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ORLICZ SPACES FOR WHICH THE HARDY–LITTLEWOOD MAXIMAL OPERATOR IS BOUNDED

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Abstract

Let M be the Hardy–Littlewood maximal operator defined by

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f|dx \quad , \quad (f \in L_{\rm loc}(\mathbb{R}^n)),$$

where the supremum is taken over all cubes Q containing x and $|Q|$ is the Lebesgue measure of Q. In this paper we characterize the Orlicz spaces L^*_ϕ, associated to N–functions ϕ, such that M is bounded in L^*_ϕ. We prove that this boundedness is equivalent to the complementary N–function ψ of ϕ satisfying the Δ_2–condition in $[0,\infty)$, that is, $\sup_{s>0} \psi(2s)/\psi(s) < \infty$.

1. Introduction

It is known that for the *Hardy–Littlewood maximal operator*, on $\mathbb{R}$, defined by

$$\theta(t;f) = \sup_{|t-\tau|>0} \frac{1}{\tau - t}\int_t^\tau |f(s)|ds, \tag{1.1}$$

to act boundedly in a symmetric space E it is necessary and sufficient that the following condition be satisfied

$$||\sigma_\tau||_E = o(\tau) \quad \text{as } \tau \to \infty, \tag{1.2}$$

where σ_τ are the dilation operators defined by $\sigma_\tau f(t) = f(\tau^{-1}t)$. (See Ch. II, Theorem 6.10 in [**2**]).

In this paper we consider the more general Hardy–Littlewood maximal operator M, on $\mathbb{R}^n$, defined by

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f|dx \quad , \quad (f \in L^1_{\rm loc}(\mathbb{R}^n)), \tag{1.3}$$

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where the supremum is taken over all cubes Q containing x and $|Q|$ is the Lebesgue measure of Q. (Cube will mean a compact cubic interval with nonempty interior).

Our aim is to characterize the Orlicz spaces L^*_ϕ, associated to N–functions ϕ, for which the operator defined by (1.3) is bounded. Such Orlicz spaces are symmetric spaces, but, for the case $n = 1$, the proof given in here represents a direct and different proof from that given in [**2**]. Moreover, our characterizing condition is more manageable than (1.2).

Except in basic questions, this paper is intended to be selfcontained.

Now, we shall present the basic definitions and results concerning to N–functions and Orlicz spaces which will be used in this paper. The proofs of most of these results can be found in [**1**] or in II–13 of [**3**].

An N–function is a continuous and convex function $\phi : [0,\infty) \to \mathbb{R}$ such that $\phi(s) > 0$, $s > 0$, $s^{-1}\phi(s) \to 0$, as $s \to 0$, and $s^{-1}\phi(s) \to \infty$, as $s \to \infty$.

As example of N–functions we have: $\phi_1(s) = s^p$, $p > 1$; $\phi_2(s) = s^p \log^k(1+s)$, $p \geq 1$ and $k > 0$; $\phi_4(s) = e^s - s - 1$; $\phi_5(s) = (1+s)\log(1+s) - s$; $\phi_5(s) = = \exp s^2 - 1$ and $\phi_6(s) = \int_0^s \rho$ where $\rho : [0,\infty) \to [0,\infty)$ is defined by $\rho(0) = 0$, $\rho(t) = 2^{-n}$ if $t \in [2^{-n}, 2^{-n+1})$ and $\rho(t) = 2^{n-1}$ if $t \in [2^{n-1}, 2^n)$, n a positive integer.

An N–function ϕ has the representation $\phi(s) = \int_0^s \varphi$ where $\varphi : [0,\infty) \to \mathbb{R}$ is continuous from the right, non decreasing such that $\varphi(s) > 0$, $s > 0$, $\varphi(0) = 0$ and $\varphi(s) \to \infty$ for $s \to \infty$. More precisely φ is the right derivate of ϕ and will be called *the density function of* ϕ.

Associated to φ we have the function $\rho : [0,\infty) \to \mathbb{R}$ defined by $\rho(t) = \sup\{s : \varphi(s) \leq t\}$ which has the same aforementioned properties of φ. We will call ρ *the generalized inverse* of ϕ.

The N–function ψ defined by $\psi(t) = \int_0^t \rho$ is called *the complementary N-function* of ϕ. Thus, if $\phi(s) = p^{-1}s^p$, $p > 1$, then $\psi(t) = q^{-1}t^q$ where $pq = p+q$.

Young's inequality asserts that $st \leq \phi(s) + \psi(t)$ for $s, t \geq 0$, equality holding if and only if $\varphi(s-) \leq t \leq \varphi(s)$ or else $\rho(t-) \leq s \leq \rho(t)$.

