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Existence of Extremal Solutions for Fuzzy Polynomials and their Numerical Solutions

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Abstract

In this paper, we consider the existence of a solution for fuzzy polynomials

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = x,$$

where a_i, $i = 0, 1, 2, \cdots, n$ and x are positive fuzzy numbers satisfying certain conditions. To this purpose, we use fixed point theory, applying results such as the well-known fixed point theorem of Tarski, presenting some results regarding the existence of extremal solutions to the above equation.
Keywords: Fixed point, Fuzzy equation, Fuzzy real number

1 Introduction

Systems of simultaneous nonlinear equations play a major role in various areas such as mathematics, statistics, engineering and social sciences. Since in many applications at least some of the system's parameters and measurements are represented by fuzzy rather than crisp numbers, it is immensely important to develop mathematical models and numerical procedures that would appropriately solve them. The concept of fuzzy numbers and arithmetic operation with these numbers were first introduced and investigated by [23]. One of the major applications of fuzzy number arithmetic is linear systems [2, 3, 6] or nonlinear systems whose parameters are all or partially represented by fuzzy numbers [14, 18, 20]. The numerical solutions of fuzzy nonlinear equation(s) by Newton's method were considered in [1, 4, 5].

Standard analytical techniques like Buckley and Qu method [7, 8, 9, 10], can not suitable for solving polynomials such as

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = x, \tag{1}$$

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where x and a_i, $i = 0, 1, \cdots, n$, are positive fuzzy numbers. These polynomials have many applications in decision theory, physics, economics, etc [11]. In [7], the authors gave two examples for applications of quadratic equations in physics and economics (see [7], examples 4.3 and 4.4). Therefore, for solving some economics problems we must solve polynomials. This fact motivated us to develop the numerical methods to find the roots of polynomials.

In [22], the authors studied the existence of extremal solutions for quadratic fuzzy equation

$$Ex^2 + Fx + G = x, \tag{2}$$

where E, F, G and x were positive fuzzy numbers satisfying certain conditions, and gave the interval which contains extremal solutions ([22], Theorem 12). In this paper we consider the equation (1) and give the extremal solutions and numerical solutions for this equation.

In Section 2, we give some basic results on fuzzy numbers and applied theorems of [12]. In Section 3, we study the existence of solution to (1) by using some fixed point theorems such as Tarski's Fixed Point Theorem. In Section 4, we propose Newton's method for solving (1) and in Section 5, we illustrate some examples and conclusions in the last section.

2 Preliminaries

Definition 1. A fuzzy number is a fuzzy set like $x : \mathbb{R} \to I = [0, 1]$ which satisfies, [17, 23, 24],

i. x is upper semi-continuous,

ii. $x(t) = 0$ outside some interval $[c, d]$,

iii. There are real numbers a, b such that $c \leq a \leq b \leq d$ and

1. $x(t)$ is monotonic increasing on $[c, a]$,
2. $x(t)$ is monotonic decreasing on $[b, d]$,
3. $x(t) = 1,\ a \leq x \leq b$.

The set of all the fuzzy numbers (as given by Definition 1) is denoted by E^1.

Definition 2. For a fuzzy number $x \in E^1$, we denote the α-level set

$$[x]^\alpha = \{t \in \mathbb{R} : x(t) \geq \alpha\}$$

by the interval $[x_{\alpha l}, x_{\alpha r}]$, for each $\alpha \in (0, 1]$, and

$$[x]^0 = \overline{\bigcup_{\alpha \in (0,1]} [x]^\alpha} = [x_{0l}, x_{0r}].$$

We consider the partial ordering $\leq$ in E^1 given by

$$x, y \in E^1,\ x \leq y \ \Leftrightarrow\ (x_{\alpha l} \leq y_{\alpha l} \ and \ x_{\alpha r} \leq y_{\alpha r}), \quad for\ all\ \alpha \in (0, 1],$$

and the distance that provides E^1 as a complete metric space is given by

$$d_\infty(x,y) = \sup_{\alpha\in[0,1]} d_H([x]^\alpha,[y]^\alpha) \ \ \forall x,y\in E^1,$$

being d_H the Hausdorff distance between nonempty compact convex subset of $\mathbb{R}$ (that is, compact intervals). It can be proved that, if $x\leq y\leq z$ for every $x,y,z,\in E^1$, then $d_\infty(z,y)\leq d_\infty(z,x)$, (see [12]).

For each fuzzy number $x\in E^1$, we define the functions $x_L : [0,1]\to\mathbb{R}$, $x_R : [0,1]\to\mathbb{R}$ given by $x_L(\alpha)=x_{\alpha l}$ and $x_R(\alpha)=x_{\alpha r}$, for each $\alpha\in[0,1]$.

Definition 3 [21]. A fuzzy number $x\in E^1$ in parametric form is a pair (x_L,x_R) of functions $x_L(\alpha),x_R(\alpha),\ 0\leq\alpha\leq 1$, which satisfy the following requirements:

1. $x_L(\alpha)$ is a bounded monotonic increasing left continuous function,
2. $x_R(\alpha)$ is a bounded monotonic decreasing left continuous function,
3. $x_L(\alpha)\leq x_R(\alpha),\ 0\leq\alpha\leq 1$.

An important class of fuzzy number is the triangular fuzzy number $x=(a,c,b)$, with the membership function

$$x(t)=\begin{cases}\dfrac{t-a}{c-a}, & a\leq t\leq c,\\[2mm] \dfrac{t-b}{c-b}, & c\leq t\leq b,\end{cases}$$

where $c\neq a$, $c\neq b$ and hence

$$x_L(\alpha)=a+(c-a)\alpha,\qquad x_R(\alpha)=b+(c-b)\alpha.$$

If $c=\frac{a+b}{2}$ or $x_L(\alpha)+x_R(\alpha)$ is independent of α, then the triangular fuzzy number is called symmetric .

