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IS THE PRODUCT OF CCC SPACES A CCC SPACE?

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This paper is dedicated to John C. Oxtoby, who sparked the author's interest in the subject.

Abstract

In this expository paper it is shown that Martin's Axiom and the negation of the Continuum Hypothesis imply that the product of *ccc* spaces is a *ccc* space. The Continuum Hypothesis is then used to construct the Laver–Galvin example of two *ccc* spaces whose product is not a *ccc* space

1. Introduction

A *ccc space* is a topological space which satisfies the *countable chain condition*: Each family of (pairwise) disjoint nonempty open sets is countable. A separable space, for instance, is a *ccc* space. This is so because given a family $\mathcal{F}$ of disjoint nonempty open sets in a separable space X, one can define an injection of $\mathcal{F}$ into a countable dense subset S of X by choosing one point of S in each member of $\mathcal{F}$. A *ccc* space, however, need not be separable. For example, if I is a set with cardinality greater than $|\mathbb{R}|$ (the cardinality of the set of real numbers) and for each $i \in I$, $X_i = \{0, 1\}$ with the discrete topology, then the product space $\Pi_{i\in I} X_i$ is *ccc* but not separable [**10**, p. 51]. A simple example of a topological space which is not *ccc* is any uncountable set with the discrete topology. In a 1947 paper [**12**], E. Marczewski gave a proof (which may also be found in [**15**]) of Pondiczery's theorem that the product of at most $|\mathbb{R}|$ separable spaces is separable; and he raised the question as to whether the product of (just) two *ccc* spaces is a *ccc* space. It turns out that special axioms are needed to answer this question: it has a negative answer in the presence of the Continuum Hypothesis (CH) and an affirmative answer when Martin's Axiom and the negation of CH are assumed. The main goal of this paper is to give fairly self–contained proofs of these two assertions, which we do in Sections

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2 and 3. Related topics (Souslin's Hypothesis, Property (K)) are touched upon in Sections 4 and 5. None of the results and proofs are new.

First, some preliminaries. Let us recall that an ordinal number may be identified with the set of all smaller ordinal numbers; thus for ordinals α and β, the statements $\alpha < \beta$, $\alpha \epsilon \beta$, and $\alpha \subset \beta$ are all equivalent. The smallest infinite ordinal and the smallest uncountable ordinal will be denoted by ω and ω_1 respectively. We will need the fact that ω and ω_1 are also cardinal numbers. (A good reference for cardinal and ordinal numbers is [**6**].) The cardinality of the set of all functions from ω to 2 is 2^ω, and $2^\omega = |\mathbb{R}|$. The *Continuum Hypothesis* (CH) is the statement $\omega_1 = 2^\omega$. The negation of CH, denoted here by $\neg CH$, is therefore the statement $\omega_1 < 2^\omega$. Gödel and Cohen proved that CH is independent of the usual axioms of set theory, namely, the Zermelo–Fraenkel axioms together with the Axiom of Choice (usually denoted by ZFC).

We review the definition of the Gleason space of a topological space X, for use several times in the sequel. An open subset U of X is called a *regular* open set if U is equal to the interior of its closure, i.e., $U = (\bar{U})^0$. The set $\mathcal{R}(X)$ of all regular open subsets of X forms a complete Boolean algebra with respect to the operations $U \wedge V = U \cap V, U \vee V = (\overline{U \cup V})^0$, and $U' = (X \backslash U)^0$. The *Gleason space* of X, denoted here by $G(X)$, is the Stone space of $\mathcal{R}(X)$. The space $G(X)$ is constructed as follows. A subset $\mathcal{F}$ of $\mathcal{R}(X)$ is called a *filter* of $\mathcal{R}(X)$ if (i) $\phi \notin \mathcal{F}$, (ii) $U \cap V \epsilon \mathcal{F}$ for all $U, V \epsilon \mathcal{F}$, and (iii) $V \epsilon \mathcal{F}$ if $U \subset V$ and $U \epsilon \mathcal{F}$. An *ultrafilter* of $\mathcal{R}(X)$ is a filter which is not properly contained in any filter of $\mathcal{R}(X)$. Then $G(X)$ is the set of all ultrafilters of $\mathcal{R}(X)$. For each open set U in X, let

$$U^* = \{\mathcal{F} \epsilon\, G(X) : (\bar{U})^0 \epsilon\, \mathcal{F}\}.$$

With the topology determined by the base $\{U^* : U \text{ is open in } X\}$, the Gleason space $G(X)$ is compact, Hausdorff and extremally disconnected; and if X is a *ccc* space, then so is $G(X)$.

