Mathware & Soft Computing 14 (2007) 171-181

Approximation of Proximities by Aggregating T-indistinguishability Operators

L. Garmendia[1] and J. Recasens[2]
[1]Ingeniería del Software e Inteligencia Artificial. Fac. de Informática
Univ. Complutense de Madrid
C/ Profesor José García Santesmases, s/n. 28040 Madrid. Spain
[2]Secció Matemàtiques i Informàtica
ETS Arquitectura del Vallès. Univ. Politècnica de Catalunya
C/ Pere Serra 1-15. 08190 Sant Cugat del Vallès. Spain
lgarmend@fdi.ucm.es, j.recasens@upc.edu

Abstract

For a continuous Archimedean t-norm T a method to approximate a proximity relation R (i.e. a reflexive and symmetric fuzzy relation) by a T-transitive one is provided.

It consists of aggregating the transitive closure $\overline{R}$ of R with a (maximal) T-transitive relation B contained in R using a suitable weighted quasi-arithmetic mean to maximize the similarity or minimize the distance to R.

Keywords: Proximity, Transitive Closure, Transitive Opening, T-indistinguishability Operator, Aggregation Operator.

1 Introduction

T-indistinguishability operators are one of the most important kind of fuzzy relations since they fuzzify the concept of crisp equality and crisp equivalence relation. They were introduced by Zadeh in 1971 [14] and have been the subject of many papers ([3],[6],[10],[13]).

Proximity matrices, also called tolerance relations (i.e., reflexive and symmetric fuzzy relations on a finite universe X) appear in many situations, such as cluster analysis, information systems, image clustering,....

Reflexivity and symmetry are important properties in Decision Making problems when the knowledge is modeled by a fuzzy relation. Tolerance relations contain information related to how close or similar the objects of a universe are and can be built from a decision table. However, when the relation must be used as a similarity or indistinguishability to compare or classify objects, some coherence between their relations is needed. T-transitivity becomes then relevant and must be imposed to the initial given information. In these cases transitivity of the proximity relation R with respect to a t-norm T is required, so that R is replaced by T-indistinguishability operator E, that should be as close to E as possible.

The most popular method to find such E is calculating the transitive closure of R which is the smallest T-indistinguishability operator greater or equal than R.

Alternative methods are calculating a transitive opening of R, which is a T-indistinguishability operator smaller or equal than R and maximal with this property, and generating an indistinguishability from the columns of R using the well known Representation Theorem of [13].

In all those three methods a matrix with either all the entries greater or equal than the entries of R or smaller or equal than them. If we want E to be as close as possible to R, algorithms allowing to find T-indistinguishability relations with some entries smaller, some equal and some greater than the entries of the original matrix are needed, since most of the times, the closest T-indistinguishability operator to R is not related to R by the inclusion relation.

Trying to find the closest E to R is very expensive. Indeed, if n is the cardinality of the universe X, the transitivity of T-indistinguishability operators can be modeled by $3\binom{n}{3}$ inequalities and they lay in the region of the $\binom{n}{2}$-dimensional space defined by them. The calculation of E becomes then a non-linear programming problem. Therefore, simpler methods to find a close E to R are desirable.

In Section 3 we propose to aggregate the transitive closure $\overline{R}$ of R and a transitive opening B or the relation obtained by the Representation Theorem $\underline{R}$ to find a better approximation of R when the t-norm is continuous Archimedean. Some examples illustrate the algorithm.

There are of course several ways to define the closeness of two fuzzy relations, many of them related to some metric. In this paper we propose a way related to the natural indistinguishability operator E_T associated to T, so that the degree of closeness or similarity between two fuzzy relations R and S is calculated aggregating the similarity of their respective entries using the quasi-arithmetic mean generated by an additive generator of T.

Also the Euclidean metric will be used as an alternative method to compare fuzzy relations.

2 Preliminaries

This Section contains some results on t-norms and indistinguishability operators that will be needed later on in the paper.

Though many results remain valid for arbitrary t-norms and especially for left continuous ones, for the sake of simplicity we will assume continuity for the t-norms throughout the paper.

