu\geq m,\nu\geq n} \Phi(\lambda M(x,m,n,\mu,\nu)) =$$

$$= b\Psi(\omega_\varphi(\lambda x; r,s)) \sum_{k,l=1}^{\infty} \sum_{\mu=km}^{(k+1)m-1} \sum_{\nu=ln}^{(l+1)n-1} \Phi(\lambda M(x,m,n,\mu,\nu)) =$$

$$= b\Psi(\omega_\varphi(\lambda x; r,s)) \sum_{k,l=1}^{\infty} \sum_{u=m}^{2m-1} \sum_{v=n}^{2n-1} \Phi(\lambda|t_{km+u,ln+v} - t_{km+u,(l-1)n+v} -$$

$$- t_{(k-1)m+u,ln+v} + t_{(k-1)m+u,(l-1)n+v}|) =$$

$$= b\Psi(\omega_\varphi(\lambda x; r,s)) \sum_{u=m}^{2m-1} \sum_{v=n}^{2n-1} \sum_{k,l=1}^{\infty} \Phi(\lambda|t_{km+u,ln+v} - t_{km+u,(l-1)n+v} -$$

$$- t_{(k-1)m+u,ln+v} + t_{(k-1)m+u,(l-1)n+v}|) \leq$$

$$\leq b\Psi(\omega_\varphi(\lambda x; r,s)) \sum_{u=m}^{2m-1} \sum_{v=n}^{2n-1} v_\Phi(\lambda x) = bmn\Psi(\omega_\varphi(\lambda x; r,s))v_\Phi(\lambda x).$$

Finally we obtain

$$\sum_{\mu,\nu=0}^{\infty} \Gamma(c\lambda|(\tau_{mn}x)_{\mu\nu} - (x)_{\mu\nu}|) \leq bmn\Psi(\omega_\varphi(\lambda x; r, s))v_\Phi(\lambda x)$$

for some positive constants $c$, $b$, $\lambda$ and for all $m \geq r$, $n \geq s$, where $r, s$ are nonnegative integers. Hence, taking $m = r$ and $n = s$, we get the inequality $(*)$. ■

**Theorem 2.** *Let $\Phi, \varphi, \Gamma$ be $\varphi$-functions ($\Phi$ convex) and let $\Psi$ have the same properties as in the previous theorem. Let $x \in \tilde{x} \in \tilde{X}(\Phi, \Psi)$ and $x \in X_d$ (or $x \in X_i$). Then $\tau_{rs}x - x \in l^\Gamma$ for all $r, s \geq 0$, and $\tau_{rs}x - x \to 0$ in the sense of modular convergence in $l^\Gamma$.*

*Proof:* First, let us remark that the condition $x \in X(\Phi, \Psi)$ implies that $v_\Phi(\lambda x) < \infty$ and $rs\Psi(\omega_\varphi(\lambda x; r, s)) < \varepsilon$ for sufficiently small $\lambda > 0$ and for sufficiently large $r$ and $s$, where $\varepsilon$ is an arbitrary positive number. But, an easy computation shows that if the $\varphi$-function $\Phi$ is convex then the conditions $x \in X_\Phi$ and $v_\Phi(kx) < \infty$ for some positive constant $k$ are equivalent. Applying this observation and Theorem 1, we conclude that $\tau_{rs}x - x \in l^\Gamma$ for all nonnegative integers $r$ and $s$. In order to get the condition $\tau_{rs}x - x \to 0$ in the sense of modular convergence in $l^\Gamma$, it will be necessary to take $r, s \to \infty$, in the inequality $(*)$. ■

**Theorem 3.** *Let $x^p = (t^p_{\mu\nu})^\infty_{\mu,\nu=0} \in X_\Phi$, $t^p_{\mu 0} = t^p_{0\nu} = 0$ for $p = 1, 2, \ldots$ where $\mu, \nu = 0, 1, 2, \ldots$, and let $x^p$, $p = 1, 2, \ldots$ belong to the $v_\Phi$-ball in $X_\Phi$, where $\Phi$ is an increasing $\varphi$-function. Then the set of sequences $(x^p)$ is uniformly bounded.*

