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INDUCTIVE LIMITS OF VECTOR–VALUED SEQUENCE SPACES

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Abstract

Let L be a normal Banach sequence space such that every element in L is the limit of its sections and let $E =$ ind E_n be a separated inductive limit of locally convex spaces. Then ind $L(E_n)$ is a topological subspace of $L(E)$.

The aim of this note is to prove the following result on the interchangeability of inductive limits and spaces of vector valued sequences: if L is a normal Banach sequence space with the property that every element of L is the limit of its sections and $E =$ ind E_n is a separated locally convex inductive limit, then the inductive limit ind $L(E_n)$ is a topological subspace of $L(E)$. The situation is completely different for the sequence space $L = 1^\infty$. In fact the first two authors showed in [**2**] that there are even strict inductive limits of Fréchet spaces $E =$ ind E_n such that the canonical injection ind $1^\infty(E_n) \subset 1^\infty(E)$ is not open.

In what follows $(L, \| \, \|)$ denotes a normal Banach sequence space, i.e., a Banach space that satisfies

(α) $\varphi \subset L \subset \omega$ algebraically and the inclusion $(L, \| \, \|) \subset \omega$ is continous.

(β) $\forall\, a = (a_k)_{k \in \mathbb{N}} \in L \,\forall\, b = (b_k)_{k \in \mathbb{N}} \in \omega$ such that $|b_k| \leq |a_k| \,\forall\, k \in \mathbb{N}$, we have that $b \in L$ and $\|b\| \leq \|a\|$.

We will also assume the following property (cf [**1**])

(ε) $\lim\limits_{n \to \infty} \|((0)_{k<n}, (a_k)_{k \geq n})\| = 0, \forall\, a = (a_k)_{k \in \mathbb{N}} \in L$.

This property is sometimes called AK-property. Clearly $(L, \| \, \|) = 1^\infty$ does not satisfy (ε), whereas $(L, \| \, \|) = 1^p, 1 \leq p \quad \infty$ or c_0 has property (ε).

We observe that there is $(\mu_k)_{k \in \mathbb{N}} \in L$ with $\mu_k > 0\, (k \in \mathbb{N})$ and $\|(\mu_k)_{k \in \mathbb{N}}\| = 1$

Given a locally convex space E, we denote by $cs(E)$ the family of all continuous seminorms on E. Given E the vector valued sequence space $L(E)$ is defined by

$$L(E) = \{x = (x_k)_{k \in \mathbb{N}} \in E^{\mathbb{N}}; (r(x_k)_{k \in \mathbb{N}}) \in L \text{ for all } r \in cs(E)\}$$

endowed with the locally convex topology defined by the seminorms

$$x \longrightarrow \|(r(x_k)_{k\in\mathbb{N}})\|$$

as r varies in $cs(E)$. Clearly if $(L, \|\ \|)$ satisfies property (ε), then the countable direct sum $\oplus\{E : n \in \mathbb{N}\} = E^{(\mathbb{N})}$ is dense in $L(E)$.

Given a separated locally convex inductive limit $E = \operatorname{ind} E_n$ we are interested in the following question: *is* ind $L(E_n)$ *a topological subspace of* $L(E)$?. If $(L, \|\ \|) = 1^1$, a positive answer follows from a classical result of Grothendieck on projective tensor products (see e.g. [4]). If $(L, \|\ \|) = c_0$ the positive answer is a particular case of a result of Mujica [5,I,7]. We prove now that the answer is positive for arbitrary $(L, \|\ \|)$ satisfying (ε).

1. Proposition. *Let E be a locally convex space, F a closed subspace of E and $q : E \longrightarrow E/F$ the canonical surjection. The mapping $Q : L(E) \longrightarrow L(E/F)$ defined by $Q((x_k)_{k\in\mathbb{N}}) := (q(x_k))_{k\in\mathbb{N}}$ is open onto its image. If E is a Fréchet space then Q is also surjective.*

Proof: Since $E^{(\mathbb{N})}$ is a dense subspace of $L(E)$ and $Q(E^{(\mathbb{N})}) = (E/F)^{(\mathbb{N})}$, according to [4, 32, 5(3)] it is enough to show that $Q : E^{(\mathbb{N})} \longrightarrow (E/F)^{(\mathbb{N})}$ is open. To do this we fix $r \in cs(E)$ and we show

$$Q(\{x \in L(E); x \in E^{(\mathbb{N})} \|(r(x_k))_{k\in\mathbb{N}}\| \leq 1\}) \supset$$
$$\{\tilde{x} \in L(E/F); \tilde{x} \in (E/F)^{(\mathbb{N})} \|(\tilde{r}(\tilde{x}_k))_{k\in\mathbb{N}}\| \leq 2^{-1}\}$$