An N–function ϕ is said to satisfy the Δ_2*–condition* in $[0,\infty)$ (or merely the Δ_2–condition) if $\sup_{s>0} \phi(2s)/\phi(s) < \infty$. If φ is the density function of ϕ, then, ϕ satisfies Δ_2 if and only if there exists a constant $\alpha > 1$ such that $s\varphi(s) < \alpha\phi(s)$, $s > 0$.

The Δ_2–condition for ϕ does not transfer necessarily to the complementary N–function; for example, ϕ defined by $\phi(s) = (1+s)\log(1+s) - s$ satisfies the Δ_2–condition but its complementary N–function ψ, defined by $\psi(t) = e^t - t - 1$, does not. In this paper the Δ_2–condition for the complementary N–function ψ of ϕ plays a fundamental role; precisely, this is the characterizing condition for the boundeddness of the Hardy–Littlewood maximal operator defined in (1.3). For this reason it is very interesting to give some characterizations of this condition, which permit to know wheter ψ satisfies Δ_2 even if we do not know explicitly the function ψ.

First, it is known that ψ satisfies the Δ_2–condition in $[0, \infty)$ if and only if there exists a constant $a > 1$ such that $\phi(s) \leq (2a)^{-1}\phi(as)$, $s \geq 0$.

The following characterization, which is used in this paper, appear in the literature with more restrictive hypothesis than the one we shall use and so we shall include its proof.

Proposition 1.4. *The complementary N-function of ϕ satisfies the Δ_2-condition in $[0, \infty)$ if and only if $\inf_{s>0} s\varphi(s)/\phi(s) > 1$, where φ is the density function of ϕ.*

Proof of (1.4): Let ρ be the generalized–inverse of φ and let ψ be the complementary N–function of ϕ. Assume that ψ satisfies Δ_2; then, there exists $\alpha > 1$ such that $t\rho(t) < \alpha\psi(t)$ for every $t > 0$. On the other hand, it follows from the equality cases in Youngs's inequality that $t\rho(t) = \phi(\rho(t)) + \psi(t)$ and therefore $\phi(\rho(\varphi(s)))/\rho(\varphi(s)) < \alpha^{-1}(\alpha - 1)\varphi(s)$, $s > 0$. Since $\rho(\varphi(s)) \geq s$ and the function $u \to u^{-1}\phi(u)$ increases strictly for $u > 0$ we obtain that $\inf_{s>0} s\varphi(s)/\phi(s) > \alpha(\alpha - 1)^{-1}$.

Assume now that $\inf_{s>0} s\varphi(s)/\phi(s) > 1$; then, there exists $\beta > 1$ such that the function $s \to s^{-\beta}\phi(s)$ increases strictly for $s > 0$ and, therefore, there exists $a > 1$ such that $\phi(s) < (2a)^{-1}\phi(as)$, $s > 0$. Thus, ψ satisfies Δ_2 and this finishes the proof. ∎

If $(X, \mathcal{M}, \mu)$ is a σ–finite measure space we denote by $\mathfrak{M}$ the space of $\mathcal{M}$–measurable and μ–a.e. finite functions from X to $\mathbb{R}$ (or to $\mathbb{C}$). If ϕ is an N–function *the Orlicz spaces* $L_\phi \equiv L_\phi(X, \mathcal{M}, \mu)$ and $L^*_\phi \equiv L^*_\phi(X, \mathcal{M}, \mu)$ are defined by $L_\phi = \{f \in \mathfrak{M} : \int_X \phi(|f|)d\mu < \infty\}$ and $L^*_\phi = \{f \in \mathfrak{M} : fg \in L_1(\mu)$ for all $g \in L_\psi\}$, where ψ is the complementary N–function of ϕ.

We have $L_\phi \subset L^*_\phi$ and L^*_ϕ coincides with the set of $\mathfrak{M}$ such that $\lambda f \in L_\phi$ for some positive real λ.

The space L^*_ϕ is a linear space with the usual operations and we may define the following norms in L^*_ϕ:

$$||f||_\phi = \sup\{\int_X |fg|d\mu : g \in S_\psi\},$$

where $S_\psi = \{g \in L_\psi : \int_X \psi(|g|)d\mu \leq 1\}$, and

$$||f||_{\phi)} = \inf\{\lambda > 0 : \int_X \phi(\lambda^{-1}|f|)d\mu \leq 1\},$$

which are called *the Orlicz norm* and *the Luxemburg norm* respectively. Both norms are equivalent, actually $||f||_{(\phi)} \leq ||f||_\phi \leq 2||f||_{(\phi)}$, and they make L^*_ϕ into a Banach space.