Definition 2.1. The residuation $\overrightarrow{T}$ of a t-norm T is defined by

$$\overrightarrow{T}(x|y) = \sup\{\alpha \in [0,1] \mid T(x,\alpha) \leq y).$$

Definition 2.2. The natural T-indistinguishability E_T associated to a given t-norm T is the fuzzy relation on [0,1] defined by

$$E_T(x, y) = T(\overrightarrow{T}\ (x|y), \overrightarrow{T}\ (y|x)).$$

E_T is indeed a special kind of T-indistinguishability operator (Definition 2.3) [3] and in a logical context where T plays the role of the conjunction, E_T is interpreted as the bi-implication associated to T [7].

Definition 2.3. Given a t-norm T, a T-indistinguishability operator E on a set X is a fuzzy relation $E : X \times X \to [0, 1]$ satisfying for all $x, y, z \in X$

1. $E(x, x) = 1$ (Reflexivity)
2. $E(x, y) = E(y, x)$ (Symmetry)
3. $T(E(x, y), E(y, z)) \leq E(x, z)$ (T-transitivity).

Example 2.4.

1. If T is the Łukasiewicz t-norm, then $E_T(x, y) = 1 - |x - y|$ for all $x, y \in [0, 1]$.
2. If T is the Product t-norm, then $E_T(x, y) = \min(\frac{x}{y}, \frac{y}{x})$ for all $x, y \in [0, 1]$ where $\frac{z}{0} = 1$.
3. If T is the Minimum t-norm, then $E_T(x, y) = \begin{cases} \min(x, y) & \text{if } x \neq y \\ 1 & \text{otherwise.} \end{cases}$

Theorem 2.5. *Representation Theorem [13]. Let R be a fuzzy relation on a set X and T a continuous t-norm. R is a T-indistinguishability operator if and only if there exists a family $(h_i)_{i \in I}$ of fuzzy subsets of X such that for all $x, y \in X$*

$$R(x, y) = \inf_{i \in I} E_T(h_i(x), h_i(y)).$$

$(h_i)_{i \in I}$ is called a generating family of R.

In particular, given a proximity matrix or relation R on X (i.e. a reflexive and symmetric fuzzy relation), we can build the T-indistinguishability operator $\underline{R}$ generated by the set of the columns of R (i.e. the fuzzy subsets $R(x, \cdot)$, $x \in X$).

Proposition 2.6. *[13] $\underline{R} \subseteq R$.*

Definition 2.7. Let R be a proximity matrix or relation (i.e. a reflexive and symmetric fuzzy relation) on X and T a continuous t-norm. The T-transitive closure $\overline{R}$ of R is the smallest T-indistinguishability operator on X satisfying $R \subseteq \overline{R}$.

Definition 2.8. Let R and S be two fuzzy relations on X and T a continuous t-norm. The Sup-T product or composition of R and S is the fuzzy relation $R \circ S$ on X defined for all $x, y \in X$ by

$$(R \circ S)(x, y) = \sup_{z \in X} T(R(x, z), S(z, y)).$$

Since the Sup-T product is associative or continuous t-norms, we can define for $n \in N$ the n^{th} power R^n_T of a fuzzy relation R:

$$R^n_T = \overbrace{R \circ ... \circ R}^{n\ times}.$$

Definition 2.9. Let R be a fuzzy relation on a set X and T a continuous t-norm. The transitive closure of R with respect to T is the fuzzy relation

$$R_T = \sup_{n \in N} R^n_T.$$

Proposition 2.10. *Let R be a proximity relation on a finite set X of cardinality n. Then*

$$R^T = \sup_{s \in \{1,...,n-1\}} R^s_T.$$

3 Aggregating the transitive closure and a T-indistinguishability contained in a proximity R

In many Decision Making problems, all the available information is contained in a tolerance relation. When its real use is comparing and classifying objects, T-transitivity becomes important, since then it generalizes (fuzzifies) the concept of classical (crisp) equivalence relation. Given a proximity relation R on X, it is therefore necessary in many cases to replace it by a T-indistinguishability operator E, when T-transitivity is required. In these cases, we want to find E as close to R as possible, where the closeness or similarity between fuzzy relations can be defined in many different ways.

This section proposes a completely new method to approximate an initially given information by means of a tolerance relation R by a T-indistinguishability operator as close as possible to R to be used in clustering and reasoning applications.