*Proof:* By assumption $v_\Phi(k_0 x^p) \leq M_0$ for $p = 1, 2, \ldots$, where $k_0, M_0$ are some positive constants. In consequence, we have

$$\Phi(k_0|t^p_{\mu\nu}|) = \Phi(k_0|t^p_{00} - t^p_{0\nu} - t^p_{\mu 0} + t^p_{\mu\nu}|) \leq v_\Phi(k_0 x^p) \leq M_0.$$

Now, applying the properties of $\varphi$-function $\Phi$ we obtain that there exists a positive constant $M$ such that $|t^p_{\mu\nu}| \leq M$ for $\mu, \nu = 0, 1, 2, \ldots$ . ■

**Theorem 4.** *Let $\Gamma, \Phi, \varphi$ be $\varphi$-functions ($\Phi$ and $\varphi$ are convex) and let $\Psi$ be a nonnegative, nondecreasing function of $u \geq 0$ such that $\Psi(u) \to 0$ as $u \to 0+$. Let us suppose that the functions $\Phi, \varphi, \Psi$ and $\Gamma$ satisfy the condition 4.2.(i). Moreover, let $(x^p)$ be a sequence such that $t^p_{\mu 0} = t^p_{0\nu} = 0$ for $\mu, \nu = 0, 1, 2, \ldots$, $p = 1, 2, \ldots$, $x^p \in \tilde{x}^p$, $\tilde{x}^p \in \tilde{X}(\Phi, \Psi)$, $\tilde{x}^p \xrightarrow{\gamma} 0$ as $p \to \infty$ in $(\tilde{X}(\Phi, \Psi), \tilde{v}_\Phi, \tilde{\rho}_\varphi)$. Then $\tau_{rs}x^p - x^p \to 0$ with respect to modular convergence in $l^\Gamma$, as $p \to \infty$, uniformly for $r \geq 0$ and $s \geq 0$.*

*Proof:* The condition $\tilde{x}^p \xrightarrow{\gamma} 0$ implies that $\tilde{x}^p \in \tilde{K}$, where $\tilde{K}$ is a $\tilde{v}_\Phi$-ball, with parameters $k_0, M_0$, and by Theorem 3 we have $|t^p_{\mu\nu}| \leq M$ for all $\mu, \nu, p$

with an $M > 0$. Choosing $u_1 = 4\lambda M$, $c = a\frac{u_0}{u_1}$, where $0 < \lambda < k_0$, and applying the inequality $(*)$, we obtain

$$(+)\qquad \sum_{\mu,\nu=0}^{\infty} \Gamma(c\lambda|(\tau_{rs}x^p)_{\mu\nu} - (x^p)_{\mu\nu}|) \leq b\rho_\varphi(\lambda x^p)v_\Phi(\lambda x^p) \leq bM_0\rho_\varphi(\lambda x^p).$$

By assumption there exists a $\lambda > 0$ such that for every $\varepsilon > 0$ there is an integer $P$ for which

$$\tilde{\rho}_\varphi(2\lambda\tilde{x}^p) = \inf\{\rho_\varphi(y) : y \in 2\lambda\tilde{x}^p\} < \varepsilon$$

for all $p > P$. In consequence there exist $y^p \in 2\lambda\tilde{x}^p$, such that

$$(++)\qquad \rho_\varphi(y^p) < \varepsilon \text{ for } p > P.$$

Since

$$\rho_\varphi(\lambda x^p) = \rho_\varphi\left(\frac{y^p + (2\lambda x^p - y^p)}{2}\right) \leq \rho_\varphi(y^p) + \rho_\varphi(2(\lambda x^p - \frac{1}{2}y^p))$$

and

$$\frac{1}{2}y^p - \lambda x^p \in \overline{c},$$

then we have

$$(+++)\qquad \rho_\varphi(\lambda x^p) \leq \rho_\varphi(y^p), \text{ for } p > P.$$

By the inequalities $(++)$ and $(+++)$ we obtain

$$\rho_\varphi(\lambda x^p) < \varepsilon$$

for sufficiently large $p$. Finally, the condition $(+)$ implies that $\tau_{rs}x^p - x^p \to 0$ with respect to modular convergence in $l^\Gamma$ as $p \to \infty$, uniformly for $r, s \geq 0$. ■

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