Let $TF(\mathbb{R})$ be the set of all triangular fuzzy numbers. The addition and scalar multiplication of fuzzy numbers are defined by the extension principle and can be equivalently represented as follows.
For arbitrary $x=(x_L,x_R)$, $y=(y_L,y_R)$ and scalar $k>0$ we define addition $(x+y)$ and multiplication as

$$(x+y)_L(\alpha)=x_L(\alpha)+y_L(\alpha),\quad (x+y)_R(\alpha)=x_R(\alpha)+y_R(\alpha), \tag{3}$$

$$(kx)_L(\alpha)=kx_L(\alpha),\quad (kx)_R(\alpha)=kx_R(\alpha). \tag{4}$$

Here, the product $x\cdot y$ of two fuzzy numbers x and y is given by the Zadeh's extension principle:

$$x\ \cdot\ y:\mathbb{R}\longrightarrow[0,1]$$

$$(x\cdot y)(t) = \sup_{s\cdot s^{'}=t}\min\{x(s),y(s^{'})\}.$$

Note that $[x \cdot y]^\alpha = [x]^\alpha \cdot [y]^\alpha$.

Theorem 1. *([12], Theorem 2.3)* Let u_0, $v_0 \in E^1$ and $u_0 < v_0$. Let

$$B \subset [u_0, v_0] = \{x \in E^1 : u_0 \leq x \leq v_0\}$$

be a closed set of E^1 such that u_0, $v_0 \in B$. Suppose that $A : B \longrightarrow B$ is an increasing operator such that

$$u_0 \leq Au_0, \quad Av_0 \leq v_0,$$

and A is condensing, that is, A is continuous, bounded and $r(A(S)) < r(S)$ for any bounded set $S \subset B$ with $r(S) > 0$, where $r(S)$ denotes the measure of non-compactness of S. Then A has maximal fixed point x^* and a minimal fixed point x_* in B, moreover

$$x^* = \lim_{n \to +\infty} v_n, \; x_* = \lim_{n \to +\infty} u_n,$$

where $v_n = Av_{n-1}$ and $u_n = Au_{n-1}$, $n = 1, 2, \cdots$ and

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0.$$

Corollary 1. *([12], Corollary 2.4)* In the hypothesis of Theorem 1, if A has only one fixed point $\overline{x}$ in B, then, for any $x_0 \in B$, the successive iterates

$$x_n = Ax_{n-1}, \; n = 1, 2, \cdots$$

converges to $\overline{x}$, that is, $d_\infty(x_n, \overline{x}) \longrightarrow 0$ as $n \longrightarrow \infty$.

Definition 4. [22] Let κ_C^1 denotes the space of nonempty compact convex subset of $\mathbb{R}$ furnished with the Hausdorff metric d_H, we say that a fuzzy number $x : \mathbb{R} \longrightarrow [0, 1]$ is continuous if the function

$$[x]^{\cdot} : [0, 1] \longrightarrow \kappa_C^1$$

given by $\alpha \longrightarrow [x]^\alpha$ is continuous on $(0, 1]$, that is, for every $\alpha \in (0, 1]$, and $\epsilon > 0$, there is a number $\delta(\epsilon, \alpha) > 0$ such that $d_H([x]^\alpha, [x]^\beta) < \epsilon$, for every $\beta \in (\alpha - \delta, \alpha + \delta) \bigcap [0, 1]$.

Theorem 2. *([22], Theorem 6)* Let x be a fuzzy number, then x is continuous if and only if functions

$$x_L : [0, 1] \longrightarrow \mathbb{R} \text{ and } x_R : [0, 1] \longrightarrow \mathbb{R}$$

are continuous.

Definition 5. [22] We say that $x \in E^1$ is a Lipschitzian (or K-Lipschitzian) fuzzy number if it is a Lipschitz function of its membership grade, in the sense that

$$d_H([x]^\alpha, [y]^\beta) \leq K \mid \alpha - \beta \mid,$$

for every α, $\beta \in [0,1]$ and some fixed, finite constant $K \geq 0$.

Theorem 3. *([22], Theorem 7)* Let $x \in E^1$. Then x is a Lipschitzian fuzzy number, with Lipschitz constant $K \geq 0$, if and only if $x_L : [0,1] \longrightarrow \mathbb{R}$ and $x_R : [0,1] \longrightarrow \mathbb{R}$ are $K-$Lipschitzian functions.

Theorem 4. *([22], Theorem 8)* Suppose that x and y are fuzzy numbers *(in the sence of Definition 3)*, then

$$d_\infty(x,y) = \max\{\|x_L - y_L\|_\infty \ , \ \|x_R - y_R\|_\infty\}.$$

Lemma 1. *([22], Lemma 2)* Suppose that $B \subset E^1$ consists of continuous fuzzy numbers, hence $B_L = \{x_L : \ x \in B\}$ and $B_R = \{x_R : \ x \in B\}$ are subsets of $C[0,1]$. If B_L and B_R are relatively compact in $(C[0,1], \| \ . \ \|_\infty)$, then B is a relatively compact set in E^1.

3 Existence results

Definition 6. For $M \geq 0$ fixed, consider the set

$$B_M = \{x \in E^1 \ : \ \chi_{\{0\}} \leq x \leq \chi_{\{1\}}\},$$

where x is M-Lipschitzian, $\chi_{\{0\}}$ and $\chi_{\{1\}}$ are , respectively, the characteristic functions of 0 and 1.

Theorem 5. Let $M > 0$ be a real number, $I = \{0,1,2,\ldots,n\}$, $J = \{1,2,\ldots,n\}$ and $a_i, \ i \in I$ are fuzzy numbers such that

1. $a_i \geq \chi_{\{0\}}$, $d_\infty(a_0, \chi_{\{0\}}) \leq \frac{n}{n+1}$, and $d_\infty(a_j, \chi_{\{0\}}) \leq \frac{1}{n(n+1)}$, for $i \in I$ and $j \in J$.
2. a_i is $\frac{M}{2(n+1)}$ -Lipschitzian, for $i \in I$.

Then (1) has a solution in B_M.