Let X be a finite set of cardinality n. Ordering its elements linearly, we can view the fuzzy subsets of X as vectors: $X = \{x_1, ..., x_n\}$ and a fuzzy set h is the vector $(h(x_1), ..., h(x_n))$. A proximity relation R on X can be represented by a matrix (also called R) determined by the $\binom{n}{2}$ entries r_{ij} $1 \leq i < j \leq n$ of R above the diagonal.

Proposition 3.1. *Let $E = (e_{ij})_{i,j=1,...,n}$ be a proximity matrix on a set X of cardinality n and T a continuous Archimedean t-norm with additive generator t. E is a T-indistinguishability operator if and only if for all i, j, k $1 \leq i < j < k \leq n$*

$$t(e_{ij}) + t(e_{jk}) \geq t(e_{ik})$$
$$t(e_{ij}) + t(e_{ik}) \geq t(e_{jk})$$
$$t(e_{ik}) + t(e_{jk}) \geq t(e_{ij})$$

Example 3.2. For the Łukasiewicz t-norm, an additive generator is $t(x) = 1 - x$ and the last inequalities become

$$e_{ij} + e_{jk} - e_{ik} \leq 1$$
$$e_{ij} + e_{ik} - e_{jk} \leq 1$$
$$e_{ik} + e_{jk} - e_{ij} \leq 1$$

Example 3.3. For the Product t-norm, an additive generator is $t(x) = -log(x)$ and the last inequalities become

$$e_{ij} \cdot e_{jk} \leq e_{ik}$$
$$e_{ij} \cdot e_{ik} \leq e_{jk}$$
$$e_{ik} \cdot e_{jk} \leq e_{ij}$$

Given a proximity matrix R, we must then search for (one of) the closest matrices E satisfying the last $3\binom{n}{3}$ inequalities which is a non-linear programming problem.

Instead of this, we propose an alternative method to obtain not the best but reasonably good approximations of proximity relations by T-indistinguishability operators.

Definition 3.4. [1], [10] Given a continuous monotonic map $t : [0, 1] \to [-\infty, \infty]$ and p, q positive integers with $p + q = 1$, the weighted quasi-arithmetic mean $m_t^{p,q}$ generated by t and weights p and q is defined for all $x, y \in [0, 1]$ by

$$m_t^{p,q}(x, y) = t^{-1}\left(p \cdot t(x) + q \cdot t(y)\right).$$

m_t is continuous if and only if $Ran\ t \neq [-\infty, \infty]$.

Proposition 3.5. *Fixed the weights p and q, the map assigning to every continuous Archimedean t-norm T with generator t the weighted mean $m_t^{p,q}$ generated by t is a bijection between the set of continuous Archimedean t-norms and the set of continuous quasi-arithmetic means with these weights.*

Proposition 3.6. *Let T be a continuous Archimedean t-norm with additive generator t, $p \in [0, 1]$ and E, F two T-indistinguishability operators on X. The weighted quasi-arithmetic mean $m_t^{p,1-p}$ with weights p and $1 - p$ of E and F is a T-indistinguishability operator.*

Thanks to this last proposition, given a proximity matrix R we can calculate its transitive closure $\overline{R}$ and a smaller T-indistinguishability operator than R, for example $\underline{R}$ and find the weights $p, 1 - p$ to obtain the closest average of $\overline{R}$ and $\underline{R}$ to R.

The similarity between two fuzzy relations on X will be calculated in the following way.

Definition 3.7. Let T be a continuous Archimedean t-norm with additive generator t and R, S two fuzzy relations on a finite set X of cardinality n. The degree $DS(R,S)$ of similarity or closeness between R and S is defined by

$$DS(R,S) = t^{-1}\left(\frac{\sum_{1\le i,j\le n} |t(r_{ij}) - t(s_{ij})|}{n}\right).$$

Proposition 3.8. *DS is a T-indistinguishability operator on the set of fuzzy relations on X.*

Proof. For fixed i, j, $t^{-1}\left(|t(r_{ij}) - t(s_{ij})|\right)$ is a T-indistinguishability operator on $[0,1]$.

