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ERGODIC RESULTS FOR CERTAIN CONTRACTIONS ON ORLICZ SPACES WITH FIXED POINTS

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Abstract

Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, $L_\phi \equiv L_\phi(X, \mathcal{M}, \mu)$ an Orlicz space associated to an N-function ϕ and let $T: L_\phi \longrightarrow L_\phi$ be a linear operator with a fixed point $h \neq 0$ a.e., such that

$$\int_X \phi(|Tf|)d\mu \leq \int_X \phi(|f|)d\mu \quad (f \in L_\phi)$$

and it is either a $\| \ \|_1$-contraction in $L_\phi \cap L_1$ or a $\| \ \|_\infty$-contraction in $L_\phi \cap L_\infty$. The main result of this paper is that for a wide class of N-functions ϕ, the ergodic maximal operator associated to T is bounded in L_ϕ. Moreover, for every $f \in L_\phi$ we have the almost everywhere convergence and the norm convergence of certain weighted averages which include the Césàro averages.

1. Introduction and preliminaries

Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and $L_\phi \equiv L_\phi(X, \mathcal{M}, \mu)$ and Orlicz space associated to an N-function ϕ (L_ϕ may be a complex Banach space). In this paper we will consider linear operators T such that

i) $\int_X \phi(|Tf|)d\mu \leq \int_X \phi(|f|)d\mu$, $f \in L_\phi$

ii) T has a fixed point h, $h \neq 0$ a.e.

iii) T is either a $\| \ \|_1$-contraction in $L_\phi \cap L_1$ or a $\| \ \|_\infty$-contraction in $L_\phi \cap L_\infty$.

The main aim of this paper is to prove that, for a wide class of N-functions ϕ, *the ergodic maximal operator* M_T defined by

$$M_T f = \sup_{n \geq 1} \left| \frac{1}{n} \sum_{k=0}^{n-1} T^k f \right| \tag{1.1}$$

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is bounded in L_ϕ (*dominated ergodic theorem*). Moreover, we shall prove that if $\{b_k\}$ is a *bounded Besicovitch sequence*, then for every $f \in L_\phi$ there exists $f^* \in L_\phi$ such that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} b_k T^k f(x) = f^*(x) \quad \text{a.e.}, \qquad \lim_{n\to\infty} \|\frac{1}{n}\sum_{k=0}^{n-1} b_k T^k f - f^*\|_{(\phi)} = 0.$$

A *sequence of complex numbers* $\{b_k\}$ *is called a Besicovitch sequence if for every* $\varepsilon > 0$ *there exists a trigonometric polynomial* α_ε *such that*

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} |b_k - \alpha_\varepsilon(k)| < \varepsilon.$$

As a special case we obtain the almost everywhere convergence (*individual ergodic theorem*) and the norm convergence (*mean ergodic theorem*) of *the Césàro-averages* $n^{-1}(f + Tf + \cdots + T^{n-1}f)$.

In the real L_p-case, with $1 < p < \infty$, and $(X, \mathcal{M}, \mu)$ a finite measure space the corresponding dominated ergodic theorem is proved by A. de la Torre in [10]. R. Sato proved in [9] that the de la Torre's result may be extended to the case $(X, \mathcal{M}, \mu)$ σ-finite and a complex L_p-space. The ergodic result for an operator which only satisfies conditions i) and iii) is an open problem even in the L_p-case, $1 < p < \infty$.

The bounded Besicovitch sequences as weights in the averages were used by J.H. Olsen in [8].

In order to obtain the dominated ergodic theorem we first need some *extrapolation theorems* which extend the ones given by M.A. Akcoglu and R.V. Chacon in [1] and R. Sato in [9], for L_p, $1 < p < \infty$.

Now, we shall present the basic definitions and results concerning to N-functions and Orlicz spaces which will be used in this paper. The proofs of most of these results can be found in [5] or in II-13 of [7].

An N-function is a continuous and convex function $\phi\colon [0,\infty) \longrightarrow \mathbf{R}$ *such that* $\phi(s) > 0$, $s > 0$, $s^{-1}\phi(s) \longrightarrow 0$ *as* $s \longrightarrow 0$ *and* $s^{-1}\phi(s) \longrightarrow \infty$ *as* $s \longrightarrow \infty$.

The function ϕ is an N-function if and only if it has the representation $\phi(s) = \int_0^s \varphi$ where $\varphi\colon [0,\infty) \longrightarrow \mathbf{R}$ is continuous from the right, non decreasing such that $\varphi(s) > 0$, $s > 0$, $\varphi(0) = 0$ and $\varphi(s) \longrightarrow \infty$ as $s \longrightarrow \infty$. More precisely φ is the right derivate of ϕ and will be called the *density function of* ϕ.

Associated to φ we have the function $\rho\colon [0,\infty) \longrightarrow \mathbf{R}$ defined by $\rho(t) = \sup\{s\colon \varphi(s) \le t\}$ which has the same aforementioned properties of φ. We will call ρ *the generalized inverse of* φ.

The N-function ψ defined by $\psi(t) = \int_0^t \rho$ is called *the complementary N-function of* ϕ. Thus, if $\phi(s) = p^{-1}s^p$, $p > 1$, then $\psi(t) = q^{-1}t^q$ where $pq = p + q$.