The cardinality of a set X will be denoted by $|X|$.

We will need the following *delta system* lemma several times. It is so called because the family $\mathcal{B}$ is a delta system.

Lemma. *Let $\mathcal{G}$ be an uncountable family of finite sets. Then there is an uncountable subfamily $\mathcal{B}$ of $\mathcal{G}$ and a fixed set R such that $A \cap B = R$ whenever A and B are distinct members of $\mathcal{B}$.*

Proof: [**8**, p. 225]. Since $\mathcal{G}$ is uncountable and the sets are finite, there must be uncountably many members of $\mathcal{G}$ with the same number of elements; therefore we may assume that for some $n, |X| = n$ for all X in $\mathcal{G}$. We proceed by induction on n. For $n = 1$, we may take $R = \phi$. Assume that the lemma holds for $n = k$ and let $\mathcal{G}$ be an uncountable family of sets each of which has $k + 1$ elements.

If there is some point a which belongs to each set in an uncountable subfamily $\mathcal{C}$ of $\mathcal{G}$, then the induction hypothesis may be applied to the family $\{X \setminus \{a\} : X \epsilon \mathcal{C}\}$ to yield an uncountable subfamily $\mathcal{B}$ of $\mathcal{C}$ and a finite set R such that

$$(X \setminus \{a\}) \cap (Y \setminus \{a\}) = R$$

for any distinct X and Y in $\mathcal{B}$. Then $X \cap Y = R \cup \{a\}$ for any distinct X and Y in $\mathcal{B}$.

Otherwise, each element a belongs to only countably many members of $\mathcal{G}$, and we construct a disjoint subfamily $\mathcal{B} = \{X_\alpha : \alpha < \omega_1\}$ of $\mathcal{G}$ by transfinite induction on α, as follows. Assume that we have constructed X_α for all $\alpha < \beta$. Then each element of the countable set $\cup_{\alpha<\beta} X_\alpha$ belongs to at most countably many members of $\mathcal{G}$, so there is some X in $\mathcal{G}$ which is disjoint from $\cup_{\alpha<\beta} X_\alpha$. Let $X_\beta = X$ and the proof is complete. ■

2. Martin's Axiom and products of *ccc* spaces

Martin's Axiom [**13**] states that *no compact Hausdorff ccc space is the union of fewer than* 2^ω *nowhere dense sets.* Observe that Martin's Axiom (henceforth denoted by MA) is implied by the Continuum Hypothesis because under CH, MA is an immediate consequence of the Baire category theorem. However, $MA \not\Rightarrow CH$ because, as is shown in [**19**], there is a model of ZFC in which MA holds and $\omega_1 < 2^\omega$.

In this section we prove that $(MA + \neg CH)$ implies that every product of *ccc* spaces is a *ccc* space, and we do it with the help of an interesting theorem about products of *ccc* spaces (Theorem 2.2) whose proof requires no special axioms.

Lemma. [**17**, p. 16]. *$(MA + \neg CH)$ implies that if X is a compact Hausdorff ccc space, then any uncountable family $g = \{U_\alpha : \alpha < \omega_1\}$ of nonempty open sets has a cardinality ω_1 subfamily with nonempty intersection.*

Proof: Consider the set S of all families of disjoint nonempty open subsets of X with the property that if $\mathcal{F} \epsilon S$, then each member of $\mathcal{F}$ meets only countable many members of G. If S is empty, we are through. So suppose that S is not empty. Then by Zorn's lemma, S has a maximal member, say $\mathcal{F}$. Since X is a *ccc* space, $\mathcal{F}$ is countable. Hence there is a member V of G which does not meet any member of $\mathcal{F}$. Then, because $\mathcal{F}$ is maximal, any open subset of $\bar{V}$ must intersect ω_1 members of G. For each $\beta < \omega_1$, define

$$H_\beta = \bar{V} \setminus (\cup_{\alpha > \beta} U_\alpha).$$

Then H_β is nowhere dense in the compact Hausdorff *ccc* space $\bar{V}$. ($\bar{V}$ is *ccc* because V is.) Hence $(MA + \neg CH)$ implies that $\bar{V} \neq \cup_{\beta<\omega_1} H_\beta$. Thus there is v in $\bar{V}$ such that $v \notin \cup_{\beta<\omega_1} H_\beta$, and therefore v belongs to ω_1 U_α's.