$$DS(R,S) = t^{-1}\left(\frac{\sum_{1\le i,j\le n} |t(r_{ij}) - t(s_{ij})|}{n}\right) =$$

$$t^{-1}\left(\frac{\sum_{1\le i,j\le n} t\circ t^{-1}\left(|t(r_{ij}) - t(s_{ij})|\right)}{n}\right)$$

is the the quasi-arithmetic mean of T-indistinguishability operators which is such an operator thanks to Proposition 3.6. □

Corollary 3.9. *Let $R = (r_{ij})$ be a proximity matrix on a finite set X of cardinality n, T a continuous Archimedean t-norm with additive generator t, $\overline{R} = (\overline{r}_{i,j})$ its transitive closure, $\underline{R} = (\underline{r}_{ij})$ the T-indistinguishability operator obtained from R with the Representation Theorem, $p \in [0,1]$ and $m_t^{p,1-p}(\overline{R}, \underline{R})$ the T-indistinguishability operator quasi-arithmetic mean of $\overline{R}$ and $\underline{R}$ with weights p and $1-p$. Then*

$$DS(R, m_t^{p,1-p}(\overline{R}, \underline{R})) =$$

$$t^{-1}\left(\frac{\sum_{1\le i,j\le n} \left|p\cdot t(\overline{r}_{ij}) + (1-p)\cdot t(\underline{r}_{ij}) - t(r_{ij})\right|}{n}\right).$$

We are looking for the value (or values) of p that maximize the last equality. Since t^{-1} is a decreasing map, this is equivalent to minimize

$$\sum_{1\le i,j\le n} \left|p\cdot t(\overline{r}_{ij}) + (1-p)\cdot t(\underline{r}_{ij}) - t(r_{ij})\right|$$

and, since R is reflexive and symmetric, is equivalent to minimize

$$f(p) = \sum_{1\le i<j\le n} \left|p\cdot t(\overline{r}_{ij}) + (1-p)\cdot t(\underline{r}_{ij}) - t(r_{ij})\right|$$

Proposition 3.10. *Let $f_1, ..., f_n : [0,1] \to R$ be n concave functions. Then $\sum_1^n f_i$ is a concave function.*

Proof. By definition, given two points x_1, x_2 of $[0,1]$, the segments joining their images by f_i $i = 1, .., n$ are above f_i. $\sum_1^n f_i$ will then be bellow the sum of all the segments. □

Corollary 3.11. *$f(p)$ is a concave function.*

Proof. Each summand $\left|p \cdot t(\overline{r}_{ij}) + (1-p) \cdot t(\underline{r}_{ij}) - t(r_{ij})\right|$ of f is a concave function. □

Proposition 3.12. *The set of minima of $f(p)$ consists of a single point or of a closed interval.*

Proof. f is a concave function and its graphic is a polygonal line. □

Proposition 3.13. *The computation of $m_t(\overline{R}, \underline{R})$ with maximum $DS(R, m_t(\overline{R}, \underline{R}))$ can be done taking $O(n^3)$ time complexity.*

Proof:

The computation of $\overline{R}$ and $\underline{R}$ can be done in $O(n^3)$ time complexity [11].

The addition (aggregation of distances) takes $O(n^2)$ time complexity.

The minimization of f(p) takes at most $O(n^2)$ time complexity.

So the most complex part of this process is the computation of $\overline{R}$ and $\underline{R}$, which still takes $O(n^3)$ time complexity.

Example 3.14. Let X be a set of cardinality 7 and R the proximity relation given by

$$R = \begin{pmatrix} 1 & 1 & 0.3 & 0.3 & 0.1 & 0.3 & 0.4 \\ 1 & 1 & 0.6 & 0.4 & 0.5 & 0.4 & 0.2 \\ 0.3 & 0.6 & 1 & 0.1 & 0.3 & 0.2 & 0.5 \\ 0.3 & 0.4 & 0.1 & 1 & 1 & 1 & 1 \\ 0.1 & 0.5 & 0.3 & 1 & 1 & 1 & 1 \\ 0.3 & 0.4 & 0.2 & 1 & 1 & 1 & 1 \\ 0.4 & 0.2 & 0.5 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Then, for T the Łukasiewicz t-norm,