Young's inequality asserts that $st \leq \phi(s) + \psi(t)$ for $s, t \geq 0$, equality holding if and only if $\varphi(s-) \leq t \leq \varphi(s)$ or else $\rho(t-) \leq s \leq \rho(t)$ (See [3]).

If ϕ_1 and ϕ_2 are N-functions with complementary N-functions given by ψ_1 and ψ_2 respectively, then, *the inequality for complementary functions* asserts that if $\phi_1(s) \leq \phi_2(s)$ for $s \geq s_0$, then $\psi_2(t) \leq \psi_1(t)$ for $t \geq \varphi_2(s_0)$, where φ_2 is the density function of ϕ_2.

An N-function ϕ is said to satisfy the *Δ_2-condition* in $[s_0, \infty)$, $s_0 \geq 0$, if there exists a constant α such that $\phi(2s) \leq \alpha\phi(s)$ for every $s \geq s_0$.

If φ is the density function of ϕ, then ϕ satisfies Δ_2 in $[s_0, \infty)$ if and only if there exists a constant $\alpha > 1$ such that $s\varphi(s) \leq \alpha\phi(s)$, $s \geq s_0$.

The Δ_2-condition for ϕ does not transfer necessarily to the complementary N-function.

If $(X, \mathcal{M}, \mu)$ is a σ-finite measure space we denote by $\mathbf{M} = \mathbf{M}(X, \mathcal{M}, \mu)$ the space of $\mathcal{M}$-measurable and μ-a.e. finite functions from X ro $\mathbb{R}$ or to $\mathbb{C}$. If ϕ is an N-function we consider *the Orlicz spaces* $L_\phi \equiv L_\phi(X, \mathcal{M}, \mu)$ and $L_\phi{}^* \equiv L_\phi{}^*(X, \mathcal{M}, \mu)$ defined by $L_\phi = \{f \in \mathbf{M} : \int_X \phi(|f|)d\mu < \infty\}$ and $L_\phi{}^* = \{f \in \mathbf{M} : fg \in L_1 \text{ for all } g \in L_\psi\}$ where ψ is the complementary N-function of ϕ. We have $L_\phi \subset L_\phi{}^*$ and if ϕ satisfies Δ_2 then $L_\phi = L_\phi{}^*$.

We have that $L_\phi{}^*$ is a linear space with the usual operations on which we may define the norms $\|f\|_\phi = \sup\{\int_X |fg|d\mu : g \in S_\psi\}$, where $S_\psi = \{g \in L_\psi : \int_X \psi(|g|)d\mu \leq 1\}$, and $\|f\|_{(\phi)} = inf\{\lambda > 0 : \int_X \phi(\lambda^{-1}|f|)d\mu \leq 1\}$ which are called *Orlicz norm and Luxemburg norm* respectively. Both norms are equivalent.

Holder's inequality asserts that for every $f \in L_\phi{}^*$ and every $g \in L_\psi{}^*$ we have $\|fg\|_1 \leq \|f\|_{(\phi)}\|g\|_\psi$ where ϕ and ψ are complementary N-functions.

If $\phi(s) = s^p$ with $p > 1$ then $L_\phi{}^* = L_\phi = L_p, \|f\|_{(\phi)} = \|f\|_p$ and $\|g\|_\psi = \|g\|_q$ where $pq = p + q$.

The convergence $f_n \longrightarrow f$ in $[L_\phi{}^*, \| \quad \|_\phi]$ implies the mean convergence $\lim_{n\to\infty} \int_X(|f_n - f|)d\mu = 0$ but, in general, mean convergence only implies norm convergence when ϕ satisfies Δ_2. Then the set $\mathcal{S}$ of simple functions (with support of finite measure) is dense in $[L_\phi, \| \quad \|_\phi]$ if ϕ satisfies Δ_2.

If ϕ verifies Δ_2, then for every continuous linear functional F over $[L_\phi, \| \quad \|_{(\phi)}]$ there exists an unique function $g \in L_\psi{}^*$ such that $F(f) = \int_X fg d\mu$, $f \in L_\phi$, and moreover $\|F\|_{(\phi)} = \|g\|_\psi$, where ψ is the complementary N-function of ϕ, but if ϕ does not satisfy Δ_2 then there exist linear functionals on $L_\phi{}^*$ which are not represented by functions of $L_\psi{}^*$.