The following theorem (more generally, Corollary 2.4) was proved independently by K. Kunen, F. Rowbottom, and R.M. Solovay. (See [**5**, p. 34].) The basic ideas in our proof are from [**17**, p. 17].

Theorem 2.1. *$(MA + \neg CH)$ implies that if X and Y are ccc spaces, then $X \times Y$ is a ccc space.*

Proof: Suppose that $\{U_\alpha : \alpha < \omega_1\}$ is an uncountable family of nonempty open subsets of $X \times Y$. We will show that there are two members of this family which intersect. By shrinking if neccessary, we may assume that each U_α is basic. Then there are open sets $V_\alpha \subset X$ and $W_\alpha \subset Y$ such that $U_\alpha = V_\alpha \times W_\alpha$. The Gleason spaces, $G(X)$ and $G(Y)$, of X and Y are compact, Hausdorff and *ccc*. Hence by the lemma above, there is an uncountable subset D of ω_1 such that

$$\cap_{\alpha \epsilon D} V_\alpha^* \neq \phi \text{ and } \cap_{\alpha \epsilon D} W_\alpha^* \neq \phi.$$

(Recall from Section 1 that U^* is the set of all ultrafilters which contain $(\bar{U})^0$.) Let $\beta, \gamma \epsilon D$ with $\beta \neq \gamma$. Then

$$(\overline{V_\beta})^0 \cap (\overline{V_\gamma}^0 \neq \phi \text{ and } (\overline{W_\beta})^0 \cap (\overline{W_\gamma}^0 \neq \phi.$$

Hence $V_\beta \cap V_\gamma \neq \phi$ and $W_\beta \cap W_\gamma \neq \phi$, and it follows easily that $U_\beta \cap U_\gamma \neq \phi$. ■

The *ccc* has the interesting property that if the product of any two *ccc* spaces is a *ccc* space, then the product of any number of *ccc* spaces is a *ccc* space. To see this, assume that *ccc* is preserved by products of two spaces. Then by induction, it is preserved by products with a finite numbers of factors. Then by the following theorem (which seems to have originated in [**14**]), it is preserved by arbitrary products.

Theorem 2.2. *Suppose that $\{X_i : i \epsilon I\}$ is a family of topological spaces such that $\Pi_{i \epsilon J} X_i$ is a ccc space for every finite $J \subset I$. Then $\Pi_{i \epsilon I} X_i$ is a ccc space.*

Proof: [**10**, p. 51]. Let $X = \Pi_{i \epsilon I} X_i$ and suppose that there exists an uncountable family $\{U_\alpha : \alpha < \omega_1\}$ of disjoint nonempty open subsets of X. As before, we may assume that each U_α is basic. Then by definition of the product topology, for each $\alpha < \omega_1$ there is a finite subset F_α of I such that $U_\alpha = \cap\{P_i^{-1}(V_i) : i \epsilon F_\alpha\}$, where P_i is the projection of X onto X_i, and V_i is open in X_i. By the delta system lemma of Section 1, there is an uncountable subset D of ω_1 and a set R such that $F_\alpha \cap F_\beta = R$ whenever $\alpha, \beta \epsilon D$ and $F_\alpha \neq F_\beta$. Note that R cannot be empty because $F_\alpha \cap F_\beta = \phi$ implies $U_\alpha \cap U_\beta = \phi$. For each α in D, let $P(U_\alpha) = \Pi_{i \epsilon R} P_i(U_\alpha)$. It is not hard to verify that $\{P(U_\alpha) : \alpha \epsilon D\}$ is an uncountable family of disjoint nonempty open subsets of $\Pi_{i \epsilon R} X_i$, which is a contradiction. ■

Since a finite product of separable spaces is separable, the following corollary is an immediate consequence of Theorem 2.2.