Proof. We define the mapping

$$A : B_M \longrightarrow B_M,$$

by $Ax = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. To check that A is well-defined, let $x \in B_M$ and then

$$|(Ax)_L(\alpha) - (Ax)_L(\beta)| =$$

$$|[(a_n)_L(\alpha)x_L^n(\alpha) + (a_{n-1})_L(\alpha)x_L^{n-1}(\alpha) + \cdots + (a_1)_L(\alpha)x_L(\alpha) + (a_0)_L(\alpha)] -$$

$$[(a_n)_L(\beta)x_L^n(\beta) + (a_{n-1})_L(\beta)x_L^{n-1}(\beta) + \cdots + (a_1)_L(\beta)x_L(\beta) + (a_0)_L(\beta)]|$$

$$\leq |(a_n)_L(\alpha) - (a_n)_L(\beta)|x_L^n(\alpha) + |(a_{n-1})_L(\alpha) - (a_{n-1})_L(\beta)|x_L^{n-1}(\alpha) + \cdots +$$

$$|(a_1)_L(\alpha)-(a_1)_L(\beta)|x_L(\alpha)+|(a_0)_L(\alpha)-(a_0)_L(\beta)|+$$
$$(a_n)_L(\beta)|x_L^n(\alpha)-x_L^n(\beta)|+(a_{n-1})_L(\beta)|x_L^{n-1}(\alpha)-x_L^{n-1}(\beta)|+\cdots+$$
$$(a_1)_L(\beta)|x_L(\alpha)-x_L(\beta)|.$$

By using

$$x_L^n(\alpha)-x_L^n(\beta)=(x_L(\alpha)-x_L(\beta))(x_L^{n-1}(\alpha)+x_L^{n-2}(\alpha)x_L(\beta)+\cdots+x_L^{n-1}(\beta))$$

we have

$$|(Ax)_L(\alpha)-(Ax)_L(\beta)|\leq$$
$$\frac{M}{2(n+1)}(n+1)|\alpha-\beta|+\frac{M}{n(n+1)}|\alpha-\beta|(1+2+\cdots+n)$$
$$=M|\alpha-\beta|,\qquad\forall\ \alpha,\ \beta\in[0,1],$$

and, analogously,

$$|(Ax)_R(\alpha)-(Ax)_R(\beta)|\leq M|\alpha-\beta|,\qquad\forall\ \alpha,\ \beta\in[0,1],$$

therefore, by Theorem 3, $Ax\in E^1$ is $M-$Lipschitzian and, using the hypotheses and $\chi_{\{0\}}\leq x\leq\chi_{\{1\}}$, we obtain

$$0\leq(a_n)_L(\alpha)x_L^n(\alpha)+(a_{n-1})_L(\alpha)x_L^{n-1}(\alpha)+\cdots+(a_1)_L(\alpha)x_L(\alpha)+(a_0)_L(\alpha)=$$
$$(Ax)_L(\alpha)\leq(Ax)_R(\alpha)=$$
$$(a_n)_R(\alpha)x_R^n(\alpha)+(a_{n-1})_R(\alpha)x_R^{n-1}(\alpha)+\cdots+(a_1)_R(\alpha)x_R(\alpha)+(a_0)_R(\alpha)$$
$$\leq\frac{1}{n(n+1)}+\cdots+\frac{1}{n(n+1)}+\frac{n}{n+1}=\frac{n+1}{n+1}=1,\quad\forall\ \alpha\ \in[0,1].$$

Therefore $Ax\in B_M$. Moreover, A is a non-decreasing and continuous mapping. A is bounded, since

$$d_1(\alpha)=$$
$$(a_n)_L(\alpha)x_L^n(\alpha)+(a_{n-1})_L(\alpha)x_L^{n-1}(\alpha)+\cdots+(a_1)_L(\alpha)x_L(\alpha)+(a_0)_L(\alpha),$$
$$d_2(\alpha)=$$
$$(a_n)_R(\alpha)x_R^n(\alpha)+(a_{n-1})_R(\alpha)x_R^{n-1}(\alpha)+\cdots+(a_1)_R(\alpha)x_R(\alpha)+(a_0)_R(\alpha)$$

then

$$d_\infty(Ax,\chi_{\{0\}})=d_\infty(a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0,\chi_{\{0\}})=$$
$$\sup d_H([a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0]^\alpha,\chi_{\{0\}})=$$
$$\sup d_H([a_n]^\alpha[x^n]^\alpha+[a_{n-1}]^\alpha[x^{n-1}]^\alpha+\cdots+[a_1]^\alpha[x]^\alpha+[a_0]^\alpha,\chi_{\{0\}})=$$
$$\sup\max\{|d_1(\alpha)-0|,|d_2(\alpha)-0|\}=\sup\{|d_2(\alpha)|\}=$$
$$\sup\{(a_n)_R(\alpha)x_R^n(\alpha)+(a_{n-1})_R(\alpha)x_R^{n-1}(\alpha)+\cdots+(a_1)_R(\alpha)x_R(\alpha)+(a_0)_R(\alpha)\}$$
$$\leq\frac{1}{n(n+1)}+\frac{1}{n(n+1)}+\cdots+\frac{1}{n(n+1)}+\frac{n}{n+1}=1,\ \ \forall\ x\ \in B_M.$$

Let $S \subset B_M$ a bounded set (consisting of continuous fuzzy numbers) with $r(S) > 0$. We prove that that $A(S)_L$ and $A(S)_R$ are relatively compact. Indeed, using that for $y \in A(S)$, $\chi_{\{0\}} \leq y \leq \chi_{\{1\}}$, we obtain that $A(S)_L$ is a bounded set in $C[0,1]$,

$$\| y_L \|_\infty \leq d_\infty(y, \chi_{\{0\}}) \leq 1, \ y \in A(S).$$

Let $f \in A(S)_L$, then f is M-Lipschitzian, and $A(S)_L$ is equicontinuous. This proves that $A(S)_L$ is relatively compact by Arzela-Ascoli Theorem, and the same for $A(S)_R$. Lemma 1 guarantees that $A(S)$ is relatively compact and so

$$r(A(S)) = 0 < r(S)$$

and, therefore, A is condensing. Besides, $\chi_{\{0\}}$ and $\chi_{\{1\}}$ are elements in B_M and $\chi_{\{0\}} \leq A\chi_{\{0\}}$, $A\chi_{\{1\}} \leq \chi_{\{1\}}$. This completes the proof. In fact, there exist extremal solutions between $\chi_{\{0\}}$ and $\chi_{\{1\}}$. □

