Mathware & Soft Computing 14 (2007) 57-66

Characterizing Ordered Semigroups by Means of Intuitionistic Fuzzy Bi-ideals

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Abstract

In this paper, we endow set S with the structure of an ordered semigroup and define "intuitionistic fuzzy" analogous for several notions that have been proved to be useful in the theory of ordered semigroups and give some equivalent conditions of these definitions. Then we characterize the completely regular and the strongly regular ordered semigroups by means of intuitionistic fuzzy bi-ideals.

1 Introduction

The concept of fuzzy set was introduced by Zadeh [9]. Fuzzy set theory has been shown to be a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situations by attributing a degree to which a certain object belongs to a set. But in fuzzy sets theory, there is no means to incorporate the hesitation or uncertainty in the membership degrees. In 1983, Antanassov [10] introduces the concept of intuitionistic fuzzy sets, which constitute a extension of fuzzy sets theory: intuitionistic fuzzy sets give both a membership degree and a non-membership degree. The only constraint on these two degrees is that the sum must smaller than or equal to 1. In the present paper, we will call the intuitionistic fuzzy sets defined by Antanassov as for Atanassov's intuitionistic fuzzy sets (A-IFSs for short). The semigroup and group theory of fuzzy sets was deeply studied by many authors [1-8, 13]. In [14], authors define the fuzzy bi-ideals in ordered semigroups and give some theorem which characterizes the bi-ideals in terms of fuzzy bi-ideals and characterize the left and right simple, the completely regular, and the strongly regular ordered semigroups by means of fuzzy bi-ideals. In the present paper, we will define "intuitionistic fuzzy" analogous for several notion that have been proved to be useful in the theory of ordered semigroups and give some equivalent conditions of these definitions. Then we characterize the completely regular and the strongly regular ordered semigroups by means of intuitionistic fuzzy bi-ideals.

2 Preliminaries

Let $\langle S,\cdot,\leq\rangle$ be an ordered semigroup, a subsemigroup $A$of S is a nonempty subset A of S such that $A^2\subseteq A$ and a left (right) ideal of S is a nonempty subset A of S such that (1) $SA\subseteq A$ $\quad(AS\subseteq A)$and (2) If $a\in A$ and $b\in S$ such that $b\leq a$, then $b\in A$.

A two-ideal (or simply ideal) is a subset of S which is both a left and a right ideal of S. S is called left (right) simple if for every left (right) ideal $A$of S, we have $A=S$.

For $H\subseteq S$, we denote

$$(H]:=\{\,t\in S|\,t\leq h \text{ for some } h\in H\}$$

$$(H]*:=\{\,t\in H|\,t\leq h \text{ for some } h\in H\}$$

An ordered semigroup $\langle S,\cdot,\leq\rangle$ is called regular if for every element a of S there exists an element x in S such that $a\leq axa$ (equivalent definition: (1) $a\in(aSa]\ \ \forall a\in S$; (2) $A\subseteq(ASA]\ \ \forall A\subseteq S$). An ordered semigroup $\langle S,\cdot,\leq\rangle$ is called left (right) regular if for every $a\in S$ there exists $x\in S$ such that $a\leq xa^2\ \left(a\leq a^2x\right)$ (equivalent definition: (1) $a\in\left(Sa^2\right]\left(a\in\left(a^2S\right]\right)\ \forall a\in S$, (2) $a\in\left(Sa^2\right]\left(a\in\left(a^2S\right]\right)\ \forall a\in S$).

An ordered semigroup $\langle S,\cdot,\leq\rangle$ is called completely regular if it is regular, left regular and right regular. A subsemigroup A of S is called a bi-ideal of S if

(1) $ASA\subseteq A$, (2) If $a\in A$ and $Sb\leq a,$then $b\in A$.

Definition 1[12] : An A-IFS A in an ordered semigroup $\langle S,\cdot,\leq\rangle$ is an object

$$A=\{\,\langle x,\mu_A(x),v_A(x)\rangle|\,x\in S\}$$

where, for all $x\in S$, $\mu_A(x)\in[0,1]$ and $v_A(x)\in[0,1]$ are called the membership degree and the non-membership degree, respectively, of x in S, and furthermore satisfy $\mu_A(x)+v_A(x)\leq 1$.