Corollary 2.3. *If $\{X_i : i \epsilon I\}$ is a family of separable spaces, then $\Pi_{i \epsilon I} X_i$ is a ccc space.*

Corollary 2.4. *$(MA + \neg CH)$ implies that if $\{X_i : i \epsilon I\}$ is a family of ccc spaces, then $\Pi_{i \epsilon I} X_i$ is a ccc space.*

Other consequences of $(MA + \neg CH)$, not all topological, may be found in [4] and [**18**].

3. The Laver–Galvin example

In this section we describe the example due to R. Laver and F. Galvin [**5**], which assumes CH, of *ccc* spaces X_0 and X_1 such that $X_o \times X_1$ is not a *ccc* space. This will lead (via Gleason spaces) to a compact Hausdorff extremally disconnected *ccc* space X whose square is not a *ccc* space. The material in this section is adapted from [**2**] and [**5**].

The following lemma can be proved by induction, A proof of a somewhat more general result may be found in [**2**, p. 190].

Lemma 3.1. *Let A be a set and for each $n < \omega$, let $\{F_{i,n} : i \epsilon I_n\}$ be a family of disjoint finite subsets of A with $| I_n | = \omega$. Then there are two subsets A_0 and A_1 of A such that*

$$| \{i \epsilon I_n : F_{i,n} \subset A_t\} | = \omega$$

for all $n < \omega$ and $t = 0, 1$.

Lemma 3.2. *Assume that $w_1 = 2^\omega$. Then there are two families $\{K_0(\alpha) : \alpha < \omega_1\}, \{K_1(\alpha) : \alpha < \omega_1\}$ of subsets of ω_1 such that*
(i) $K_0(\alpha) \cup K_1(\alpha) = \alpha$ for all $\alpha < \omega_1$, and
(ii) If $\mathcal{F} = \{F_i : i < \omega\}$ is a countably infinite family of disjoint finite subsets of ω_1, then there is an ordinal λ (depending on $\mathcal{F}$) with $\lambda < \omega_1$, such that if $\lambda < \alpha < \omega_1$, $\cup \mathcal{F} \subset \alpha, X$ is a finite subset of $\alpha, t \epsilon \{0, 1\}$ and

$$| \{i < \omega : F_i \subset \beta \backslash K_t(\beta) \ \forall \beta \epsilon X\} | = \omega,$$

then

$$| \{i < \omega : F_i \subset \beta \backslash K_t(\beta) \ \forall \beta \epsilon X \cup \{\alpha\} | = \omega.$$

Proof: We begin by computing, under CH, the cardinality of the set S of all countably infinite families of disjoint finite subsets of ω_1. Clearly $| S | \geq \omega_1$ since for each ordinal α such that $\omega \leq \alpha < \omega_1$, the family of α singletons is countably infinite and disjoint. To see that $| S | \leq \omega_1$, first note that the set of all finite subsets of ω_1 has cardinality

$$\omega_1 + (\omega_1)^2 + (\omega_1)^3 + \cdots = \omega_1 + \omega_1 + \omega_1 + \cdots = \omega_1 \omega = \omega_1.$$

Hence

$$| S | \leq \omega_1^\omega = (2^\omega)^\omega = 2^{\omega^2} = 2^\omega = \omega_1..$$

Thus $|S| = \omega_1$. Let $\{\mathcal{F}_\lambda : \lambda < \omega_1\}$ be a well–ordering of S in such a way that for each $\lambda < \omega$, the members of $\mathcal{F}_\lambda$ are subsets of ω.

In what follows, $F_{i,\lambda}$ denotes the ith member of the family $\mathcal{F}_\lambda$.