Theorem 6. Let a_i be Lipschitzian fuzzy numbers with $a_i \geq \chi_{\{0\}}$ for $i = 0, 1, \ldots, n$. Moreover, suppose that there exist $k > 0$, $S \geq 0$ such that

$$(a_n)_R(0)k^n + (a_{n-1})_R(0)k^{n-1} + \cdots + (a_1)_R(0)k + (a_0)_R(0) \leq k, \tag{5}$$

and

$$M_n k^n + \cdots + M_1 k + M_0 + S\sum_{i=1}^{n} ik^{i-1}(a_i)_R(0) \leq S, \tag{6}$$

where M_i are, respectively, the Lipschitz constants of a_i, $i = 0, 1, \ldots, n$. Then (1) has a solution in

$$B_{k,S} = \{x \in E^1 \ : \ \chi_{\{0\}} \leq x \leq \chi_{\{k\}}, \quad x \ \ is \ S - Lipschitzian\}.$$

Proof. Define

$$A : B_{k,S} \longrightarrow E^1,$$

by $Ax = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. We show that $A(B_{k,S}) \subseteq B_{k,S}$. Indeed, for $x \in B_{k,S}$, and every $\alpha \in [0,1]$,

$$0 \leq (a_n)_L(\alpha)x_L^n(\alpha) + (a_{n-1})_L(\alpha)x_L^{n-1}(\alpha) + \cdots + (a_1)_L(\alpha)x_L(\alpha) + (a_0)_L(\alpha)$$

$$= (Ax)_L(\alpha) \leq (Ax)_R(\alpha) =$$

$$(a_n)_R(\alpha)x_R^n(\alpha) + (a_{n-1})_R(\alpha)x_R^{n-1}(\alpha) + \cdots + (a_1)_R(\alpha)x_R(\alpha) + (a_0)_R(\alpha)$$

$$\leq (a_n)_R(\alpha)k^n + (a_{n-1})_R(\alpha)k^{n-1} + \cdots + (a_1)_R(\alpha)k + (a_0)_R(\alpha)$$

$$\leq (a_n)_R(0)k^n + (a_{n-1})_R(0)k^{n-1} + \cdots + (a_1)_R(0)k + (a_0)_R(0) \leq k,$$

which proves that $\chi_{\{0\}} \leq Ax \leq \chi_{\{k\}}$. Besides, for $x \in B_{k,S}$, and $\alpha, \ \beta \in [0,1]$,

$$|(Ax)_L(\alpha) - (Ax)_L(\beta)|$$

$$\leq |(a_n)_L(\alpha) - (a_n)_L(\beta)|x_L^n(\alpha) + |(a_{n-1})_L(\alpha) - (a_{n-1})_L(\beta)|x_L^{n-1}(\alpha) + \cdots +$$

$$|(a_1)_L(\alpha)-(a_1)_L(\beta)|x_L(\alpha)+|(a_0)_L(\alpha)-(a_0)_L(\beta)|+$$
$$(a_n)_L(\beta)|x_L^n(\alpha)-x_L^n(\beta)|+(a_{n-1})_L(\beta)|x_L^{n-1}(\alpha)-x_L^{n-1}(\beta)|+\cdots+$$
$$(a_1)_L(\beta)|x_L(\alpha)-x_L(\beta)|$$
$$\leq \{M_nk^n+\cdots+M_1k+M_0+S\sum_{i=1}^{n}ik^{i-1}(a_i)_L(\beta)\}|\alpha-\beta|$$
$$\leq \{\sum_{i=0}^{n}M_ik^i+S\sum_{i=1}^{n}ik^{i-1}(a_i)_R(0)\}|\alpha-\beta|\leq S|\alpha-\beta|,$$

and, similarly,

$$|(Ax)_R(\alpha)-(Ax)_R(\beta)|\leq S|\alpha-\beta|,$$

proving $Ax\in B_{k,S}$. The proof is completed in the same of Theorem 5.□

Remark 1. Inequalities (5) and (6) in Theorem 6 are equivalent to

$$\sum_{i=0}^{n}d_\infty(a_i,\chi_{\{0\}})k^i\leq k, \tag{7}$$

and

$$\sum_{i=0}^{n}M_ik^i+S\sum_{i=1}^{n}ik^{i-1}d_\infty(a_i,\chi_{\{0\}})\leq S, \tag{8}$$

since, for $x\in E^1$, $x\geq\chi_{\{0\}}$,

$$d_\infty(x,\chi_{\{0\}})=\sup_{\alpha\in[0,1]}\max\{|x_L(\alpha)|,|x_R(\alpha)|\}=x_R(0).$$

Corollary 2. In Theorem 6, take $(a_i)_R(0)\leq\frac{1}{n(n+1)}$ for $i=1,2,...,n$ and $(a_0)_R(0)\leq\frac{n}{n+1}$, and $M_0=M_1=\cdots=M_n=\frac{M}{2(n+1)}$, with $M>0$, to obtain Theorem 5.

Proof. Conditions in Theorem 6 are valid for $k=1$ and $S=M$. Indeed,

$$(a_n)_R(0)k^n+(a_{n-1})_R(0)k^{n-1}+\cdots+(a_1)_R(0)k+(a_0)_R(0)\leq 1=k,$$

and

$$M_nk^n+\cdots+M_1k+M_0+S\sum_{i=1}^{n}ik^{i-1}(a_i)_R(0)\leq\frac{M}{2}+\frac{M}{n(n+1)}\sum_{i=1}^{n}i=M=S.\square$$

Theorem 7. *([22], Theorem 11)* Let X be a complete lattice and

$$F:X\longrightarrow X$$

a non-decreasing function, i,e., $F(x)\leq F(y)$ whenever $x\leq y$. Suppose that there exists $x_0\in X$ such that $F(x_0)\geq x_0$. Then F has at least one fixed point in X.