where $\tilde{r}(z + F) := \inf\{r(z + y); y \in F\}\,(z \in E)$ is the quotient seminorm. We fix $(\mu_k)_{k\in\mathbb{N}} \in L$, $\mu_k > 0 (k \in \mathbb{N})$, $\|(\mu_k)_{k\in\mathbb{N}})\| = 1$. Given $\tilde{x} \in (E/F)^{(\mathbb{N})}$ with $\|(\tilde{r}(\tilde{x}_k))_{k\in\mathbb{N}}\| \leq 2^{-1}$ we find $1 \in \mathbb{N}$ such that $\tilde{x}_k = 0$ for $1 \leq k$. For each $k < 1$ we select $y \in F$ such that $r(x_k + y_k) < \tilde{r}(x_k + F) + 2^{-1}\mu_k$. Then $x = ((x_k + y_k)_{k<1}, (0)_{1\leq k}))$ belongs to $E^{(\mathbb{N})} \subset L(E), Q(x) = \tilde{x}$ and $\|((r(x_k + y_k))_{k<1}, (0)_{1\leq k}))\| \leq 1$.

If E is also a Fréchet space, then $Q(L(E))$ is a Fréchet space dense in $L(E/F)$. Consequently Q is surjective. ∎

2. Proposition. *Let $(E_n)_{n\in\mathbb{N}}$ be a sequence of locally convex spaces. Then the map $\psi : L(\oplus\{E_n : n \in \mathbb{N}\}) \longrightarrow \oplus\{L(E_n) : n \in \mathbb{N}\}$ defined by*

$$\psi(((x_n^k)_{n\in\mathbb{N}})_{k\in\mathbb{N}}) := ((x_n^k)_{k\in\mathbb{N}})_{n\in\mathbb{N}}$$

is a topological isomorphism.

Proof: Given $x = ((x_n^k)_{n\in\mathbb{N}})_{k\in\mathbb{N}}$ in $L(\oplus\{E_n : n \in \mathbb{N}\})$, to show that $\psi(x) \in \oplus\{L(E_n) : n \in \mathbb{N}\}$ it is enough to see that there is $m \in \mathbb{N}$ such that $x_n^k = 0$ for all $n \geq m$, $k \in \mathbb{N}$. If we assume the contrary we can find two strictly increasing sequences $(k(j))_{j\in\mathbb{N}}$ and $(n(j))_{j\in\mathbb{N}}$ such that $x_{n(j)}^{k(j)} \neq 0$ for all $j \in \mathbb{N}$ (recall that each $(x_n^k)_{n\in\mathbb{N}}$ belongs to $\oplus\{E_n : n \in \mathbb{N}\}$). We select $(\lambda_k)_{k\in\mathbb{N}} \in \omega\backslash L$ with $\lambda_{k(j)} > 0$ for all $j \in \mathbb{N}$ and $\lambda_k = 0$ if $k \notin \{k(j); j \in \mathbb{N}\}$. For all $j \in \mathbb{N}$ we find $r_j \in cs(E_{n(j)})$ with $r_j(x_{n(j)}^{k(j)})$ greater than $\lambda_{k(j)}$. It is clear that $r((z_n)_{n\in\mathbb{N}}) = = \sum_{j=1}^{\infty} r_j(z_{n(j)})$ defines a continous seminorm on $\oplus\{E_n : n \in \mathbb{N}\}$. Therefore for $x^k := (x_n^k)_{n\in\mathbb{N}}$ $(k \in \mathbb{N})$, we have $(r(x^k))) \in L$. But $r(x^{k(j)}) \geq r_j(x_{n(j)}^{k(j)}) > \lambda_{k(j)}$, for all $j \in \mathbb{N}$ and $0 = \lambda_k \leq r(x^k)$ if $k \notin \{k(j); j \in \mathbb{N}\}$. Consequently $(\lambda_k)_{k\in\mathbb{N}} \in L$, a contradiction. Therefore ψ is well defined. Clearly ψ is linear and injective. To show that ψ is surjective, we take $x = ((x_n^k)_{k\in\mathbb{N}})_{n\in\mathbb{N}}$ in $\oplus\{L(E_n) : n \in \mathbb{N}\}$. Clearly $(x_n^k)_{n\in\mathbb{N}} \in \oplus\{E_n; n \in \mathbb{N}\}$ for all $k \in \mathbb{N}$, since $x_n^k = 0$ for all $n \geq m$ and $k \in \mathbb{N}$. Given $r \in cs(\oplus\{E_n; n \in \mathbb{N}\})$ we can find $r_n \in cs(E_n)$ $n \in \mathbb{N}$, with $r(z) \leq \max\,(r_n(z_n); n \in \mathbb{N})$ for all $z = (z_n) \in \oplus\{E_n; n \in \mathbb{N}\}$. Therefore for all $k \in \mathbb{N}$

$$r((x_n^k)_{n\in\mathbb{N}}) \leq \max\,(r_n(x_n^k); 1 \leq n \leq m) \leq \sum_{n=1}^{m} r_n(x_n^k)$$

Since $(r_n(x_n^k)_{k\in\mathbb{N}}) \in L$ for $1 \leq n \leq m$, we conclude $y = ((x_n^k)_{n\in\mathbb{N}})_{k\in\mathbb{N}} \in L(\oplus\{E_n; n \in \mathbb{N}\})$ and $\psi(y) = x$.