$$\overline{R} = \begin{pmatrix} 1 & 1 & 0.6 & 0.4 & 0.5 & 0.4 & 0.4 \\ 1 & 1 & 0.6 & 0.5 & 0.5 & 0.5 & 0.5 \\ 0.6 & 0.6 & 1 & 0.5 & 0.5 & 0.5 & 0.5 \\ 0.4 & 0.5 & 0.5 & 1 & 1 & 1 & 1 \\ 0.5 & 0.5 & 0.5 & 1 & 1 & 1 & 1 \\ 0.4 & 0.5 & 0.5 & 1 & 1 & 1 & 1 \\ 0.4 & 0.5 & 0.5 & 1 & 1 & 1 & 1 \end{pmatrix}$$

and

$$\underline{R} = \begin{pmatrix} 1 & 0.6 & 0.3 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.6 & 1 & 0.3 & 0.2 & 0.1 & 0.2 & 0.2 \\ 0.3 & 0.3 & 1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0.1 & 1 & 0.8 & 0.9 & 0.6 \\ 0.1 & 0.1 & 0.1 & 0.8 & 1 & 0.8 & 0.7 \\ 0.1 & 0.2 & 0.1 & 0.9 & 0.8 & 1 & 0.7 \\ 0.1 & 0.2 & 0.1 & 0.6 & 0.7 & 0.7 & 1 \end{pmatrix}.$$

$$f(p) = |0.4p| + |0.3p - 0.3| + |0.3p - 0.1| + |0.4p - 0.4| +$$

$$|0.3p-0.1|+|0.3p|+|0.3p|+|0.3p-0.1|+|0.4p|+|0.3p-0.1|+$$
$$|0.3p-0.3|+|0.4p-0.4|+|0.4p-0.2|+|0.4p-0.3|+|0.4p|+$$
$$|0.2p|+|0.1p|+|0.4p|+|0.2p|+|0.3p|+|0.3p|$$

which attains its minimum for $p=\frac{1}{3}$.

A good T-transitive approximation of R (for T the Łukasiewicz t-norm) is then

$$\begin{pmatrix} 1 & 0.733 & 0.4 & 0.2 & 0.233 & 0.2 & 0.2 \\ 0.733 & 1 & 0.4 & 0.3 & 0.233 & 0.3 & 0.3 \\ 0.4 & 0.4 & 1 & 0.233 & 0.233 & 0.233 & 0.233 \\ 0.2 & 0.3 & 0.233 & 1 & 0.867 & 0.933 & 0.733 \\ 0.233 & 0.233 & 0.233 & 0.867 & 1 & 0.867 & 0.8 \\ 0.2 & 0.3 & 0.233 & 0.933 & 0.867 & 1 & 0.8 \\ 0.2 & 0.3 & 0.233 & 0.733 & 0.8 & 0.8 & 1 \end{pmatrix}.$$

Example 3.15. Let X be a set of cardinality 7 and R the proximity relation given by

$$R=\begin{pmatrix} 1 & 0.5 & 0.7 & 0.7 & 0.5 & 0.7 & 0.8 \\ 0.5 & 1 & 1 & 0.8 & 0.9 & 0.8 & 0.6 \\ 0.7 & 1 & 1 & 0.5 & 0.7 & 0.6 & 0.9 \\ 0.7 & 0.8 & 0.5 & 1 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.9 & 0.7 & 0.5 & 1 & 0.5 & 0.5 \\ 0.7 & 0.8 & 0.6 & 0.5 & 0.5 & 1 & 0.5 \\ 0.8 & 0.6 & 0.9 & 0.5 & 0.5 & 0.5 & 1 \end{pmatrix}.$$

Then, for T the Product t-norm,

$$\overline{R}=\begin{pmatrix} 1 & 0.7 & 0.72 & 0.7 & 0.5 & 0.7 & 0.8 \\ 0.7 & 1 & 1 & 0.8 & 0.9 & 0.8 & 0.9 \\ 0.72 & 1 & 1 & 0.8 & 0.9 & 0.8 & 0.9 \\ 0.7 & 0.8 & 0.8 & 1 & 0.72 & 0.64 & 0.56 \\ 0.5 & 0.9 & 0.9 & 0.72 & 1 & 0.72 & 0.63 \\ 0.7 & 0.8 & 0.8 & 0.64 & 0.72 & 1 & 0.56 \\ 0.8 & 0.9 & 0.9 & 0.56 & 0.63 & 0.56 & 1 \end{pmatrix}$$