Lemma 2. If $a, x, y \in E^1$ are such that $a \geq \chi_{\{0\}}$ and $\chi_{\{0\}} \leq x \leq y$, then $a^n \geq \chi_{\{0\}}$ and $\chi_{\{0\}} \leq a^n x \leq a^n y$ for $n \in \mathbb{N}$.

Proof. By hypotheses, for all $\alpha \in [0,1]$,

$$0 \leq x_L(\alpha) \leq y_L(\alpha), \quad 0 \leq x_R(\alpha) \leq y_R(\alpha),$$

$$0 \leq a_L(\alpha) \leq a_R(\alpha),$$

so that

$$[a^n]^\alpha = [a_L^n, a_R^n],$$

$$[a^n x]^\alpha = [a_L^n(\alpha)x_L(\alpha), a_R^n(\alpha)x_R(\alpha)], [a^n y]^\alpha = [a_L^n(\alpha)y_L(\alpha), a_R^n(\alpha)y_R(\alpha)],$$

where

$$0 \leq a_L^n(\alpha)x_L(\alpha) \leq a_L^n(\alpha)y_L(\alpha),$$

$$0 \leq a_R^n(\alpha)x_R(\alpha) \leq a_R^n(\alpha)y_R(\alpha), \quad \forall \alpha \in [0,1],$$

hence $a^n \geq \chi_{\{0\}}$ and

$$\chi_{\{0\}} \leq a^n x \leq a^n y.$$

□

Theorem 8. Let a_i, $i = 0, 1, \ldots, n$ be fuzzy numbers such that

$$a_i \geq \chi_{\{0\}},$$

and suppose that there exist $p > 0$ such that

$$(a_n)_R(0)p^n + (a_{n-1})_R(0)p^{n-1} + \cdots + (a_1)_R(0)p + (a_0)_R(0) \leq p.$$

Then (1) has extremal solutions in the interval

$$[\chi_{\{0\}}, \chi_{\{p\}}] = \{x \in E^1 \ : \ \chi_{\{0\}} \leq x \leq \chi_{\{p\}}\}.$$

Proof. Since $p > 0$, $\chi_{\{0\}} < \chi_{\{p\}}$. Define

$$A : [\chi_{\{0\}}, \chi_{\{p\}}] \longrightarrow E^1,$$

by $Ax = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. We show that $A([\chi_{\{0\}}, \chi_{\{p\}}]) \subseteq [\chi_{\{0\}}, \chi_{\{p\}}]$. Indeed,

$$A\chi_{\{0\}} = a_n(\chi_{\{0\}})^n + a_{n-1}(\chi_{\{0\}})^{n-1} + \cdots + a_1(\chi_{\{0\}}) + a_0 =$$

$$\chi_{\{0\}} + \chi_{\{0\}} + \cdots + \chi_{\{0\}} + a_0 = a_0 \geq \chi_{\{0\}},$$

so that, using the conditions, for all $\alpha \in [0,1]$, we have

$$[A\chi_{\{p\}}]^\alpha = [(a_n)_L(\alpha), (a_n)_R(\alpha)]p^n + [(a_{n-1})_L(\alpha), (a_{n-1})_R(\alpha)]p^{n-1}$$

$$+ \cdots + [(a_1)_L(\alpha), (a_1)_R(\alpha)]p + [(a_0)_L(\alpha), (a_0)_R(\alpha)] = [L(\alpha), R(\alpha)],$$

where

$$L(\alpha) = (a_n)_L(\alpha)p^n + (a_{n-1})_L(\alpha)p^{n-1} + \cdots + (a_1)_L(\alpha)p + (a_0)_L(\alpha),$$

and

$$R(\alpha) = (a_n)_R(\alpha)p^n + (a_{n-1})_R(\alpha)p^{n-1} + \cdots + (a_1)_R(\alpha)p + (a_0)_R(\alpha).$$

By hypotheses and using the properties of $(a_i)_L$ and $(a_i)_R$, we obtain, for all $\alpha \in [0,1]$,

$$L(\alpha) \leq R(\alpha) \leq R(0) \leq p.$$

This proves that $A\chi_{\{p\}} \leq \chi_{\{p\}}$. Moreover, A is non-decreasing operator. Indeed, for $\chi_{\{0\}} \leq x \leq y$, we have for all $\alpha \in [0,1]$,

$$0 \leq x_L(\alpha) \leq y_L(\alpha),\ 0 \leq x_R(\alpha) \leq y_R(\alpha),$$

and thus

$$0 \leq (x_L(\alpha))^n \leq (y_L(\alpha))^n,\ 0 \leq (x_R(\alpha))^n \leq (y_R(\alpha))^n.$$

Hence

$$\chi_{\{0\}} \leq x^n \leq y^n.$$

This fact could have also been deduced from application of Lemma 2. Using that $a_i \geq \chi_{\{0\}}$, for $i = 1, \ldots, n$, and applying Lemma 2, we obtain

$$Ax = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \leq a_n y^n + a_{n-1}y^{n-1} + \cdots + a_1 y + a_0 = Ay.$$

Therefore, $A : [\chi_{\{0\}}, \chi_{\{p\}}] \longrightarrow E^1$ is non-decreasing and $[\chi_{\{0\}}, \chi_{\{p\}}]$ is a complete lattice. Tarski's Fixed Point Theorem provides the existence of extremal fixed points for A in $[\chi_{\{0\}}, \chi_{\{p\}}]$, that is, extremal solutions to (1) in the same interval.□

Remark 2. To find an appropriate $p > 0$, we can solve inequality

$$(a_n)_R(0)p^n + (a_{n-1})_R(0)p^{n-1} + \cdots + ((a_1)_R(0) - 1)p + (a_0)_R(0) \leq 0.$$

If $(a_1)_R(0) > 1$, there is no such value of p.

