Mathware & Soft Computing 14 (2007) 165-170

Short Note: Counting Conjectures

Adolfo R. de Soto[1]*, A. Alvárez[2] and E. Trillas[2]†
[1]Industrial and Computer Science Engineering School
University of León, Spain
adolfo.rdesoto@unileon.es
[2,3]European Centre for Soft Computing, Spain
ana.alvarez@neo-metrics.com, enric.trillas@softcomputing.es

Abstract

This paper only goal is to study what is, in some finite ortholattices, the number of conjectures, refutations, consequences, hypotheses and speculations.

Keywords: Conjectures, consequences, hypotheses, speculations. CHC Models.

1 Introduction

The concept of conjecture in an ortholattice was introduced in [3], and there were also defined the particular cases of consequences, hypotheses, and speculations. In [4], speculations were classified in two types.

This paper deals with the number of, respectively, conjectures, refutations, consequences, hypotheses and type-1 and type-2 speculations, in some finite ortholattices and, in particular, in finite boolean algebras.

Since in boolean algebras all elements are the union of atoms and these decompositions are unique, the number of elements in the before mentioned classes of conjectures are exactly computed. Nevertheless, in general finite ortholattices, the elements are neither the union of atoms nor, if this decomposition exists for some elements, is unique. Because of this, only bounding inequalities are reached in the case of some of finite ortholattices.

2 Basic Concepts

2.1 Given a set of premise $P = \{p_1, \ldots, p_n\}$ represented by elements in an ortholaticce $(L, \cdot, +, '; 0, 1)$ -L for short- such that $p_\wedge = p_1 \cdot \ldots \cdot p_n \neq 0$ to avoid

*This author has been partially supported by spanish CYCIT with the project TIN2005-08943-C02-01 and the regional council Junta de Castilla y León with the project LE051A05

†This author has been partially supported by spanish CYCIT with the project TIN2005-08943-C02-01

contradictions, in [3] there were defined the sets:

$Conj(P) = \{q \in L : p_\wedge \not\leq q'\} = \{q \in L \ : \ p_\wedge \leq q'\}^c$, of the **conjectures** of P,

$Ref(P) = \{q \in L \ : \ p_\wedge \leq q'\} = Conj(P)^c$, of the **refutations** of P, (not in [3])

$Hyp(P) = \{q \in L : q \neq 0, q < p_\wedge\}$, of the **hypotheses** of P,

$Cons(P) = \{q \in L : p_\wedge \leq q\}$, of the **consequences** of P,

$Sp(P) = \{q \in Conj(P) : p_\wedge NCq\}$, of the **speculations** of P,

verifying

$$Conj(P) = Cons(P) \cup Hyp(P) \cup Sp(P),$$

and denoting by $xNCy$ that x and y are not comparable under the natural order $\leq$ of L ($a \leq b$ iff $a \cdot b = a$), and by $NC(p_\wedge)$ the set of elements which are not comparable with $p_\wedge$.
In [4], the set $Sp(P)$ was decomposed in

$$Sp(P) = Sp1(P) \cup Sp2(P),$$

with

$$Sp1(P) = \{q \in Sp(P) : q' < p_\wedge\}, \ \ Sp2(P) = \{q \in Sp(P) : q'NCp_\wedge\},$$

called, respectively, type-1 speculations, and type-2 speculations.

2.2 An atom in a lattice is an element $0 \neq a \in L$ such as there is not any element beetwen 0 and a. A lattice is atomic when for all $0 \neq q \in L$ there exists an atom $a \in L$ such as $a \leq q$. Any finite lattice is atomic. A lattice is univocally complemented when the complement of each element is unique. It is interesting to remark that an atomic and univocally complemented lattice is a boolean algebra [2]. A lattice is said atomistic if each element is the sum of the atoms contained in it[1], and atomically-independent if any combination of different atoms gives different elements. Notice that a finite ortholattice is an atomistic and atomically-independent lattice if and only if is a boolean algebra, since atomistic and atomically-independent properties implies univocally complemented property.