For a measurable set A, with $0 < \mu(A) < \infty$, $||\chi_A||_\phi = \mu(A)\psi^{-1}(1/\mu(A))$, where χ_A denotes the characteristic function of A.

If $\phi(s) = s^p$, $p > 1$, then, $L^*_\phi = L_\phi = L_p$, $||f||_{(\phi)} = ||f||_p$ and $||g||_\psi = ||g||_q$, where $pq = p + q$.

2. The main result

Theorem 2.1. *Let ϕ an N-function, ψ the complementary N-function of ϕ, L^*_ϕ the Orlicz space associated to ϕ and let M be the Hardy–Littlewood maximal operator defined in (1.3). The following conditions are equivalent:*

(a) There exist positive constants A and b such that

$$\int_{\mathbb{R}^n} \phi(bMf)dx \leq A \int_{\mathbb{R}^n} \phi(|f|)dx \quad , \quad (f \in L^*_\phi).$$

(b) There exists a positive constant C such that

$$||Mf||_{(\phi)} \leq C||f||_{(\phi)} \quad , \quad (f \in L^*_\phi).$$

(c) There exists a positive constant K such that

$$||M\chi_A||_{(\phi)} \leq K||\chi_A||_{(\phi)} \quad , \quad (|A| < \infty).$$

(d) ψ satisfies the Δ_2-condition in $[0, \infty)$.

Proof: It suffices to prove $(a) \Rightarrow (b)$, $(c) \Rightarrow (d)$ and $(d) \Rightarrow (a)$.

The proof of $(a) \Rightarrow (b)$ is easy. In fact, if $f \in L^*_\phi$ there exists $\lambda > 0$ such that $\lambda f \in L_\phi$; hence $\lambda bMf \in L_\phi$ and therefore $Mf \in L^*_\phi$. Moreover, we have

$$\int_{\mathbb{R}^n} \phi(Mf/(b^{-1}\max(1, A)||f||_{(\phi)}))dx <$$

$$< A(\max(1, A))^{-1} \int_{\mathbb{R}^n} \phi(|f|/||f||_{(\phi)})dx \leq 1,$$

for $f \neq 0$ and thus we get $||Mf||_{(\phi)} \leq b^{-1}\max(1, A)||f||_{(\phi)}$ for every $f \in L^*_\phi$. ∎

The proof of $(d) \Rightarrow (a)$ follows from the following *interpolation* result, taking into account that M is of weak type (1,1) and bounded in L_∞.

Theorem 2.2. *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{F}, v)$ be two σ-finite measure spaces, ϕ an N-function whose complementary N-function satisfies the Δ_2-condition and let T: $L_1(\mu) + L_\infty(\mu) \to \mathfrak{M}(Y)$ be a quasi-additive operator which is simultaneously of weak type (1,1) and of type (∞, ∞). Then, T is defined on $L^*_\phi(\mu)$ and there exist positive constants A and m such that*

$$\int_Y \phi(|Tf|)dv \leq A \int_X \phi(m|f|)d\mu \quad , \quad (f \in L^*_\phi(\mu)). \tag{2.3}$$

Proof of Theorem 2.2: By hypothesis there exists a constant C such that

$$v\{y \in Y : |Tg(y)| > \lambda\} \leq C\lambda^{-1} \int_X |g| d\mu,$$
$$||Th||_\infty \leq C||h||_\infty \quad , \quad |T(g+h)| \leq C(|Tg| + |Th|)$$

for every $g \in L_1(\mu)$, $h \in L_\infty(\mu)$ and $\lambda > 0$.

For $f \in L^*_\phi$ and $\lambda > 0$ let $f_\lambda = f\chi_{A(\lambda/2C^2)}$ and $f^\lambda = f - f_\lambda$ where $A(\alpha) = \{x \in X : |f(x)| > \alpha\}$. We have $f_\lambda \in L_1(\mu)$ and $f^\lambda \in L_\infty(\mu)$ and therefore

$$\int_Y \phi(|Tf|)dv = \int_0^\infty \varphi(\lambda)v\{y \in Y :\ |Tf(y)| > \lambda\}d\lambda \leq$$
$$\leq 2C^2 \int_0^\infty \lambda^{-1}\varphi(\lambda)(\int_X |f_\lambda| d\mu)d\lambda =$$
$$= 2C^2 \int_X |f(x)|(\int_0^{2C^2|f(x)|} \lambda^{-1}\varphi(\lambda)d\lambda)d\mu(x),$$

where φ is the density function of ϕ.