If ϕ and ψ satisfy Δ_2 then $[L_\phi, \| \quad \|_{(\phi)}]$ is *reflexive.*

In the following, we shall always assume that $(X, \mathcal{M}, \mu)$ is a *σ-finite measure space* and ϕ, *together with its complementary N-function ψ, satisfy the Δ_2-condition in* $[0, \infty)$. The Δ_2-condition for ϕ is a very important condition that plays fundamental roles in many questions and the best known Orlicz spaces are associated to functions which satisfy Δ_2. The Δ_2-condition for ψ may seem to be a restrictive assumption. Some know Orlicz spaces as, for example, the Zygmund Orlicz space $L \operatorname{Log} L$ and the $L \operatorname{Log}^k L$ spaces, $k > 0$, are associated to N-functions which satisfy Δ_2 but their complementary N-functions do not; but the above spaces do not satisfy our dominated ergodic result. In fact *the* Δ_2*-condition for the complementary N-function is necessary for such result.*

Precisely, let $([0,1], \mathcal{B}, \lambda)$ be the Lebesgue-space and let τ an invertible λ-measure preserving transformation from $[0,1]$ into itself. In [2] B. Bru and H. Heinich characterize the Orlicz spaces, associated to Young's functions, for which the ergodic maximal operator associated to the operator T, defined by $Tf = f \circ \tau^{-1}$, is bounded in L_ϕ (*classical dominated ergodic theorem*) (the Young's functions in [2] are our N-functions). The characterizing condition given in [2] is the condition of comoderation on ϕ.

The function ϕ is said to be *comoderated* if there exist s_0, a and $b > 1$ such that $\varphi(as) \geq b\varphi(s)$ for $s \geq s_0$, where φ is the density function of ϕ or, equivalently, if *there exist* s_0, *a and* $b > 1$ *such that* $\phi(as) \geq ab\phi(s)$ *for* $s \geq s_0$ (in [2] a function continuous from the left is taken as density function of ϕ whereas our density function is right continuous).

The paper [2] does not establisch the equivalence between the comoderation of ϕ and the *moderation* (Δ_2-condition in some $[t_0, \infty)$) of the complementary N-function ψ unless φ be continuous. However, we observe that the comoderation of ϕ is equivalent to the moderation of ψ. At the same time, we shall prove another characterization of the moderation of ψ, which is used in this paper, and which appear in [2], [5] and in the rest of the literature with more restrictive hypothesis. Exactly:

Proposition 1.2. *Let ϕ be an N-function and ψ the complementary N-function of ϕ. The following conditions are equivalent:*

a) ϕ is comoderated.

b) ψ is moderated.

c) There exist s_0 and $\beta > 1$ such that $\beta\phi(s) \leq s\varphi(s)$ for $s \geq s_0$.

Proof: *a)* $\Longrightarrow$ *b)*. If ϕ is comoderated then $\phi(s) \leq \phi_1(s)$ for $s \geq s_0$ where ϕ_1 is the N-function given by $\phi_1(s) = (ab)^{-1}\phi(as)$. The complementary function of ϕ_1 is given by $\psi_1(t) = (ab)^{-1}\psi(bt)$. Taking into account the inequality for complementary N-functions we obtain that $\psi(bt) \leq ab\psi(t)$ for $t \geq t_0 = \varphi_1(s_0)$, where $b > 1$, which equivales to condition Δ_2 of ψ for $t \geq t_0$.

b) $\Longrightarrow$ *c)*. Let ρ be the generalized-inverse of φ. Since ψ is moderated there exist t_0 and $\alpha > 1$ such that $t\rho(t) \leq \alpha\psi(t)$ for every $t \geq t_0$. On the other hand, it follows from the equality cases in Young's inequality that $t\rho(t) = \phi(\rho(t)) + \psi(t)$ and therefore

$$\phi(\rho(t)) \leq \alpha^{-1}(\alpha - 1)t\rho(t), \quad t \geq t_0.$$

Then, since $\rho(\varphi(s)) \geq s$ and the function $u \longrightarrow u^{-1}\phi(u)$ increases for $u > 0$ we obtain

$$s^{-1}\phi(s) \leq \phi(\rho(\varphi(s)))/\rho(\varphi(s)) \leq \alpha^{-1}(\alpha - 1)\varphi(s), \quad s \geq \rho(t_0)$$

and thus we obtain c) with $s_0 = \rho(t_0)$ and $\beta = \alpha(\alpha-1)^{-1} > 1$.

c) $\Longrightarrow$ *a)*. Condition c) implies that there exist s_0 and $\beta > 1$ such that the function $s \longrightarrow s^{-\beta}\phi(s)$ increases for $s \geq s_0$ (or for $s > s_0$ if $s_0 = 0$). Then, if $a > 1$ is such that $a^{\beta-1} \geq 2$ we have $\phi(as) \geq a^{\beta}\phi(s) \geq 2a\phi(s)$ for $s \geq s_0$ and thus we obtain the comoderation of ϕ.

Note. Since $\varphi(0) = \rho(0) = 0$, if some of the conditions of Proposition 1.2 is satisfied for every $s \geq 0$, then the others two conditions are also valids for every $s \geq 0$.

In this way, the moderation of ψ is necessary for the classical dominated ergodic result and, therefore, for our dominated ergodic result since that the operator T, defined by $Tf = f \circ \tau^{-1}$ satisfies conditions i), ii) and iii), whatever the N-function ϕ may be. On the other hand, the space $([0,1], \mathcal{B}, \lambda)$ is of finite measure and our spaces can be of infinite measure. For this reason we shall assume the Δ_2-condition in $[0,\infty)$, but un the case $\mu(X) < \infty$ the argument which we shall use can be adapted if only we suppose the Δ_2-condition in some $[s_0, \infty)$.