3 Results for some proper ortholattices

In this work only finite ortholattice with cardinal N will be considered. We will denote by $a_1, \ldots, a_p$ the atoms such that $a_i \leq p_\wedge$ for all $i \in \{1, \ldots, p\}$, and by $a_{p+1}, \ldots, a_n$ the rest of atoms.

[1] In [2], a atomistic lattice is just called atomic lattice

Proposition 1 *Let* L *be a finite atomically-independent ortholattice, then it holds that* $2^p - 2 \leq |Hyp(P)| \leq N - 2^n + 2^p - 1$.

Proof: The sum of atoms smaller than $p_\wedge$ is also less or equal than $p_\wedge$. Therefore, there are at least $\sum_{i=1}^{p} \binom{p}{i} = 2^p - 2$ posible hypotheses as sum of atoms. 0 and the sum of all atoms are eliminated, because it can happen that $\sum_{i=i}^{p} a_i = p_\wedge$. Hence $2^p - 2 \leq |Hyp(P)|$.

On the other hand, the sum of atoms greater than or not comparable with $p_\wedge$ is not a hypothesis. Therefore, there are at least $\sum_{i=0}^{n-p} \binom{n-p}{i} = 2^{n-p}$ elements that are not hypotheses. 0 is eliminated.

Also the sum of at least an atom smaller than $p_\wedge$ with at least an atom greater than or not comparable with $p_\wedge$ is not an hypotheses. Therefore, there are at least $\sum_{i=1}^{p} \binom{p}{i} \cdot \sum_{i=1}^{n-p} \binom{n-p}{i} = (2^p - 1)(2^{n-p} - 1) = 2^n - 2^{n-p} - 2^p + 1$ elements that are not hypotheses.

Hence $|Hyp(P)| \leq N - [2^{n-p} + 2^n - 2^{n-p} - 2^p + 1] = N - 2^n + 2^p - 1$.□

Proposition 2 *In all ortholattice, if* $p_\wedge \neq 1$ *then* $|Sp1(P)| = |Hyp(P)|$.

Proof: Let $p_\wedge \neq 1$. If $q \neq 1$ and $q' < p_\wedge$, then $p_\wedge NCq$, since, if $q \leq p_\wedge$ then $p_\wedge = 1$; and, if $q \geq p_\wedge$ then $q > q'$, that is absurd. So when $p_\wedge \neq 1$ and $q \neq 1$, $q' < p_\wedge$ implies $p_\wedge NCq$. Hence, $|Sp1(P)| = |\{q \in L : q' < p_\wedge, qNCp_\wedge\}| = |\{q \in L : q' < p_\wedge\}| - 1 = |\{q' \in L : q' < p_\wedge\}| - 1 = |\{q \in L : 0 \neq q < p_\wedge\}| = |Hyp(P)|$.□

When $p_\wedge = 1$, it will be $Hyp(P) = L - \{0, 1\}$ and $Sp1(P) = \emptyset$.

Proposition 3 *In all ortholattice,* $|Ref(P)| = |Cons(P)|$.

Proof: $|Ref(P)| = |\{q \in L : q' \geq p_\wedge\}| = |\{q' \in L : q' \geq p_\wedge\}| = |Cons(P)|$.□

Corollary 1 *In all finite ortholattice,* $|Conj(P)| = N - |Cons(P)|$.

Proof: $|Conj(P)| = N - |Conj(P)^c| = N - |Ref(P)| = N - |Cons(P)|$.□

Proposition 4 *In all ortholattice,* $|Cons(P)| = |Hyp(\{p'_\wedge\})| + 2$.

Proof: $|Cons(P)| = |\{q \in L : q \geq p_\wedge\}| = |\{q \in L : q' \leq p'_\wedge\}| = |\{q \in L : 0 \neq q' < p'_\wedge\} \cup \{1, p_\wedge\}| = |\{q \in L : 0 \neq q < p'_\wedge\}| + |\{0, p_\wedge\}| = |Hyp(\{p'_\wedge\})| + 2$.□

Proposition 5 *In all ortholattice, if* $p_\wedge \neq 1$ *and* a_i *is an atom such that* $a_i < p_\wedge$, *then* $a'_i \in Sp1(P)$.