and

$$\underline{R}=\begin{pmatrix} 1 & 0.5 & 0.5 & 0.625 & 0.5 & 0.625 & 0.714 \\ 0.5 & 1 & 0.625 & 0.5 & 0.625 & 0.555 & 0.555 \\ 0.5 & 0.625 & 1 & 0.5 & 0.555 & 0.555 & 0.6 \\ 0.625 & 0.5 & 0.5 & 1 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.625 & 0.555 & 0.5 & 1 & 0.5 & 0.5 \\ 0.625 & 0.555 & 0.555 & 0.5 & 0.5 & 1 & 0.5 \\ 0.714 & 0.555 & 0.6 & 0.5 & 0.5 & 0.5 & 1 \end{pmatrix}.$$

$f(p)$ attains its minimum for $p=0.521$.

A good T-transitive approximation of R (for T the Product t-norm) is then

$$\begin{pmatrix} 1 & 0.587 & 0.595 & 0.660 & 0.5 & 0.660 & 0.754 \\ 0.587 & 1 & 0.783 & 0.626 & 0.744 & 0.662 & 0.700 \\ 0.595 & 0.783 & 1 & 0.626 & 0.700 & 0.662 & 0.729 \\ 0.660 & 0.626 & 0.626 & 1 & 0.595 & 0.563 & 0.528 \\ 0.5 & 0.744 & 0.700 & 0.595 & 1 & 0.595 & 0.559 \\ 0.660 & 0.662 & 0.662 & 0.563 & 0.595 & 1 & 0.528 \\ 0.754 & 0.700 & 0.729 & 0.528 & 0.559 & 0.528 & 1 \end{pmatrix}.$$

The degree of closeness between two fuzzy relations can also be calculated using the Euclidean distance.

Definition 3.16. Let $R = (r_{ij})$ and $S = (s_{ij})$ be two fuzzy relations on a finite set X of cardinality n. The Euclidean distance D between R and S is

$$D(R,S) = \left(\sum_{1 \le i,j \le n} (r_{ij} - s_{ij})^2 \right)^{\frac{1}{2}}$$

Corollary 3.17. *Let $R = (r_{ij})$ be a proximity matrix on a finite set X of cardinality n, T a continuous Archimedean t-norm with additive generator t, $\overline{R} = (\overline{r}_{i,j})$ its transitive closure, $\underline{R} = (\underline{r}_{ij})$ the T-indistinguishability operator obtained from R with the Representation Theorem, $p \in [0,1]$ and $m_t(\overline{R}, \underline{R})$ the T-indistinguishability operator quasi-arithmetic mean of $\overline{R}$ and $\underline{R}$ with weights p and $1-p$. Then*

$$D(R, m_t(\overline{R}, \underline{R})) =$$

$$\left(\sum_{1 \le i,j \le n} \left(t^{-1} \left(p \cdot t\left(\overline{r}_{ij}\right) + (1-p) \cdot t\left(\underline{r}_{ij}\right) \right) - t(r_{ij}) \right)^2 \right)^{\frac{1}{2}}.$$

Proposition 3.18. *Let T be the Łukasiewicz t-norm and R a proximity on a set X of cardinality n. The closest $m_t(\overline{R}, \underline{R})$ to R is attained for*

$$p = \frac{\sum_{1 \le i < j \le n} \left(\overline{r}_{ij} - \underline{r}_{ij}\right)\left(r_{ij} - \underline{r}_{ij}\right)}{\sum_{1 \le i < j \le n} \left(\overline{r}_{ij} - \underline{r}_{ij}\right)^2}$$

Proof. Due to symmetry and reflexivity, it is enough to minimize

$$f(p) = \sum_{1 \le i < j \le n} \left(p\left(\overline{r}_{ij} - \underline{r}_{ij}\right) + \underline{r}_{ij} - r_{ij} \right)^2.$$

$$f'(p) = 2 \sum_{1 \le i < j \le n} \left(p\left(\overline{r}_{ij} - \underline{r}_{ij}\right) + \underline{r}_{ij} - r_{ij} \right)\left(\underline{r}_{ij} - r_{ij}\right) = 0$$

and

$$p = \frac{\sum_{1 \le i < j \le n} \left(\overline{r}_{ij} - \underline{r}_{ij}\right)\left(r_{ij} - \underline{r}_{ij}\right)}{\sum_{1 \le i < j \le n} \left(\overline{r}_{ij} - \underline{r}_{ij}\right)^2}.$$