Our results are valid, for example, for the known $L^p \operatorname{Log}^k L$ spaces, with $p > 1$ and $k \geq 0$, since the N-functions of the form $\phi(s) = s^p \log^k(1+s)$ satisfy that $1 < p < \phi(s)/s\varphi(s) \leq p + b$ for every $s > 0$ and certain constant b.

2. Extrapolation Theorems

We first observe that the convexity theorem for positive operators given by M.A. Akcoglu and R.V. Chacon in [1] can be easily extended to Orlicz spaces, following the same type of arguments, as follows

Proposition 2.1. *Let ϕ be an N-function strictly convex in some interval and let T be a conservative positive contraction in L_1 such that*

$$\int_X \phi(|Tf|)d\mu \leq \int_X \phi(|f|)d\mu\,, \quad (f \in L_1 \cap L_\phi). \tag{2.2}$$

Then, $\|Tf\|_\infty \leq \|f\|_\infty$ for every $f \in L_1 \cap L_\infty$.

Proof: The operator T is said to be conservative when $\mu(D) = 0$, where D is the dissipative part of X with respect to T.

First assume that $\mu(X) < \infty$. It is enough to prove that $Tc \leq c$ almost everywhere for some constant $c \neq 0$.

We have that φ increases strictly in some interval I, where φ is the density function of ϕ. Let $c \in I$ with $c \neq 0$. Then, we get that

$$\phi(c+s) > \phi(c) + s\varphi(c) \quad (0 \neq s \geq -c). \tag{2.3}$$

Since T is conservative we have $\int_X Tf d\mu = \int_X f d\mu$ for every $f \in L_1$.

Let $Tc(x) = c + g(x)$; then $\int_X g d\mu = 0$ and therefore if $\mu\{x \in X : g(x) > 0\} > 0$ we have

$$\int_X \phi(|Tc|)d\mu > \int_X \phi(c)d\mu,$$

which contradicts (2.2). This proves that $Tc \leq c$.

The general case follows from the preceding by a method similar to the one given in [1] using the following result:

Lemma 2.4. *Let ϕ be an N-function and T a positive contraction in L_1 satisfing (2.2). Then, for every $A \in \mathcal{M}$ there exists a linear operator*
$T_A: L_1(A) \longrightarrow L_1(A)$ such that
a) T_A is a positive contraction in $L_1(A)$ and

$$\int_X \phi(|T_A f|)d\mu \leq \int_X \phi(|f|)d\mu\ ,\ (f \in L_1(A) \cap L_\phi(A)).$$

b) For every $f \in {L_1}^+(A)$ and every $n \geq 1$

$$\sum_{k=0}^{n} T^k f(x) \leq \sum_{k=0}^{n} T_A^k f(x) \quad a.e.\ in\ A.$$

The proof of Lemma 2.4 can be obtained easily following the arguments of [1].

Remarks.

1. The conservative condition of T cannot be eliminated from the hypothesis of Proposition 2.1 since in $\mathbb{R}$ with Lebesgue-measure if $Tf(x) = \sqrt{2}f(2x)$ then T is a positive contraction in L_1, an isometry in L_2 but $\|Tf\|_\infty = \sqrt{2}\|f\|_\infty$.

2. There exist N-function which are strictly convex over no interval. An example is the following. We consider the dyadic intervals $I_n = [2^{n-1}, 2^n)$ and $J_n = [2^{-n}, 2^{-n+1})$ where n is a positive integer and let $\varphi: [0,\infty) \longrightarrow [0,\infty)$ be defined by $\varphi(0) = 0$, $\varphi(t) = 2^{-n}$ if $t \in J_n$ and $\varphi(t) = 2^{n-1}$ if $t \in I_n$. Then ϕ defined by $\phi(s) = \int_0^s \varphi$ is an N-function. Since $\phi(2s) = 4\phi(s)$ we have that ϕ, as well as its complementary N-function, satisfy the Δ_2-condition. However ϕ is not strictly convex over any interval. Furthemore there is no constant $c \neq 0$ such that (2.3) holds.

However most of N-functions are strictly convex in some interval.

In the following results the operators are not necessarily positive but they have a fixed point h with $h \neq 0$ a.e.

Theorem 2.5. *Let ϕ be an N-function, strictly convex in some interval and let $T: L_\phi \longrightarrow L_\phi$ be a linear operator such that*

i) $\int_X \phi(|Tf|)d\mu \leq \int_X \phi(|f|)d\mu$, $(f \in L_\phi)$.

ii) $\|Tf\|_1 \leq \|f\|_1$, $(f \in L_1 \cap L_\phi)$.

iii) There exists $h \in L_\phi$, $h \neq 0$ *a.e., such that* $Th = h$.

Then, $\|Tf\|_\infty \leq \|f\|_\infty$ *for every* $f \in L_1 \cap L_\infty$ *and consequently for every* $f \in L_\phi \cap L_\infty$.

Proof: In this proof we follows the idea given by Sato in [9].

Let k be such that $\phi(s) < s$ for $0 < s < k$. Given $f \in L_1 \cap L_\infty$ let $B = \{x \in X\colon\ |f(x)| \geq k\}$; then $\mu(B) < \infty$ and therefore $\int_X \phi(|f|)d\mu \leq \|f\|_1 + \mu(B)\phi(\|f\|) < \infty$. Consequently $L_1 \cap L_\infty \subset L_\phi$.