Obviously, each ordinary fuzzy set may be written as

$$A=\{\,\langle x,\mu_A(x),1-\mu_A(x)\rangle|\,x\in S\}\,.$$

Definition 2[12] :Let A,B be two A-IFSs in S, then

$$A\subseteq B \text{ iff } (\forall x\in S)\left(\mu_A(x)\leq\mu_B(x)\ \&\ v_A(x)\leq v_B(x)\right)$$

$$A \cap B = \{ \langle x, \min \{\mu_A(x), \mu_B(x)\}, \max\{ v_A(x), v_B(x)\}\rangle | x \in S\}$$
$$A \circ B \quad = \quad \{ \langle x, \mu_{A\circ B}(x), v_{A\circ B}(x)\rangle | x \in S\}$$

where

$$\mu_{A\circ B}(x) = \begin{cases} \sup\limits_{x=yz} \{\min\{\mu_A(y), \mu_B(z)\}\} & \text{if } x \text{ is expressible as } x = yz \\ 0 \text{ otherwise} \end{cases}$$

$$v_{A\circ B}(x) = \begin{cases} \inf\limits_{x=yz} \{\max\{v_A(y), v_B(z)\}\} & \text{if } x \text{ is expressible as } x = yz \\ 1 \text{ otherwise} \end{cases}$$

Definition 3: If $\langle S, \cdot, \leq \rangle$ be an ordered semigroup, an A-IFS A in S is called an intuitionistic fuzzy subsemigroup of S if $\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\}$ and $v_A(xy) \leq \max\{v_A(x), v_A(y)\}$ for all $x, y \in S$. And is called a left (right) ideal of S if
(1) $\mu_A(xy) \geq \mu_A(y)$ and $v_A(xy) \leq v_A(y)$,
($\mu_A(xy) \geq \mu_A(x)$ and $v_A(xy) \leq v_A(x)$),
(2) $x \leq y \Rightarrow \mu_A(x) \geq \mu_A(y)$ and $v_A(x) \leq v_A(y)$ for all $x, y \in S$.

An A-IFS A in S is called an intuitionistic fuzzy two-sided ideal (or simply ideal) of S if it is both an intuitionistic fuzzy left and an intuitionistic right ideal of S.

Definition 4: An intuitionistic fuzzy semigroup A in an ordered semigroup$S$ is called an intuitionistic fuzzy bi-ideal of S if (1) $x \leq y \Rightarrow \mu_{A(x)} \geq \mu_A(y)$ and $v_A(x) \leq v_A(y)$ and (2)$\mu_A(xyz) \geq \min\{\mu_A(x), \mu_A(z)\}$ and $v_A(xyz) \leq \max\{v_A(x), v_A(z)\}$ for all $x, y, z \in S$.

Let A be a subset of an ordered semigroup $\langle S, \cdot, \leq \rangle$, then we denote

$$\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x), v_{\tilde{A}}(x)\rangle | x \in S\}$$

where

$$\mu_{\tilde{A}}(x) = \begin{cases} 1 & x \in A \\ 0 & \text{otherwise} \end{cases}, v_{\tilde{A}}(x) = \begin{cases} 0 & x \in A \\ 1 & \text{otherwise} \end{cases}.$$

Obviously $\tilde{A}$ is an A-IFS in S. Obviously, an ordered semigroup $\langle S, \cdot, \leq \rangle$ also can be seen as an A-IFS $\tilde{S} = \{ \langle x, 1, 0\rangle | x \in S\}$. In the present paper, we will use S represent S and $\tilde{S}$, it is easy to see their means from context.

Lemma 1: Let A be a nonempty subset of an ordered semigroup $\langle S, \cdot, \leq \rangle$, then

(1) A is a subsemigroup of S if and only if $\tilde{A}$ is an intuitionistic fuzzy subsemigroup of S.