We now define the sets $K_0(\alpha), K_1(\alpha)$ by recursion. We set $K_t(\alpha) = \alpha$ for $t = 0,1$ and $\alpha \leq \omega$. Let $\omega < \alpha < \omega_1$ and suppose that $K_t(\alpha')$ has been defined for $t = 0,1$ and $\alpha' < \alpha$. Consider the set of all triples (t, λ, X) such that $t\epsilon\{0,1\}$, $\lambda < \alpha$, $\cup\mathcal{F}_\lambda \subset \alpha, X$ is a finite subset of α, and

$$|\{i < \omega : F_{i,\lambda} \subset \beta \backslash K_t(\beta)\ \forall \beta \epsilon X\}| = \omega.$$

This set of triples is countably infinite because X can be the empty set, and if $\lambda < \omega$ then certainly $\cup\mathcal{F}_\lambda \subset \alpha$. Let $\{(t_n, \lambda_n, X_n) : n < \omega\}$ be an enumeration of this set, and for each $n < \omega$ let

$$I_n = \{i < \omega : F_{i,\lambda_n} \subset \beta \backslash K_{t_n}(\beta)\ \forall \beta \epsilon X_n\}$$

Then $|I_n| = \omega$, hence we may apply Lemma 3.1 with A and $F_{i,n}$ replaced by α and F_{i,λ_n} respectively; then there are disjoint subsets $L_0(\alpha)$ and $L_1(\alpha)$ of α such that

$$|\{i\epsilon I_n : F_{i,\lambda_n} \subset L_t(\alpha)\}| = \omega$$

for $n < \omega$ and $t = 0,1$. We set

$$K_t(\alpha) = \alpha \backslash L_t(\alpha) \text{ for } t = 0,1.$$

Let us verify that properties (i) and (ii) are satisfied.

(i) Clearly $K_0(\alpha) \cup K_1(\alpha) = \alpha$ because $L_0(\alpha)$ and $L_1(\alpha)$ are disjoint subsets of α.

(ii) Let $\mathcal{F} = \{F_i : i < \omega\}$ be a countably infinite family of disjoint finite subsets of ω_1. Then $\mathcal{F} = \mathcal{F}_\lambda$ for some $\lambda < \omega_1$. Suppose that $\lambda < \alpha < \omega_1$, $\cup\mathcal{F}_\lambda \subset \alpha, X$ is a finite subset of $\alpha, t\epsilon\{0,1\}$, and

$$|\{i < \omega : F_{i,\lambda} \subset \beta \backslash K_t(\beta)\ \forall \beta \epsilon X\}| = \omega.$$

Then $(t, \lambda, X) = (t_n, \lambda_n, X_n)$ for some $n < \omega$. Hence if we let

$$J = \{i\epsilon I_n : F_{i,\lambda} \subset L_t(\alpha)\},$$

then $|J| = \omega$. Let

$$H = \{i < \omega : F_{i,\lambda} \subset \beta \backslash K_t(\beta)\ \forall \beta \epsilon X \cup \{\alpha\}\}.$$

Our goal is to show that $|H| = \omega$, and we will do this by proving that $J \subset H$. Let $i\epsilon J$. Then $F_{i,\lambda} \subset L_t(\alpha)$, hence $F_{i,\lambda} \subset \alpha \backslash K_t(\alpha)$. Because $i\epsilon I_n$ we have

$$F_{i,\lambda} \subset \beta \backslash K_t(\beta)\ \forall \beta \epsilon X$$

Thus

$$F_{i,\lambda} \subset \beta \backslash K_t(\beta)\ \forall \beta \epsilon X \cup \{\alpha\},$$

and so $i\epsilon H$. Therefore $J \subset H$ and the proof is complete. ■

Theorem 3.3. *Assume that $\omega_1 = 2^\omega$. Then there are ccc spaces X_0 and X_1 such that $X_0 \times X_1$ is not a ccc space.*

Proof: For $t\epsilon\{0,1\}$, let $\{K_t(\alpha) : \alpha < \omega_1\}$ be the family of Lemma 3.2 and let X_t be the set of all functions on ω_1 to $\{0,1\}$. Set

$$V_t(\alpha) = \{x\epsilon X_t : x \mid K_t(\alpha) \equiv 0,\ x(\alpha) = 1\} \text{ for } \alpha < \omega_1,$$

and give X_t the topology determined by the subbase $\{V_t(\alpha) : \alpha < \omega_1\}$. The space $X_0 \times X_1$ is not *ccc* because the uncountable family

$$\{V_0(\alpha) \times V_1(\alpha) : \alpha < \omega_1\}$$

consists of disjoint nonempty open sets. To see that they are disjoint, suppose that there are α, α' with $\alpha < \alpha' < \omega_1$ and

$$(x_0, x_1)\epsilon(V_0(\alpha) \times V_1(\alpha)) \cap (V_0(\alpha') \times V_1(\alpha')).$$