Let $\hat{T}: L_1 \longrightarrow L_1$ be the linear extension of $T: [L_1 \cap L_\phi, \|\ \|_1] \longrightarrow L_1$ and P *the linear modulus of* $\hat{T}$. (See Theorem 4.1.1 in [6]). We shall prove that P satisfies the hypotheses of Proposition 2.1 and therefore $\|Pf\|_\infty \leq \|f\|_\infty$ $f \in L_1 \cap L_\infty$; in this way, since $|\hat{T}f| \leq P|f|$, $f \in L_1$, and $L_1 \cap L_\infty \subset L_1 \cap L_\phi$ we obtain that $\|Tf\|_\infty \leq \|f\|_\infty$, $f \in L_1 \cap L_\infty$, and consequently for every $f \in L_\phi \cap L_\infty$ since $L_1 \cap L_\infty$ is dense in $L_\phi \cap L_\infty$ with the L_∞-norm.

Now, we show that P satisfies the conditions of Proposition 2.1. The Δ_2-condition implies that $L_1 \cap L_\phi$ is dense in $[L_\phi, \|\ \|_{(\phi)}]$. On the other hand, it follows from i) that $\|Tf\|_{(\phi)} \leq \|f\|_{(\phi)}$, $f \in L_\phi$, and consequently given $\varepsilon > 0$ there is $f_\varepsilon \in L_1 \cap L_\phi$ such that for every $n \geq 1$

$$\|h - \frac{1}{n}\sum_{k=0}^{n-1} T^k f_\varepsilon\|_{(\phi)} \leq \varepsilon/2. \tag{2.6}$$

If T is a power bounded linear operator in a reflexive Banach space V, that is, the powers T^k, $k \geq 0$, are uniformly bounded in V, then the Césàro-averages.

$$R_n f = \frac{1}{n}\sum_{k=0}^{n-1} T^k f$$

converge in norm to a T-invariant limit for all $f \in V$ (See Theorem 2.1.2 in [6]).

Lef $f_\varepsilon{}^*$ be the limit in $[L_\phi, \|\ \|_{(\phi)}]$ of $R_n f_\varepsilon$. It follows from (2.6) that for $0 < \varepsilon < 1$ we have $\|h - f_\varepsilon^*\|_{(\phi)} < \varepsilon$ and consequently

$$\int_X \phi(|h - f_\varepsilon{}^*|)d\mu < \varepsilon. \tag{2.7}$$

On the other hand, $f_\varepsilon{}^*(x) = 0$ for a.e. $x \in D$, where D is the dissipative part of X with respect to P, since (Theorem 3.1.6 in [6]) $\sum_{k\geq 0} P^k f(x) < \infty$

on D for all $f \in L_1{}^+$. Since $\phi(|h|) > 0$ a.e. (2.7) shows that $\mu(D) = 0$ and thus P is conservative.

Now, in order to prove that P satisfies condition (2.2) we consider the Akcoglu and Brunel's theorem related with the structure of $\hat{T}$ on the conservative part C of X with respect to P (see Theorem 4.1.10 in [6]). Let $\mathcal{F}$ be the family of P-absorbing subsets of C; there exists a set $\Gamma \in \mathcal{F}$ and a function $s \in L_\infty(\Gamma)$, with $|s| = 1$ on Γ, such that $\hat{T}f = \bar{s}P(sf)$ for any $f \in L_1(\Gamma)$, where $\bar{s}$ is the complex conjugate of s, and if $\Delta = C - \Gamma$ then $(I - T)L_1(\Delta)$ is dense in $L_1(\Delta)$.

We have that $\operatorname{supp} T(\chi_\Gamma h) \subset \Gamma$ and $\operatorname{supp} T(\chi_\Delta h) \subset \Delta$; therefore $Tg = g$ where $g = \chi_\Delta h$. Carryng out a similar reasoning to the used for h we have that for every $\varepsilon > 0$ there exist $f_\varepsilon \in L_1(\Delta) \cap L_\phi(\Delta)$ and $f_\varepsilon{}^* \in L_\phi(\Delta)$ such that $\|g - f_\varepsilon{}^*\|_{(\phi)} < \varepsilon$ and $\lim_{n\to\infty} \|R_n f_\varepsilon - f_\varepsilon{}^*\|_{(\phi)} = 0$.

Given $\eta > 0$ there is $u_\eta \in L_1(\Delta)$ such that $\|u_\eta - Tu_\eta - f_\varepsilon\|_1 < \eta/2$ and therefore for every $n \geq 1$ we have $\|n^{-1}(u_\eta - T^n u_\eta) - R_n f_\varepsilon\|_1 = \|R_n(u_\eta - Tu_\eta - f_\varepsilon)\|_1 < \eta/2$, which proves that $\lim_{n\to\infty} \|R_n f_\varepsilon\|_1 = 0$ and so $f_\varepsilon{}^*(x) = 0$ a.e. This shows that $\|g\|_{(\phi)} = 0$ and consequently $\mu(\Delta) = 0$. Then, we have $\hat{T}f = \bar{s}P(sf)$ for every $f \in L_1$ and therefore it follows from i) that $\int_X \phi(|Pf|)d\mu = \int_X \phi(|\bar{s}\hat{T}(\bar{s}f)|)d\mu \leq \int_X \phi(|f|)d\mu$ for every $f \in L_1 \cap L_\phi$ and this finishes the proof. ■

Now, our aim is to prove that the roles of L_1 and L_∞ in Theorem 2.5 can be interchanged. For this we shall considerer the adjoint operator of T.