Remark 3. If $0 \leq (a_n)_R(0) + (a_{n-1})_R(0) + \cdots + (a_1)_R(0) < 1$, $(a_n)_R(0) > 0$, $(a_i)_R(0) \geq 0$ for $i = 1, \ldots, n-1$, and

$$\frac{(a_0)_R(0)}{1 - (a_n)_R(0) - (a_{n-1})_R(0) - \cdots - (a_1)_R(0)} \leq 1,$$

then we can take $0 < p \leq 1$ such that

$$p \geq \frac{(a_0)_R(0)}{1 - (a_n)_R(0) - (a_{n-1})_R(0) - \cdots - (a_1)_R(0)}.$$

In this case,

$$(a_n)_R(0)p^n + (a_{n-1})_R(0)p^{n-1} + \cdots + (a_2)_R(0)p^2 \leq$$

$$(a_n)_R(0)p + (a_{n-1})_R(0)p + \cdots + (a_2)_R(0)p$$

and

$$(a_0)_R(0) \leq p[1 - (a_n)_R(0) - (a_{n-1})_R(0) - \cdots - (a_1)_R(0)],$$

hence

$$(a_n)_R(0)p^n + (a_{n-1})_R(0)p^{n-1} + \cdots + (a_1)_R(0)p + (a_0)_R(0) \leq$$
$$p[(a_n)_R(0) + (a_{n-1})_R(0) + \cdots + (a_1)_R(0)] + (a_0)_R(0) \leq p.$$

Theorem 9. Let a_0 and a_i, $i = 1, \ldots, n$, be fuzzy numbers such that

$$a_i \geq \chi_{\{0\}},$$

and suppose that there exist b, $c \in E^1$ with $c > b \geq \chi_{\{0\}}$ and

$$a_n b^n + a_{n-1} b^{n-1} + \cdots + a_1 b + a_0 \geq b,$$

$$a_n c^n + a_{n-1} c^{n-1} + \cdots + a_1 c + a_0 \leq c.$$

Then (1) has extremal solutions in the interval

$$[b, c] = \{x \in E^1 \ : \ b \leq x \leq c\}.$$

Moreover, if $b = c$, then b is a solution to (1).

Proof. Define

$$A : [b, c] \longrightarrow E^1,$$

by $Ax = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$. We show that $A([b, c]) \subseteq [b, c]$. Indeed, by hypotheses

$$Ab = a_n b^n + a_{n-1} b^{n-1} + \cdots + a_1 b + a_0 \geq b,$$

$$Ac = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_1 c + a_0 \leq c.$$

Then $Ab \leq Ac$. Moreover, A is non-decreasing operator. Indeed, for $\chi_{\{0\}} \leq x \leq y$, we have

$$Ax \leq Ay.$$

Therefore, $A : [b, c] \longrightarrow E^1$ is non-decreasing and $[b, c]$ is a complete lattice. Tarski's Fixed Point Theorem provides the existence of extremal fixed points for A in $[b, c]$, that is, extremal solutions to (1) in the same interval.□

4 The Newton's method

In this section, we suppose that $\underline{x} = x_L$ and $\overline{x} = x_R$. Now our aim is to obtain a solution for quadratic fuzzy equations (1), i.e.

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = x,$$

where x and a_i, $i = 0, 1, \ldots, n$, are positive fuzzy numbers. The parametric form for all $\alpha \in [0, 1]$ is as follows:

$$\begin{cases} \underline{a}_n(\alpha)\underline{x}^n(\alpha) + \underline{a}_{n-1}(\alpha)\underline{x}^{n-1}(\alpha) + \cdots + \underline{a}_1(\alpha)\underline{x}(\alpha) + \underline{a}_0(\alpha) = \underline{x}(\alpha), \\ \overline{a}_n(\alpha)\overline{x}^n(\alpha) + \overline{a}_{n-1}(\alpha)\overline{x}^{n-1}(\alpha) + \cdots + \overline{a}_1(\alpha)\overline{x}(\alpha) + \overline{a}_0(\alpha) = \overline{x}(\alpha). \end{cases} \tag{9}$$

Definition 7. The solution of (9) for all $\alpha \in [0, 1]$, is called analytical solution of (1).

Suppose that $x = (\underline{z}, \overline{z})$ be the solution of (9), i.e., for all $\alpha \in [0, 1]$

$$\begin{cases} \underline{a}_n(\alpha)\underline{z}^n(\alpha) + \underline{a}_{n-1}(\alpha)\underline{z}^{n-1}(\alpha) + \cdots + (\underline{a}_1(\alpha) - 1)\underline{z}(\alpha) + \underline{a}_0(\alpha) = 0, \\ \overline{a}_n(\alpha)\overline{z}^n(\alpha) + \overline{a}_{n-1}(\alpha)\overline{z}^{n-1}(\alpha) + \cdots + (\overline{a}_1(\alpha) - 1)\overline{z}(\alpha) + \overline{a}_0(\alpha) = 0. \end{cases}$$

Now we suppose that the functions $\underline{H}$ and $\overline{H}$ are defined for all $\alpha \in [0, 1]$ as follows:

$$\begin{cases} \underline{H}(\underline{z}, \alpha) = \underline{a}_n(\alpha)\underline{z}^n(\alpha) + \underline{a}_{n-1}(\alpha)\underline{z}^{n-1}(\alpha) + \cdots + (\underline{a}_1(\alpha) - 1)\underline{z}(\alpha), \\ \overline{H}(\overline{z}, \alpha) = \overline{a}_n(\alpha)\overline{z}^n(\alpha) + \overline{a}_{n-1}(\alpha)\overline{z}^{n-1}(\alpha) + \cdots + (\overline{a}_1(\alpha) - 1)\overline{z}(\alpha). \end{cases}$$

Therefore, if $x_0 = (\underline{x}_0, \overline{x}_0)$ is approximation solutions for this system, then for all $\alpha \in [0, 1]$, there are $h_1(\alpha), k_1(\alpha)$ such that

$$\begin{cases} \underline{z}(\alpha) = \underline{x}_0(\alpha) + h_1(\alpha), \\ \overline{z}(\alpha) = \overline{x}_0(\alpha) + k_1(\alpha). \end{cases}$$