Proof: $(a'_i)' = a_i < p_\wedge$, hence, it suffices prove that $a'_i NCp_\wedge$ to have $a'_i \in Sp1(P)$. if it were $a'_i \leq p_\wedge$ then $1 = a_i + a'_i \leq a_i + p_\wedge = p_\wedge$ or $p_\wedge = 1$, and that is absurd. If $a'_i \geq p_\wedge$ then $0 = a_i \cdot a'_i \geq a_i \cdot p_\wedge = a_i$ or $a_i = 0$, and that is also imposible.□

Proposition 6 *In all ortholattice, the number* $|Sp2(P)|$ *is even.*

Proof: If $q \in Sp2(P)$, then $qNCp_\wedge$ and $q'NCp_\wedge$, so $q' \in Sp2(P)$. Hence, $|Sp2(P)|$ is even.□

4 Counting in finite boolean algebras

Now let L be a finite boolean algebra ($N = 2^n$) and $P = \{p_1, \ldots, p_m\} \subset L$ with $p_\wedge = p_1 \cdot \ldots \cdot p_m \neq 0$. We denote by $a_1, \ldots, a_p$ the atoms such that $a_i \leq p_\wedge$ for all $i \in \{1, \ldots, p\}$, and by $a_{p+1}, \ldots, a_n$ the rest of atoms.

Proposition 7 *The total number of consequences of P is 2^{n-p}.*

Proof: As $q \in Cons(P)$ implies $p_\wedge \leq q$, every consequence has the form $a_1 + \ldots + a_p + a_{p+j_1} + \ldots + a_{p+j_r}$ with $j_1, \ldots, j_r \in \{1, \ldots, n-p\}$ and $r \geq 0$. Hence, there are $\sum_{i=0}^{n-p} \binom{p}{i} = 2^{n-p}$ posible consequences. □

Corollary 2 *The number of conjectures is $2^n - 2^{n-p}$.*

Proof: By corollary 1, $|Conj(P)| = N - |Cons(P)| = 2^n - 2^{n-p}$.□

Corollary 3 *The number of refutation is 2^{n-p}.*

Proposition 8 *The number of hypotheses of P is $2^p - 2$.*

Proof: All hypothesis can be written as $a_{i_1} + \ldots + a_{i_s}$ with $i_1, \ldots, i_s \in \{1, \ldots, p\}$, but $a_1 + \ldots + a_p = p_\wedge$ and 0 are not hypothesis. Hence, there are $\sum_{i=1}^{p-1} \binom{p}{i} = 2^p - 2$ posible hypotheses.□

Corollary 4 *When $p_\wedge \neq 1$, the number of speculations of type 1 is $2^p - 2$.*

Proof: By proposition 2, if $p_{wedge} \neq 1$, then $|Sp1(P)| = |Hyp(P)|$.

Corollary 5 *The number of speculations of P is $2^n - 2^{n-p+1} - 2^p + 2$.*

Proof: It is obvious, since it is $Conj(P) = Cons(P) \cup Hyp(P) \cup Sp(P)$ and the three sets are pairwise disjoint. □

Corollary 6 *In boolean algebras, $|Sp2(P)| = 2^n - 2^{n-p+1} - 2^{p+1} + 2^2$.*

Proof: By proposition 2 $|Sp1(P)| = |Hyp(P)| = 2^p - 2$. As $Sp(P) = Sp1(P) \cup Sp2(P)$ with $Sp1(P) \cap Sp2(P) = \varnothing$, then $|Sp2(P)| = |Sp(P)| - |Sp1(P)| = 2^n - 2^{n-p+1} - 2^p + 2 - [2^p - 2] = 2^n - 2^{n-p+1} - 2^{p+1} + 2^2$.□

5 Examples

5.1 Let us show two examples concerning the non-orthomodular ortholattices in figure 1. Both of them are atomically-independent and not atomistic lattices. Computing the number of hypothesis, it is possible to valuate the bounds of proposition 1.
In the case of figure 1(1), with $P = \{b\}$, it is $p = 2$, $n = 3$, $N = 10$ and $2 = 2^2 - 2 \leq |Hyp(P)| = 3 \leq 10 - 2^3 + 2^2 - 1 = 5$.