Then since

$$\alpha\epsilon\alpha' = K_0(\alpha') \cup K_1(\alpha').$$

$\alpha\epsilon K_t(\alpha')$ for some t in $\{0,1\}$; for this t, we have $x_t(\alpha) = 0$ and $x_t(\alpha) = 1$, a contradiction.

We now verify that X_0 and X_1 are *ccc* spaces. Let $t\epsilon\{0,1\}$ and let $\{U_i : i < \omega_1\}$ be an uncountable family of nonempty basic open subsets of X_t. For each $i < \omega$ there is a finite subset G_i of ω_1 such that

$$U_i = \cap_{\alpha\epsilon G_i} V_t(\alpha).$$

Then

$$(\cup_{\alpha\epsilon G_i} K_t(\alpha)) \cap G_i = \phi \text{ for all } i < \omega_1.$$

This is so because if we choose x in U_i, then $x\epsilon V_t(\alpha)$ for all α in G_i, hence $x \mid G_i \equiv 1$ and $x \mid \cup_{\alpha\epsilon G_i} K_t(\alpha) \equiv 0$. Thus the intersection is empty. Applying now the delta system lemma of Section 1 to the family $\{G_i\}$, we may assume without loss of generality that there is a set R such that $G_i \cap G_{i'} = R$ whenever $i < i' < \omega_1$. Let $F_i = G_i \backslash R$ for each $i < \omega_1$. Then if $i < i' < \omega_1$ and $\gamma \notin K_t(\beta)$ for all $\gamma\epsilon F_i$ and $\beta\epsilon F_{i'}$, it follows that $U_i \cap U_{i'} \neq \phi$. We outline the verification. First show that

$$G_i \cap (\cup_{\alpha\epsilon G_{i'}} K_t(\alpha)) = \phi$$

and

$$G_{i'} \cap (\cup_{\alpha\epsilon G_i} K_t(\alpha)) = \phi.$$

it then follows that

$$(G_i \cup G_{i'}) \cap (\cup_{\alpha\epsilon G_i \cup G_{i'}} K_t(\alpha)) = \phi.$$

Now choose x in X_t such that $x \mid G_i \cup G_{i'} \equiv 1$ and

$$x \mid \cup_{\alpha\epsilon G_i \cup G_{i'}} K_t(\alpha) \equiv 0.$$

Then $x \epsilon U_i \cap U_{i'}$. Thus to prove that $\{U_i : i < \omega_1\}$ is not a disjoint family, it is enough to show that there are i, i' such that $i < i' < \omega_1$ and $\gamma \notin K_t(\beta)$ for all $\gamma \epsilon F_i$ and $\beta \epsilon F_{i'}$. Set $\mathcal{F} = \{F_i : i < \omega\}$ and let $\lambda < \omega_1$ be an ordinal satisfying condition (ii) of Lemma 3.2 for $\mathcal{F}$. We may assume that $\cup \mathcal{F} \subset \lambda$. Because $\lambda + 1$ is countable and $\{F_i : \omega < i < \omega_1\}$ is uncountable, there is an ordinal i' such that $\omega < i' < \omega_1$ and $\lambda < \alpha$ for all α in $F_{i'}$. Then $\cup \mathcal{F} \subset \alpha$ for all α in $F_{i'}$. If $F_{i'} = \phi$, then for any $i < i'$ we have that $\gamma \notin K_t(\beta)$ for all $\gamma \epsilon F_i$ and $\beta \epsilon F_{i'}$. Therefore we assume that $F_{i'} \neq \phi$ and we let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be the elements of $F_{i'}$ in the order inherited from ω_1. We note that

$$\mid \{i < \omega : F_i \subset \beta \backslash K_t(\beta) \; \forall \beta \epsilon \phi\} \mid = \omega$$

and we apply condition (ii) of Lemma 3.2 successively n times with

$$\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n$$

and

$$X = \phi, \{\alpha_1\}, \{\alpha_1, \alpha_2\}, \ldots, \{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$$

correspondingly. Hence

$$\mid \{i < \omega : F_i \subset \beta \backslash K_t(\beta) \; \forall \beta \epsilon F_{i'}\} \mid = \omega.$$

In particular, there is $i < \omega$ such that $\gamma \notin K_t(\beta)$ for all $\gamma \epsilon F_i$ and $\beta \epsilon F_{i'}$. Thus $U_i \cap U_{i'} \neq \phi$ and so X_t is a *ccc* space.