Let $T\colon L_\phi \longrightarrow L_\phi$ be a bounded linear operator; more precisely, we suppose that there is a constant C such that $\|Tf\|_{(\phi)} \leq C\|f\|_{(\phi)}$, $f \in L_\phi$. Then, if $g \in L_\psi{}^*$, where ψ is the complementary N-function of ϕ, the linear functional F over $[L_\phi\,, \|\|_{(\phi)}]$ defined by $F(f) = \int_X gTf d\mu$ is continuous since by Holder's inequality we have $|F(f)| \leq C\|g\|_\psi \|f\|_{(\phi)}$ and therefore, since ϕ satisfies Δ_2, there exists an unique function $g^* \in L_\psi{}^*$ such that $\int_X gTf d\mu = \int_X fg^* d\mu$, $f \in L_\phi$. Then, we can define the bounded linear operator $T^*\colon L_\psi{}^* \longrightarrow L_\psi{}^*$, $g \longrightarrow T^*g$, where T^*g is the function in $L_\psi{}^*$ such that

$$\int_x gTf d\mu = \int_X fT^*g d\mu\,, \quad f \in L_\phi.$$

We shall call T^* *the adjoint operator of* T. T^* satisfies $\|T^*g\|_\psi \leq C\|g\|_\psi$. In our case we have

Lemma 2.8. *Let* $T\colon L_\phi \longrightarrow L_\phi$ *be a linear operator such that*

$$\int_X \phi(|Tf|)d\mu \leq \int_X \phi(|f|)d\mu \quad (f \in L_\phi).$$

Then, its adjoint operator T^* *satisfies*

$$\int_X \psi(|T^*g|)d\mu \leq \int_X \psi(|g|)d\mu \quad (g \in L_\psi) \tag{2.9}$$

and moreover, if T admits an invariant function h with $h \neq 0$ a.e., then there exists $g \in L_\psi$ with $g \neq 0$ a.e., such that $T^*g = g$.

Proof: We write $\operatorname{sig} z$ for $z/|z|$ and by $\bar{u}$ we denote the complex conjugate of u. For $g \in L_\phi{}^+$ we have

$$\int_X f|T^*g|d\mu = |\int_X f(\operatorname{sig}\overline{T^*g})T^*g d\mu| \leq \int_X |T(f\operatorname{sig}\overline{T^*g})||g|d\mu \leq \\ \leq \int_X \phi(f)d\mu + \int_X \psi(|g|)d\mu. \tag{2.10}$$

Let φ be the density function of ϕ and ρ the generalized inverse of φ. Since ψ satisfies Δ_2 there exists $\alpha > 1$ such that $s\rho(s) \leq \alpha\psi(s)$ and therefore $\phi(\rho(s)) = s\rho(s) - \psi(s) \leq (\alpha - 1)\psi(s)$. Therefore, for every $g \in L_\psi$ the function $\rho(|T^*g|)$ belongs to $L_\phi{}^+$ and so (2.9) follows from (2.10) for $f = \rho(|T^*g|)$.

Now, let us assume that $Th = h$ with $h \neq 0$ a.e. If φ is not continuous then there exists an at most countable set of positive reals $s_1, s_2, \ldots, s_n$ where φ is not continuous; in this situation, since $h \in L_\phi$, it is easy to see that $\{c > 0 : \mu\{x \in X : |s_i^{-1}h(x)| = c\} > 0\}$ is at most countable and therefore there exists $\lambda > 0$ such that for every s_i we have

$$\mu\{x \in X : |\lambda^{-1}h(x)| = s_i\} = 0. \tag{2.11}$$

In the case φ continuous (2.11) holds trivially with $\lambda = 1$.

Let $u = \lambda^{-1}h$ and $g = \varphi(|u|)\operatorname{sig}\bar{u}$. We have that $g \neq 0$ a.e. and $g \in L_\phi$ since ϕ satisfies Δ_2. It follows from (2.9) that

$$\int_X |u|\varphi(|u|)d\mu = |\int_X uT^*g d\mu| \leq \int_X |u||T^*g|d\mu \leq \int_X \phi(|u|)d\mu + \\ + \int_X \psi(|T^*g|)d\mu \leq \int_X \phi(|u|)d\mu + \int_X \psi(\varphi(|u|))d\mu = \int_X |u|\varphi(|u|)d\mu \tag{2.12}$$

and therefore

$$\int_X |u||T^*g|d\mu = \int_X (\phi(|u|) + \psi(|T^*g|))d\mu.$$

Then, Young's inequality shows that

$$|uT^*g| = \phi(|u|) + \psi(|T^*g|) \quad \text{a.e.} \tag{2.13}$$

It follows from (2.11) and (2.13) that $|T^*g| = \varphi(|u|)$ a.e. On the other hand we obtain from (2.12) that $(\operatorname{sig}\bar{u})\operatorname{sig}\overline{T^*u} = 1$ and therefore $T^*g = g$ which finishes the proof of the Lemma.