In 1(2), with $P = \{d'\}$, it is $p = 1$, $n = 2$, $N = 10$ and $0 = 2^1 - 2 \leq |Hyp(P)| = 3 \leq 10 - 2^2 + 2^1 - 1 = 7$.

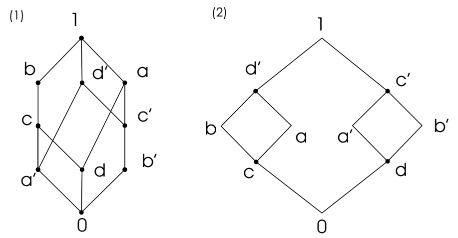

Figure 1: Two atomically-independent and non-atomistic ortholattices

5.2 In figure 2, two atomistic and not atomically-independent ortholattice are shown. The left one is not orthomodular and it does not give the lower bound given by proposition 1 because the set $Hyp(\{c'\}) = \{a, b, d, e\}$ has four element but c' contains four atoms so the lower bound is $2^4 - 2 = 14$.

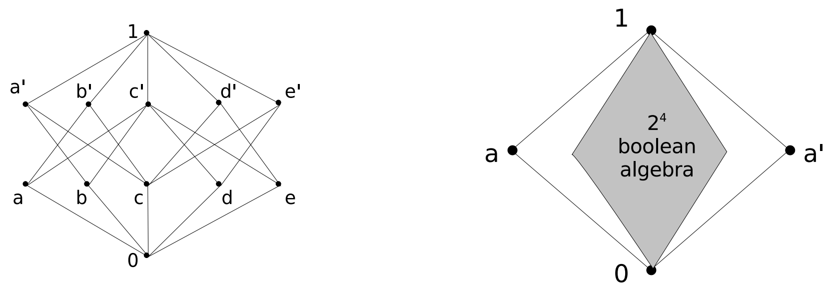

Figure 2: Two non-atomically-independent and atomistic ortholattices

The lattice in the right side of figure 2 (with a total of $2^4 + 2$ elements) is orthomodular and it verifies the bounds of proposition 1 because the boolean subalgebra satisfies this proposition that also is verified with the addtion of the elements a, a'.

5.3 Let us show a last example concerning the boolean algebra 2^3 in figure 3:

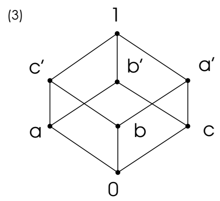

Figure 3: The 2^3 boolean algebra

In figure 3, with $P = \{b'\}$ it is $p = 2$, $n = 3$, $N = 2^3 = 8$ and $2 = 2^2 - 2 \leq |Hyp(P)| = 2 \leq 2^3 - 2^3 + 2^2 - 1 = 3$.

6 Conclusions

6.1 It should be pointed out that the proportion of consequences, hypothesis and speculations among the total number of conjectures in finite boolean algebras is:

$$\frac{|Cons(P)|}{|Conj(P)|} = \frac{1}{2^p - 1}$$
$$\frac{|Hyp(P)|}{|Conj(P)|} = \frac{2^p}{2^n}(1 - \frac{1}{2^p - 1})$$
$$\frac{|Spec(P)|}{|Conj(P)|} = 1 - \frac{2^{-(n-p)} - 2^{-(p+1)} - 2^{-(n-1)}}{2^{-p} - 1}$$

Notice that the first proportion only depends on p, the number of atoms in the descomposition of $p_\wedge$ ($p_\wedge = a_1 + \ldots + a_p$, with $p \leq n$).

References

[1] L. Beran. *Orthomodular Lattices*. D. Reidel Publishing Company, 1985

[2] G. Birkhoff. *Lattice Theory*. American Mathematical Society, Providence, RI, 1973.

[3] E. Trillas, S. Cubillo, E.Castiñeira. “On conjectures in orthocomplemented lattices”. Artificial Intelligence 117 (2000), 255–275

[4] E. Trillas, A. Pradera. An orthogonality-based classification of conjectures in ortholattices. Mathware& Soft Computing XIII, n.1 (2006), 71 - 87