Corollary. *Assuming CH, there is a compact Hausdorff extremally disconnected ccc space X such that $X \times X$ is not a ccc space.*

Proof: Let X_0 and X_1 be the *ccc* spaces of the above theorem and let X be the disjoint union of the Gleason spaces $G(X_0)$ and $G(X_1)$. Then X is compact, Hausdorff, extremally disconnected and *ccc* because both $G(X_0)$ and $G(X_1)$ are. To prove that $X \times X$ is not a *ccc* space, it is enough to show that $G(X_0) \times G(X_1)$ is not a *ccc* space because $G(X_0) \times G(X_1)$ is homeomorphic to an open (and closed) subset of $X \times X$. Let $\{U_i \times V_i : i < \omega_1\}$ be an uncountable family of disjoint nonempty basic open subsets of $X_0 \times X_1$. Then, as is not too difficult to verify, $\{U_i^* \times V_i^* : i < \omega_1\}$ is an uncountable family of disjoint nonempty open subsets of $G(X_0) \times G(X_1)$. Thus $G(X_0) \times G(X_1)$ is not a *ccc* space. ■

Before leaving this section we remark that S. Argyros, S. Mercourakis and S. Negrepontis have proved the existence, under CH, of a Corson–compact *ccc* space whose square is not *ccc* [**1**].

4. The Souslin problem

Suppose that X is a totally ordered set satisfying

(a) X has no first or last element,

(b) X is connected in the order topology, and

(c) X is separable in the order topology.

Then X is linearly isomorphic to $\mathbb{R}$. (A proof is in [**3**, p.8].) In 1920, the Russian mathematician M. Souslin [**20**] asked whether (c) can be replaced by

(c') X is *ccc* in the order topology.

A positive answer to Souslin's question has come to be known as *Souslin's Hypothesis* (SH). The axiom SH is undecidable in ZFC and is implied by $(MA + \neg CH)$ ([**7**],[**21**], [**19**]). Indeed, to quote D. Fremlin [**4**, p. 184], *Souslin's problem was the original stimulus for the invention of Martin's axiom.*

A *Souslin line* is a totally ordered set satisfying properties (a), (b) and (c'), but not (c). Thus the existence of a Souslin line is equivalent to the negation of SH. G. Kurepa proved [**11**] that if X is a Souslin line, then $X \times X$ is not a *ccc* space. (A proof may be found in [**10**, p. 66].) There is also the notion of a Souslin tree; it is shown in [**16**] that the existence of a Souslin tree is equivalent to the existence of a Souslin line.

5. Property (K)

A topological space is said to have *property* (K) (after Knaster [**9**]) if each uncountable family of open sets contains an uncountable subfamily in which every two sets have nonempty intersection. Clearly every space with property (K) is a *ccc* space. The converse is true under $(MA + \neg CH)$ [**4**, Theorem 41 A] and is false under CH. To verify the latter, we make use of Marczewski's theorem [**12**] that a product of spaces has property (K) if (and only if) each space has property (K). Recall from Section 3 that there exists, under CH, a *ccc* space X such that $X \times X$ is not a *ccc* space. Then X does not have property (K) because otherwise $X \times X$ would have property (K) and thus be a *ccc* space.

The reader has probably noticed that another approach (in Section 2) to proving that $(MA + \neg CH)$ implies that a product of *ccc* spaces is a *ccc* space would be to combine Fremlin's theorem that under $(MA + \neg CH)$ every *ccc* space has property (K), with Marczewski's theorem quoted above.

For open problems and additional references involving *ccc* spaces, the reader may consult [**2**, Chapter 7] and [**4**, § 44].

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