Theorem 2.5 and Lemma 2.8 imply easy

Theorem 2.14. *Let ϕ be an N-function whose complementary N-function is strictly convex in some interval and let $T: L_\phi \longrightarrow L_\phi$ be a linear operator such that*

i) $\int_X \phi(|Tf|)d\mu \leq \int_X \phi(|f|)d\mu \quad , \quad (f \in L_\phi)$.

ii) $\|Tf\|_\infty \leq \|f\|_\infty \quad , \quad (f \in L_\infty \cap L_\phi)$.

iii) There exists $h \in L_\phi$, $h \neq 0$ a.e., such that $Th = h$.

Then, $\|Tf\|_1 \leq \|f\|_1$ for every $f \in L_1 \cap L_\phi$.

Proof: Let ψ be the complementary N-function of ϕ, T^* the adjoint operator of T and let $\{A_n\}$ be an increasing sequence of measurable sets with $\mu(A_n) < \infty$ and $X = \cup A_n$. Then, for every $g \in L_1 \cap L_\psi$ we have

$$\int_X |T^*g|d\mu = \lim_{n\to\infty} \Big| \int_X gT(\chi_{A_n} \operatorname{sig} \overline{T^*g})d\mu \Big| \leq \|g\|_1 .$$

Consequently, $\|T^*g\|_\infty \leq \|g\|_\infty$ for every $g \in L_\psi \cap L_\infty$ and therefore for any $f \in L_1 \cap L_\phi$ and $n \geq 1$ we get $\big| \int_X fT^*(\chi_{A_n} \operatorname{sig} \overline{Tf})d\mu \big| \leq \|f\|_1$ and thus $\|Tf\|_1 \leq \|f\|_1$.

3. Ergodic results

Theorem 3.1. *(Dominated, individual and mean weighted ergodic theorem). Let ϕ and T satisfy the hypotheses of the extrapolation theorem 2.5 or 2.14. Then*

a) The ergodic maximal operator M_T-defined by (1.1) is bounded in $[L_\phi , \| \ \|_{(\phi)}]$.

b) If $\{b_k\}$ is a bounded Besicovitch sequence, then for every $f \in L_\phi$ there exists $f^* \in L_\phi$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} b_k T^k f(x) = f^*(x) \quad a.e. , \qquad \lim_{n\to\infty} \Big\| \frac{1}{n} \sum_{k=0}^{n-1} b_k T^k f - f^* \Big\|_{(\phi)} = 0.$$

Proof: Since $L_1 \cap L_\infty \subset L_\phi$ it follows from Theorem 2.5 or 2.14 that $T: L_1 \cap L_\phi \longrightarrow L_1$ admits an unique extension $\hat{T}: L_1 \longrightarrow L_1$ which is a Dunford-Schwartz operator, that is, $\|\hat{T}f\|_1 \leq \|f\|_1$, $f \in L_1$, and $\|\hat{T}f\|_\infty \leq \|f\|_\infty$, $f \in L_1 \cap L_\infty$. Therefore the linear modulus P of $\hat{T}$ is also a Dunfort-Schwartz operator.

Consequently, for every $f \in L_1$ and $\lambda > 0$ we have (see Theorem 2.3.2 in [4])

$$\mu\{x \in X : M_P f(x) > \lambda\} \leq \lambda^{-1} \int_X |f|d\mu ,$$

where M_P is the maximal operator associated to P. Moreover, trivially, $\|M_P f\|_\infty \leq \|f\|_\infty$ for $f \in L_1 \cap L_\infty$.

For $f \in L_1 \cap L_\phi$ set $f_\lambda = f\chi_{A(\lambda)}$ and $f^\lambda = f - f_\lambda$ where $A(\lambda) = \{x \in X : |f(x)| > \lambda/2\}$. We have $f_\lambda \in L_1$, $f^\lambda \in L_1 \cap L_\infty$ and therefore

$$(3.2)\quad \int_X \phi(M_P f)d\mu = \int_0^\infty \varphi(\lambda)\mu\{x \in X : M_P f(x) > \lambda\}d\lambda \leq$$
$$\leq 2\int_0^\infty \lambda^{-1}\varphi(\lambda)\Big(\int_X |f_\lambda| d\mu\Big)d\lambda = 2\int_X |f(x)|\Big(\int_0^{2|f(x)|} \lambda^{-1}\varphi(\lambda)d\lambda\Big)d\mu(x),$$

where φ is the density function of ϕ.