Now the continuity of $\psi^{-1} : \oplus\{L(E_n); n \in \mathbb{N}\} \longrightarrow L(\oplus\{E_n; n \in \mathbb{N}\})$ follows from the fact that its restriction to each $L(E_n)$ is clearly continuous. Finally we show that ψ is continuous. To do this we consider $r_n \in cs(E_n)$ $(n \in \mathbb{N})$ and we observe that

$$\sup_{n\in\mathbb{N}} \|(r_n(x_n^k))_{k\in\mathbb{N}})\| \leq \|(\sup_{n\in\mathbb{N}} (r_n(x_n^k))_{k\in\mathbb{N}})\|$$

holds for every $((x_n^k)_{n\in\mathbb{N}})_{k\in\mathbb{N}} \in L(\oplus\{E_n; n \in \mathbb{N}\})$. ■

3. Theorem. *Let $(L, \|.\|)$ be a normal Banach sequence space with property (ε). Let $E =$ ind E_n be a separated locally convex inductive limit. Then [ind $L(E_n)$] is a topological subspace of $L($ ind $E_n)$.*

Proof: We consider the following diagram

$$\begin{array}{ccc} L(\oplus\{E_n; n \in \mathbb{N}\}) & \xrightarrow{Q_1} & L(E) \\ \psi\uparrow & & \uparrow\varphi \\ \oplus\{L(E_n); n \in \mathbb{N}\} & \xrightarrow{Q_2} & \text{ind } L(E_n) \end{array}$$

where, for $q_1 : \oplus\{E_n; n \in \mathbb{N}\} \longrightarrow E$ the canonical quotient map $q_1((z_n)_{n\in\mathbb{N}}) = = \sum_{n=1}^{\infty} z_n$ we define $Q_1((x_k)_{k\in\mathbb{N}}) = (q_1(x_k))_{k\in\mathbb{N}}$ for all $(x_k)_{k\in\mathbb{N}}$ in

$L(\oplus\{E_n; n \in \mathbb{N}\})$. Q_2 is the canonical quotient map and φ is the canonical injection which is continous. According to proposition 1, Q_1 is open onto its image. Certainly Q_2 is open and ψ^{-1} is a topological isomorphism, according to proposition 2. Since the diagram is commutative, it follows that φ is also open onto its image. Thus ind $L(E_n)$ is a topological subspace of $L(E)$. ∎

4. Corollary. *Let $(L, \| \|)$ be a normal Banach sequence space with property (ε). Let $E =$ ind E_n be a strict inductive limit of locally convex spaces with E_n closed in E_{n+1} for all $n \in \mathbb{N}$. Then $L(E) =$ ind $L(E_n)$ holds algebraically and topologically.*

Proof: Only the algebraic identity needs a proof. It is a clearly enough to show that for any $x = (x_k)_{k\in\mathbb{N}} \in L(E)$ there is $n \in \mathbb{N}$ with $x_k \in E_n$. If this is not satisfied we can find an increasing sequence $(k(n))_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that $x_{k(n)} \notin E_n$, for all n in $\mathbb{N}$. We select $(\gamma_k)_{k\in\mathbb{N}} \in \omega\backslash L$ with $\gamma_{k(n)} > 0\,(n \in \mathbb{N})$ and $\gamma_k = 0$ if $k \notin \{k(n); n \in \mathbb{N}\}$. Now since E_n is closed , there is $u_n \in E'$ with $u_n(x_{k(n)}) = \gamma_{k(n)}$ and $u_n|E_n = 0$. The equicontinous sequence $(u_n)_{n\in\mathbb{N}}$ defines a continous seminorm as follows:

$$p(x) = \sup\{|u_n(x)|; n \in \mathbb{N}\}$$

Thus $(p(x_k))_{k\in\mathbb{N}} \in L$, a contradiction, since $\gamma_k \leq p(x_k)$ for all $k \in \mathbb{N}$. ∎

5. Remark: For an inductive limit $E =$ ind E_n and a normal Banach sequence space $(L, \| \|)$, the algebraic coincidence $L(E) =$ ind $L(E_n)$ is a clearly equivalent to $\forall x \in L(E)\, \exists n \in \mathbb{N}$ with $x \in L(E_n)$. For instance if $(L, \| \|) = c_0$, then $L(E) =$ ind $L(E_n)$ if and only if E is a sequentially retractive (cf [**3**]).

References

1. J. BONET, S. DIEROLF, On (LB)-spaces of Moscatelli type, *DOGA Tr. J. Math. (to appear in 1989).*
2. J. BONET, S. DIEROLF, Countable inductive limits and the bounded decomposition property, *Proc. Roy Ir. Acad. (To appear).*
3. K. FLORET, Folgenretraktive Sequenzen Lokalkonvexer Räume, *J. reine angew Math* **259** (1973), 65–85.
4. G. KÖTHE, "*Topological vector spaces II*," Springer, Berlin, Heidelberg, New York, 1979.
5. J. SCHMETS, "*Spaces of vector valued continuous functions*," Springer, Berlin, Heidelberg, New York, 1983.

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