□

Example 3.19. Let X be a set of cardinality 4 and R the proximity relation on X given by

$$R = \begin{pmatrix} 1 & 0.8 & 0.2 & 0.4 \\ 0.8 & 1 & 0.7 & 0.1 \\ 0.2 & 0.7 & 1 & 0.6 \\ 0.4 & 0.1 & 0.6 & 1 \end{pmatrix}.$$

If T is the Łukasiewicz t-norm, the closest T-indistinguishability operator of the type $m_t(\overline{R}, \underline{R})$ (with respect to the Euclidean distance) is attained for $p = 0.6388889$.

A good T-approximation of R is then

$$\begin{pmatrix} 1 & 0.6917 & 0.3917 & 0.3639 \\ 0.6917 & 1 & 0.5917 & 0.2278 \\ 0.3917 & 0.5917 & 1 & 0.5278 \\ 0.3639 & 0.2278 & 0.5278 & 1 \end{pmatrix}$$

4 Concluding Remarks

In this paper an algorithm to find good approximations of a proximity relation by T-transitive ones (T Archimedean) in a reasonable computational way is given.

One of the interesting features of it is its simplicity, that allows it to be applied in real problems at low computational cost.

Simple examples show that in general this approximation is better than the transitive closure or transitive openings of the proximity R, and can be reached in the same time complexity..

This method can also be applied using different T-transitive openings. Even though we know how to compute some T-transitive openings of a proximity, the computation of all of them is still an open problem.

The method of the paper cannot be applied to the Minimum t-norm. Other ways to obtain similar results for this t-norm are therefore needed and the authors will work on it in a forthcoming paper.

Acknowledgments

Research partially supported by DGICYT projects number TIN2006-14311 and TIN2006-06190.

References

[1] J. Aczél (1966) *Lectures on functional equations and their applications.* Academic Press. New York/London.

[2] G. Beliakov, T. Calvo, A. Pradera (2007) Aggregation Functions: A Guide for Practitioners. Springer.

[3] D. Boixader, J. Jacas, J. Recasens (2000). Fuzzy Equivalence Relations: Advanced Material. In Dubois, Prade Eds. *Fundamentals of Fuzzy Sets*, Kluwer, 261-290.

[4] T. Calvo, A. Kolesárova, M. Komorníková, R. Mesiar (2002). Aggregation Operators: Properties, Classes and Construction Methods. In Mesiar, Calvo, Mayor Eds. *Aggregation Operators: New Trends and Applications.* Studies in Fuzziness and Soft Computing. Springer, 3-104.

[5] J. Fodor, M. Roubens (1995). Structure of transitive valued binary relations. *Math. Social Sci.* 30, 71–94.

[6] S. Gottwald (2001). *A Treatise on Many-Valued Logics.* Research Studies Press Ltd. Baldock.

[7] P. Hájek (1998) *Metamathematics of Fuzzy Logic.* Kluwer. Dordrecht.

[8] J. Jacas, J. Recasens (2003) Aggregation of T-Transitive Relations. *Int J. of Intelligent Systems* 18, 1193-1214.

[9] C. M. Ling (1965) Representation of associative functions *Publ. Math. Debrecen* 12, 189-212.

[10] E. P. Klement, R. Mesiar, E. Pap (2000). *Triangular norms.* Kluwer. Dordrecht.

[11] Naessens, H., De Meyer, H., De Baets, B., Algorithms for the Computation of T-Transitive Closures, *IEEE Trans Fuzzy Systems* 10 :4, 2002, 541-551.

[12] B. Schweizer, A. Sklar (1983) *Probabilistic Metric Spaces.* North-Holland. Amsterdam.

[13] L.Valverde (1985). On the structure of F-indistinguishability operators, *Fuzzy Sets and Systems* 17, 313-328.

[14] L.A.Zadeh (1971). Similarity relations and fuzzy orderings, *Information Science* 3, 177-200.