Integrating by parts, we obtain

$$\int_0^s \lambda^{-1}\varphi(\lambda)d\lambda = s^{-1}\phi(s) + \int_0^s \lambda^{-2}\phi(\lambda)d\lambda \quad , \quad (s > 0). \tag{2.4}$$

Since ψ satisfies Δ_2, it follows from Proposotion 1.4 that there exists $\beta > 1$ such that $\beta\phi(s) < s\varphi(s)$, $s > 0$; then, if $0 < \lambda < 1$ we have that $\lambda^{-2}\phi(\lambda) \leq \phi(1)\lambda^{\beta-2}$ and therefore $\int_0^s \lambda^{-2}\phi(\lambda)d\lambda$ is finite. then (2.4) shows that

$$\int_0^s \lambda^{-1}\varphi(\lambda)d\lambda < \beta(\beta-1)^{-1}s^{-1}\phi(s) \quad , \quad (s > 0)$$

and thus, we obtain (2.3) with $A = \beta(\beta-1)^{-1}$ and $m = 2C^2$. ■

Note. If T is also positively homogenous then it follows from (2.3) that

$$\int_Y \phi(m^{-1}|Tf|)dv \leq A \int_X \phi(|f|)d\mu \quad , \quad (f \in L^*_\phi(\mu))$$

and T applies $L^*_\phi(X, \mathcal{M}, \mu)$ in $L^*_\phi(Y, \mathcal{F}, v)$.

Proof of $(c) \Rightarrow (d)$*:* It follows from (c) that there exists a constant $K > 0$ such that $||M_E\chi_A||_\phi \leq K||\chi_A||_\phi$ for every A with $|A| < \infty$,with M_E being the maximal operator on balls, defined by $M_Ef(x) = \sup_{x\in B} |B|^{-1} \int_B |f|dx$, where the supremum is taken over all euclideans balls B containing x and $|| \ ||_\phi$ is the Orlicz norm.

We denote by $B(x; r)$ the ball with center x and radius r and let a_n be the measure of $B(0; 1)$. For every pair of reals (v, s), with $v > 0$ and $s > 1$, we denote by $A(v,s)$ and $D(v,s)$ the balls $B(0;(a_n vs)^{-1/n})$ and $B(0;(a_n v)^{-1/n})$ respectively.

If $x \notin A(v,s)$ then $A(v,s) \subset B(x; 2\|x\|_2)$ and therefore

$$M_E \chi_{A(v,s)}(x) \geq (2^n a_n vs \|x\|_2^n)^{-1}.$$

On the other hand, if $g = \psi^{-1}(v)\chi_{D(v,s)}$ we have $\int_{\mathbb{R}^n} \psi(|g|)dx = 1$ and consequently

$$\|M_E \chi_{A(v,s)}\|_\phi \geq \psi^{-1}(v) \int_{D(v,s)} M_E \chi_{A(v,s)} dx \geq$$

$$\geq (2^n a_n vs)^{-1} \psi^{-1}(v) \int_{(a_n vs)^{-1/n} < \|x\|_2 < (a_n v)^{-1/n}} \|x\|_2^{-n} dx =$$

$$= (2^n vs)^{-1} \psi^{-1}(v) \log s.$$

Since $\|\chi_{A(v,s)}\|_\phi = (vs)^{-1}\psi^{-1}(vs)$ we conclude that there exists $K > 0$ such that

$$2^{-n}\psi^{-1}(v)\log s \leq K\psi^{-1}(sv) \qquad (v > 0)\ (s > 1).$$

Therefore $2\psi^{-1}(v) \leq \psi^{-1}(v \exp(K2^{n+1}))$, $v > 0$, which proves that $\psi(2t) \leq$ $\leq \exp(K2^{n+1})\psi(t)$ for every $t > 0$ and so ψ satisfies Δ_2 in $[0,\infty)$. ∎

Final Remark. It follows from Theorem 2.1 that *the Δ_2-condition on the complementary N-function of ϕ cannot be eliminated from the hypothesis of the interpolation theorem 2.2* since if the complementary N–function of ϕ does not satisfy Δ_2 (for example, in the case of ϕ defined by $\phi(s) = (1+s)\log(1+$ $+s) - s$) the result of the interpolation theorem does not hold for ϕ and the Hardy–Littlewood maximal operator in spite of the fact that this operator is of weak type $(1,1)$ and of type (∞,∞). (Moreover, observe that in the above-mentioned case the N–function ϕ satisfies the Δ_2–condition).

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