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INFINITE LOCALLY FINITE GROUPS OF TYPE $PSL(2,K)$ OR $Sz(K)$ ARE NOT MINIMAL UNDER CERTAIN CONDITIONS

JAVIER OTAL AND JUAN MANUEL PEÑA

Abstract

In classifying certain infinite groups under minimal conditions it is needed to find non-simplicity criteria for the groups under consideration. We obtain some of such criteria as a consequence of the main result of the paper and the classification of finite simple groups.

Introduction. If X is a class of groups by a *minimal non-X- group* we mean a group G which is not a X-group but in which every proper subgroup is a X-group. In [**1**], [**2**] and [**3**] Bruno and Phillips have studied various minimality conditions, which have been extended in [**7**], [**8**] and [**9**] by the authors of the present paper. Roughly speaking, our extension has consisted in replacing the term "finite group" in Bruno-Phillips cases by "Černikov group" in ours, although the results we have found have a rather different nature. In order to obtain the main results of the mentioned papers it has been needed to give some non-simplicity criteria for the groups under consideration. These criteria are now a consequence of some results obtained with the aid of the classification of finite simple groups (see [**6**] and [**12**]), which reduce the simple cases to projective special linear groups $PSL(2,K)$ and Suzuki groups $Sz(K)$, and of the main result of this paper, which we state as follows:

Theorem. *Let K be an infinite locally finite field, and let C be one of the following classes of groups: (i) CC-groups; (ii) Černikov-by-hypocentral groups; (iii) hypercentral-by-Černikov groups. Then no group of type $PSL(2,K)$ or $Sz(K)$ is a minimal non-C-group.*

If X is one of the classes of groups considered in [**1**], [**2**], [**3**], [**7**], [**8**] or [**9**] and G is a locally graded minimal non-X-group, then it is proved that G is locally finite and, in the Černikov-by-nilpotent and nilpotent-by-Černikov cases, the proper subgroups of G are soluble-by-finite, so that [**6**] or [**12**] and the present Theorem ensure that G cannot be simple. Another application of our Theorem also gives information on the structure of a (locally graded) minimal non-CC-group (see [**8**, 4.2]).

Throughout the paper we use the standard group-theoretic notation from [**11**]. The proof of our Theorem will be carried out in sections 3 and 4, where we shall construct proper subgroups of such simple group which are not C-groups for each C considered. Some of these examples are similar to those given in [**10**] and in Bruno-Phillips papers. However our cases need somewhat more than the infinitude of the field K, namely the structure of its multiplicative group K^*, which will be given in section 1, and, moreover, we have needed to look for certain Frobenius subgroups in such simple groups in order to solve the hypercentral-by-Černikov case, whose details will be done in section 2.

1. The multiplicative group of a locally finite field K. We recall that K is exactly an algebraic extension of a finite field. If p is the characteristic of K, then the additive group K^+ of K is p-elementary abelian whereas the multiplicative group K^* is locally cyclic and can be embedded in a direct product of Prüfer groups, one for each prime different of p (see [**4**]). The key point in what follows is the following well-known fact, of which we give a proof for the reader's convenience.

(1.1). *K^* is a Černikov group if and only if K is finite.*

Its proof needs an auxiliary result:

(1.2). *If p is a prime, $m \geq 1$ is an integer and q is either an odd prime or 4, then the factorization of $p^{mq} - 1$ at least contains one more prime than that of $p^m - 1$.*

Proof: Suppose that the result is false. If r is a prime divisor of $N :=$ $= p^{m(q-1)} + p^{m(q-2)} + \cdots + p^m + 1$, then r divides $p^m - 1$. Thus r divides each one of $p^{m(q-1)} - 1, \ldots, p^m - 1$ so that r divides q. Therefore $r = q$, if q is odd, or $r = 2$, otherwise.

In the first case $N = q^s$, $s \geq 1$, $p^m - 1 = q^t n$, $t \geq 1$ and $(n, q) = 1$. Thus $q^{t+s}n = p^{mq} - 1 = (q^t n + 1)^q - 1 = q^{t+1}n(a+1)$ and so $a + 1 = q^{s-1}$, where $a = \sum_{i=0}^{q-2} \binom{q}{i} q^{t(q-i-1)} n^{(q-i-1)}$. Since $q \neq 2$, we have $a > 1$ and $s - 1 \geq 1$. Clearly q divides a and so q divides 1, a contradiction.

If $r = 2$, then $(p^{2m} + 1)(p^m + 1) = N = 2^s$ and so $p^m + 1 = 2^t$ and $p \neq 2$. Thus $(p^m + 1)^2 - 2p^m = p^{2m} + 1 = 2^{s-t}$ and it follows that $s - t \geq 2$. Therefore 4 divides $2p^m$, a contradiction. ∎

Proof of (1.1): Suppose that K is infinite. Let $K_1 < K_2 < \ldots$ be an infinite ascending chain of finite subfields of K. If $|K_i| = p^{r_i}$, then r_i divides r_{i+1} for every $i \geq 1$, so that $r_{i+2} = r_i q n_i$ where $n_i \geq 1$ and q is either an odd prime or 4. By applying the Lemma we deduce that the set of all primes occurring as divisors of the $|K_i^*|$, $i \geq 1$, is infinite and therefore K^* cannot be Černikov. ∎

2. Locally finite hypercentral-by-Černikov Frobenius groups. We refer to the section 1.J of [**5**], where the abstract definition and the basic properties of a locally finite Frobenius group can be found. In what follows, we shall frequently use the following characterization of Frobenius groups ([**5**, 1.J.3]), which is an extension of that of the finite case: *If G is a locally finite group, then G is a Frobenius group if and only if G has a proper normal subgroup N such that $C_G(x) \leq N$ for every $1 \neq x \in N$.* In fact, it is not hard to show that such N is exactly the Frobenius kernel of G.