Now if we use the Taylor series of $\underline{H}, \overline{H}$ about $(\underline{x}_0, \overline{x}_0)$, we have

$$\begin{cases} \underline{H}(\underline{z}, \alpha) = \underline{H}(\underline{x}_0, \alpha) + h_1(\alpha)\underline{H}_{\underline{x}}(\underline{x}_0, \alpha) + O(h_1^2(\alpha)) = -\underline{a}_0(\alpha), \\ \overline{H}(\overline{z}, \alpha) = \overline{H}(\overline{x}_0, \alpha) + k_1(\alpha)\overline{H}_{\overline{x}}(\overline{x}_0, \alpha) + O(k_1^2(\alpha)) = -\overline{a}_0(\alpha). \end{cases}$$

If $\underline{x}_0$ and $\overline{x}_0$ are near to $\underline{z}$ and $\overline{z}$, respectively, then $h_1(\alpha)$ and $k_1(\alpha)$ are small. We assume, of course, that all needed partial derivatives exist and bounded. Therefore for enough small $h_1(\alpha)$ and $k_1(\alpha)$ we have

$$\begin{cases} \underline{H}(\underline{x}_0, \alpha) + h_1(\alpha)\underline{H}_{\underline{x}}(\underline{x}_0, \alpha) \simeq -\underline{a}_0(\alpha), \\ \overline{H}(\overline{x}_0, \alpha) + k_1(\alpha)\overline{H}_{\overline{x}}(\overline{x}_0, \alpha) \simeq -\overline{a}_0(\alpha), \end{cases}$$

and hence $h_1(\alpha)$ and $k_1(\alpha)$ are unknown quantities which can be obtained by solving the following equations:

$$J(\underline{x}_0, \overline{x}_0, \alpha) \begin{bmatrix} h_1(\alpha) \\ k_1(\alpha) \end{bmatrix} = \begin{bmatrix} -\underline{a}_0(\alpha) - \underline{H}(\underline{x}_0, \alpha) \\ -\overline{a}_0(\alpha) - \overline{H}(\overline{x}_0, \alpha) \end{bmatrix},$$

where

$$J(\underline{x}_0, \overline{x}_0, \alpha) = \begin{bmatrix} \underline{H}_{\underline{x}}(\underline{x}_0, \alpha) & 0 \\ 0 & \overline{H}_{\overline{x}}(\overline{x}_0, \alpha) \end{bmatrix}.$$

Hence, the next approximations for $\underline{x}(\alpha)$ and $\overline{x}(\alpha)$ are as follows

$$\begin{cases} \underline{x}_1(\alpha) = \underline{x}_0(\alpha) + h_1(\alpha), \\ \overline{x}_1(\alpha) = \overline{x}_0(\alpha) + k_1(\alpha), \end{cases}$$

for all $\alpha \in [0, 1]$.

We can obtain approximated solution by using the recursive scheme

$$\begin{cases} \underline{x}_n(\alpha) = \underline{x}_{n-1}(\alpha) + h_{1,n-1}(\alpha), \\ \overline{x}_n(\alpha) = \overline{x}_{n-1}(\alpha) + k_{1,n-1}(\alpha), \end{cases}$$

where $h_{1,0}(\alpha) = h_1(\alpha)$ and $k_{1,0}(\alpha) = k_1(\alpha)$ for $n = 1, 2, \ldots$ and for initial guess, one can use Theorem 8. Indeed, we solve the inequality

$$\varphi(p) = (a_n)_R(0)p^n + (a_{n-1})_R(0)p^{n-1} + \cdots + ((a_1)_R(0) - 1)p + (a_0)_R(0) \leq 0.$$

Let $p_1 > 0$ be the smallest number such that $\varphi(p_1) \leq 0$. Now for initial guess we can use the triangular fuzzy number x_0 such that $0 \leq \underline{x}_0 \leq \overline{x}_0 \leq p_1$.

Remark 4. Sequence $\{(\underline{x}_n, \overline{x}_n)\}_{n=0}^{\infty}$ convergent to $(\underline{z}, \overline{z})$ if and only if $\forall \alpha \in [0, 1]$, $\lim_{n\to\infty} \underline{x}_n(\alpha) = \underline{z}(\alpha)$ and $\lim_{n\to\infty} \overline{x}_n(\alpha) = \overline{z}(\alpha)$.

5 Numerical application

Here we present two examples to illustrating the Newton's method for fuzzy nonlinear systems.

Example 1. Consider the fuzzy equation

$$(0, .5, 1)x^3 + (.5, \frac{2}{3}, 1)x^2 + (0, .1, .25)x + (0, .06, .1) = x.$$

Suppose that x be positive, therefore the parametric form of this equation is as follows

$$\begin{cases} .5\alpha\underline{x}^3(\alpha) + (.5 + (\frac{2}{3} - .5)\alpha)\underline{x}^2(\alpha) + (.1\alpha - 1)\underline{x}(\alpha) + .06\alpha = 0, \\ (1 - .5\alpha)\overline{x}^3(\alpha) + (1 - (1 - \frac{2}{3})\alpha)\overline{x}^2(\alpha) + (.25 - .15\alpha - 1)\overline{x}(\alpha) + \\ \qquad\qquad (.1 - .04\alpha) = 0. \end{cases} \tag{10}$$