Integrating by parts, we obtain

$$(3.3)\qquad \int_0^s \lambda^{-1}\varphi(\lambda)d\lambda = s^{-1}\phi(s) + \int_0^s \lambda^{-2}\phi(\lambda)d\lambda \quad , \quad (s > 0).$$

Since the N-function complementary of ϕ satisfies Δ_2 there exists a constant $\beta > 1$ such that $\beta\phi(s) \leq s\varphi(s)$, $s \geq 0$; then, if $0 < \lambda < 1$ we have that $\lambda^{-2}\phi(\lambda) \leq \phi(1)\lambda^{\beta-2}$ and therefore $\int_{(0,s]} \lambda^{-2}\phi(\lambda)d\lambda < \infty$. Then, (3.3) shows that

$$\int_0^s \lambda^{-1}\varphi(\lambda)d\lambda < \beta(\beta-1)^{-1}s^{-1}\phi(s), \quad (s > 0).$$

Hence, it follows from (3.2) that

$$(3.4)\qquad \int_X \phi(M_P f)d\mu \leq \alpha\beta(\beta-1)^{-1}\int_X \phi(|f|)d\mu \quad (f \in L_1 \cap L_\phi),$$

where α is a constant in the Δ_2-condition for ϕ.

Since $|\hat{T}f| \leq P|f|$ for $f \in L_1$, (3.4) shows that there exists a constant $C_1 > 0$ such that $\int_X \phi(M_T f)d\mu \leq C_1 \int_X \phi(|f|)d\mu$, $f \in L_1 \cap L_\phi$, which proves that $\|M_T f\|_{(\phi)} \leq C\|f\|_{(\phi)}$, $f \in L_1 \cap L_\phi$, where $C = \max(1, C_1)$. Since $L_1 \cap L_\phi$ is a dense linear subspace of $[L_\phi, \| \ \|_{(\phi)}]$ it follows that $\|M_T f\|_{(\phi)} \leq C\|f\|_{(\phi)}$ for every $f \in L_\phi$, which proves a).

Now, let $\{b_k\}$ be a bounded Besicovitch sequence; then a) and the Banach principle show that for almost everywhere convergence it is enough to prove that the weighted averages

$$T_n f = \frac{1}{n}\sum_{k=0}^{n-1} b_k T^k f$$

converges a.e. for all f in a dense subset of $[L_\phi, \| \ \|_{(\phi)}]$.

Let $m \in \mathbb{N}$ and $S : L_\phi \longrightarrow L_\phi$ defined by $Sf = e^{im}Tf$. Since L_ϕ is reflexive and the powers S^k, $k \geq 0$, are uniformly bounded, exactly $\|S^k f\|_{(\phi)} \leq \|f\|_{(\phi)}$ for every $f \in L_\phi$ and $k \geq 0$, then , the Césàro averages $R_n f = n^{-1}(f + Sf + \ldots S^{n-1})$ converge in norm for every $f \in L_\phi$. Therefore L_ϕ is the closure of the direct sum of the set of fixed points of S and the space $(I-S)L_\phi$ (see 2.1 in [6]).

On the other hand, given $\beta > 1$ such that $\beta\phi(s) \leq s\varphi(s)$, $s \geq 0$, the function $s \longrightarrow s^{-\beta}\phi(s)$ increases for $s > 0$ and consequently $\phi(st) \leq s^\beta \phi(t)$ for $0 \leq s \leq 1$ and $t \geq 0$. Therefore, if $g \in L_\phi$ we have

$$\int_X \sum_{n=1}^{\infty} \phi(|n^{-1}S^n g|)d\mu \leq \sum_{n=1}^{\infty} n^{-\beta} \int_X \phi(|S^n g|)d\mu \leq$$

$$\leq \int_X \phi(|g|)d\mu \sum_{n=1}^{\infty} n^{-\beta} < \infty.$$

Hence $n^{-1}S^n g(x) \longrightarrow 0$ a.e. as $n \longrightarrow \infty$ and thus $R_n f \longrightarrow 0$ a.e. if $f = g - Sg$.

Since the maximal operator M_S is bounded in $[L_\phi, \|\ \|_{(\phi)}]$ we obtain that, for any $f \in L_\phi$, $n^{-1}\sum_{k=0}^{n-1} e^{\mathrm{im}\,k} T^k f$ converges a.e. and therefore for every trigonometric polynomial α and $f \in L_\phi$ we have that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \alpha(k) T^k f(x)$$

exists and is finite a.e.

Then, for every $f \in L_\phi \cap L_\infty$, $T_n f$ converges a.e. since for every $\varepsilon > 0$ there exists a trigonometric polynomial α_ε such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |b_k - \alpha_\varepsilon(k)| < \varepsilon$$

and consequently

$$\limsup_{n\to\infty} \Big|T_n f(x) - \frac{1}{n}\sum_{k=0}^{n-1} \alpha_\varepsilon(k) T^k f(x)\Big| < \varepsilon \|f\|_\infty \quad \text{a.e.}$$

In this way, since $L_\phi \cap L_\infty$ is dense in L_ϕ, we conclude that $T_n f$ converges almost everywhere for every $f \in L_\phi$.

Finally, let $f^*(x) = \lim_{n\to\infty} T_n f(x)$. It follows from a) that $f^* \in L_\phi$ and $\phi(|T_n f - f^*|)$ is dominated by $\phi(M_T f) \in L_1$; thus, taking into account the Lebesgue's dominated theorem, we get that $\lim_{n\to\infty} \int_X \phi(|T_n f - f^*|)d\mu = 0$ which proves that $\lim_{n\to\infty} \|T_n f - f^*\|_{(\phi)} = 0$.

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