The result we shall need in the next sections is the following:

(2.1). *Let G be a locally finite Frobenius group with complement C. Then the following are equivalent: (1) C is Černikov. (2) G is nilpotent-by-Černikov. (3) G is hypercentral-by-Černikov.*

Proof: Let N be the kernel of G. Since N is always nilpotent, (1) implies (2). That (2) implies (3) is trivial so that it suffices to show that (3) implies (1).

Let L be a hypercentral normal subgroup of G such that G/L is Černikov. Clearly, we may assume that $L \neq 1$. If $L \cap N = 1$, by [**5**, 1.J.2] we have that L is contained in some conjugate of C so that L is contained in every conjugate of C, which gives $L = 1$. Therefore $L \cap N \neq 1$ and so $\varsigma(L) \cap N \neq 1$ since L is hypercentral. If $1 \neq x \in \varsigma(L) \cap N$, then $L \leq C_G(x)$ and, since $C_G(x) \leq N$, it follows that $L \leq N$. Therefore $G/N \simeq C$ is a Černikov group. ∎

3. The $PSL(2,K)$ case, K infinite locally finite. Let H be the image in $PSL(2,K)$ of the group of lower triangular matrices in $SL(2,K)$. $H = UD$ is the semidirect product of the image U of the group of lower unitriangular matrices by the image D of the group of diagonal matrices in $SL(2,K)$. We remark that $U \simeq K^+$ and that D is isomorphic to a quotient of K^* by a subgroup of order at most two so that D is not a Černikov group by (1.1).

If $a \in K$ and $b \in K^*$, then we have

$$\begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ b^2 a & 1 \end{pmatrix}. \tag{1}$$

Thus x^H is infinite for every $1 \neq x \in U$ and, since $x^H \leq U$, it follows that H is not a CC-group.

Moreover,

$$\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ b^2 a & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a(b^2 - 1) & 1 \end{pmatrix}. \tag{2}$$

From (1) and (2) we readily obtain that $U = [U,D] = [U,H] = H'$. By induction it is easy to show that $U = \gamma_\alpha(H)$ for every ordinal $\alpha \geq 2$. Therefore H is not Černikov-by-hypocentral.

Finally, from (1) we obtain $C_H(x) \leq U$ for every $1 \neq x \in U$. Therefore H is a Frobenius group with complement D and so H cannot be hypercentral-by-Černikov by (2.1).

4. The $Sz(K)$ case, K infinite locally finite of characteristic 2. As in the finite case (see [**13**]) the group $Sz(K)$ is a subgroup of $SL(4,K)$ defined in terms of an automorphism θ of K satisfying $a^{\theta^2} = a^2, a \in K$, and generated by:

(i) a group Q of matrices having the form

$$(a,b) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ a^{1+\theta} + b & a^\theta & 1 & 0 \\ a^{2+\theta} + ab + b^\theta & b & a & 1 \end{pmatrix} \qquad a, b \in K.$$

(ii) the group D of diagonal matrices of the form

$$\bar{f} := diag[f^{1+\theta^{-1}}, f^{\theta^{-1}}, f^{-\theta^{-1}}, f^{-1-\theta^{-1}}] \qquad f \in K^*.$$

(iii) the permutation matrix

$$\tau = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Clearly $D \simeq K^*$, so that D is not Černikov by (1.1), and τ is an involution inverting every element of D. Thus $L := < D, \tau >$ is metabelian and $L' \leq D$. If $\bar{f} \in D$, then $[\bar{f}, \tau] = \bar{f}^{-2}$ and since $(|\bar{f}|, 2) = 1$, it follows that $D = [D,\tau] = [L,\tau]$. Hence L is not a CC-group. Moreover $D = [D,L] = D'$ and by induction it is easy to show that $D = \gamma_\alpha(L)$ for every ordinal $\alpha \geq 2$. Therefore L is not Černikov-by-hypocentral.

Let $G := QD$ be the semidirect product of Q by D. Clearly we have that $(a_1, b_1)(a_2, b_2) = (a_1 + a_2, a_1 a_2^\theta + b_1 + b_2)$ and $(a,b)^{\bar{f}} = (af, bf^{1+\theta})$. Let $1 \neq x = (a,b) \in Q$. If $\bar{f}(c,d) \in C_G(x)$, then $\bar{f}(c + a, ca^\theta + d + b) = \bar{f}(af + c, afc^\theta + bf^{1+\theta} + d)$ so that $af = a$ and $ca^\theta + b = afc^\theta + bf^{1+\theta}$. If $f \neq 1$, then $f^{1+\theta} \neq 1$ and we find $a = 0$ and $b = 0$, a contradiction. Therefore $\bar{f} = 1$ and our argument has just showed that $C_G(x) \leq Q$ for every $1 \neq x \in Q$. Thus G is a Frobenious group with complement D and hence G cannot be hypercentral-by-Černikov by (2.1).

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Departamento de Matemáticas
Facultad de Ciencias
Universidad de Zaragoza
50009–Zaragoza, SPAIN.

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