Since $p_1 = .0.191773 \simeq .192$, therefore we choose initial guess as follows:

$$x_0 = (0, .092, .192),$$

hence $\underline{x}_0(\alpha) = .092\alpha$ and $\overline{x}_0(\alpha) = .192 - .1\alpha$. Suppose that $x = (\underline{x}, \overline{x}) = (\underline{z}, \overline{z})$ be the solution of (10), then

$$\begin{cases} \underline{H}(\underline{z}, \alpha) = .5\alpha\underline{z}^3(\alpha) + (.5 + (\frac{2}{3} - .5)\alpha)\underline{z}^2(\alpha) + (.1\alpha - 1)\underline{z}(\alpha) + .06\alpha, \\ \overline{H}(\overline{z}, \alpha) = (1 - .5\alpha)\overline{z}^3(\alpha) + (1 - (1 - \frac{2}{3})\alpha)\overline{z}^2(\alpha) + (.25 - .15\alpha - 1)\overline{z}(\alpha) + \\ \qquad\qquad (.1 - .04\alpha), \end{cases}$$

and

$$\begin{bmatrix} h_1(\alpha) \\ k_1(\alpha) \end{bmatrix} = (J(\underline{x}_0, \overline{x}_0, \alpha))^{-1} \begin{bmatrix} -.06\alpha - \underline{H}(\underline{x}_0, \alpha) \\ -(.1 - .04\alpha) - \overline{H}(\overline{x}_0, \alpha) \end{bmatrix},$$

$$J(\underline{x}_0, \overline{x}_0, \alpha) = \begin{bmatrix} \underline{H}_{\underline{x}}(\underline{x}_0, \alpha) & 0 \\ 0 & \overline{H}_{\overline{x}}(\overline{x}_0, \alpha) \end{bmatrix} \quad \text{for all } \alpha \in [0, 1].$$

Now determine $\underline{x}_1(\alpha) = \underline{x}_0(\alpha) + h_1(\alpha)$ and $\overline{x}_1(\alpha) = \overline{x}_0(\alpha) + k_1(\alpha)$. After 2 iterations, we obtain the solutions of x which the maximum error would be about $5 * 10^{-5}$. For more details see Figures 1 and 2.

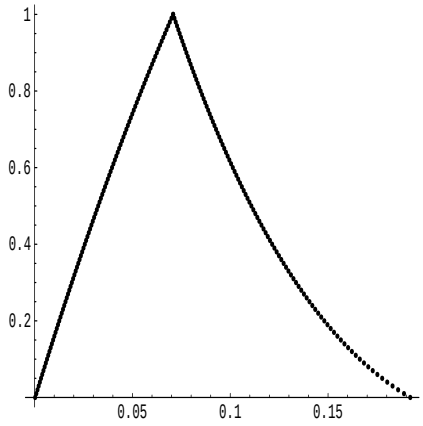

Figure 1: Analytical solution for Example 1

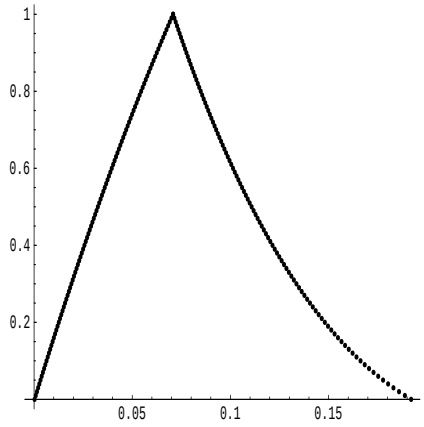

Figure 2: Newton's Method for Example 1

Example 2. Consider fuzzy equation

$$(0,1,3)x^7 + (1,2,5)x^6 + (1,2,3)x^3 + (0,.5,1)x^2 + (0,.06,.1)x + (0,.06,.1) = x,$$

with positive solution. The parametric form of this equation is as follows

$$\begin{cases} \alpha\underline{x}^7(\alpha) + (1+\alpha)\underline{x}^6(\alpha) + (1+\alpha)\underline{x}^3(\alpha) + .5\alpha\underline{x}^2(\alpha) + (.06\alpha - 1)\underline{x}(\alpha) + .06\alpha = 0 \\ (3-2\alpha)\overline{x}^7(\alpha) + (5-3\alpha)\overline{x}^6(\alpha) + (3-\alpha)\overline{x}^3(\alpha) + (1-.5\alpha)\overline{x}^2(\alpha) + \\ \qquad\qquad (.1-.04\alpha-1)\overline{x}(\alpha) + (.1-.04\alpha) = 0. \end{cases}$$

Since $p_1 = 0.1443369682692823 \simeq 0.144$, therefore we choose initial guess as $x_0 = (0, 0.044, 0.144)$, hence $\underline{x}_0(\alpha) = 0.044\alpha$ and $\overline{x}_0(\alpha) = 0.144 - .1\alpha$. One can determine $k_1(\alpha)$, $k_2(\alpha)$, $\underline{x}_1(\alpha)$ and $\overline{x}_1(\alpha)$ as example 1. If we apply two iterations from Newton's method, the maximum error would be less than $2 * 10^{-5}$, see Figures 3 and 4.

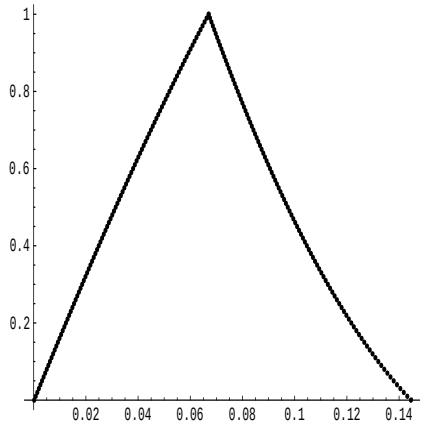

Figure 3: Analytical solution for Example 2

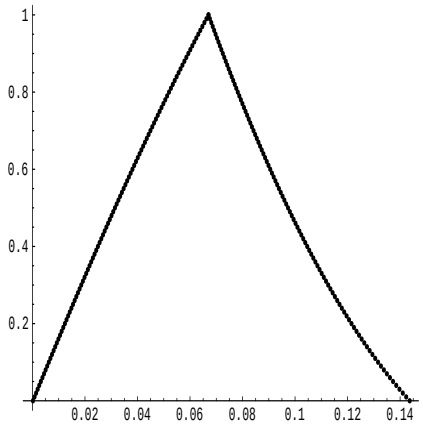

Figure 4: Newton's Method for Example 2

6 Conclusions

In this paper, we obtained extremal solutions for fuzzy polynomials and suggested numerical method for these polynomials instead of standard analytical techniques which are not suitable everywhere. In Section 4 we wrote fuzzy polynomials in parametric form and then solve them by Newton's method. Finally, examples were presented to illustrate proposed method.

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