(2) A is a left (right, two-sided) ideal of S if and only if $\tilde{A}$ is an intuitionistic fuzzy left (right, two-sided) ideal of S.

Proof: (1) For any element x, y of A, since A is a subsemigroup of S, then $xy \in S$,

$$\mu_{\tilde{A}}(xy) = 1 = \mu_{\tilde{A}}(x) = \mu_{\tilde{A}}(y) = \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\},$$
$$v_{\tilde{A}}(xy) = 0 = v_{\tilde{A}}(x) = v_{\tilde{A}}(y) = \max\{v_{\tilde{A}}(x), v_{\tilde{A}}(y)\}.$$

So $\tilde{A}$ is an intuitionistic fuzzy semigroup of S.

Conversely, for any element x, y of A, since $\tilde{A}$ is an intuitionistic fuzzy semigroup of S, then$\mu_{\tilde{A}}(xy) \geq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\} = 1, v_{\tilde{A}}(xy) \leq \max\{v_{\tilde{A}}(x), v_{\tilde{A}}(y)\} = 0$.So $xy \in A$, thus $A^2 \subseteq A$.

(2) We only prove the case when$A$ is a left ideal of S. Since A is a left ideal of S, then for arbitrary $xy \in S$, in the case when $y \in A$, then

$$\mu_{\tilde{A}}(xy) = 1 \geq \mu_{\tilde{A}}(y), v_{\tilde{A}}(xy) = 0 \leq v_{\tilde{A}}(y).$$

In the case when $y \notin A$,then

$$\mu_{\tilde{A}}(xy) \geq \mu_{\tilde{A}}(y) = 0, v_{\tilde{A}}(xy) \leq v_{\tilde{A}}(y) = 1.$$

Let $x, y \in S, x \leq y$. Then $\mu_{\tilde{A}}(x) \geq \mu_{\tilde{A}}(y)$ and $v_{\tilde{A}}(x) \leq v_{\tilde{A}}(y)$. Indeed, if $y \in A$, then $\mu_{\tilde{A}}(y) = 1, v_{\tilde{A}}(y) = 0$. Since $Sx \leq y \in A$, we have $x \in A$, then $\mu_{\tilde{A}}(x) = 1, v_{\tilde{A}}(x) = 0$, thus $\mu_{\tilde{A}}(x) \geq \mu_{\tilde{A}}(y)$ and $v_{\tilde{A}}(x) \leq v_{\tilde{A}}(y)$. If $y \notin A$, then $\mu_{\tilde{A}}(y) = 0, v_{\tilde{A}}(y) = 1$. Since $x \in A$, we have

$$\mu_{\tilde{A}}(x) \geq 0 = \mu_{\tilde{A}}(y) \text{ and } v_{\tilde{A}}(x) \leq 1 = v_{\tilde{A}}(y).$$

So $\tilde{A}$ is an intuitionistic fuzzy left ideal of S.

Conversely, for any element x, y of S, since $\tilde{A}$ is an intuitionistic fuzzy left ideal of S, then $\mu_{\tilde{A}}(xy) \geq \mu_{\tilde{A}}(y)$ and $v_{\tilde{A}}(xy) \leq v_{\tilde{A}}(y)$ in the case when $y \in A$, and

$$\mu_{\tilde{A}}(xy) \geq \mu_{\tilde{A}}(y) = 1 \text{ and } v_{\tilde{A}}(xy) \leq v_{\tilde{A}}(y) = 0,$$

so $xy \in A$, this implies that $SH \subseteq A$.

Let $x \in A, Sy \leq x$. Then $y \in A$. Indeed, since $x \in A$, so$\mu_{\tilde{A}}(x) = 1, v_{\tilde{A}}(x) = 0$. Since $\tilde{A}$ is an intuitionistic fuzzy left ideal of S and $y \leq x$, we have

$$\mu_{\tilde{A}}(y) \geq \mu_{\tilde{A}}(x) = 1 \text{ and } v_{\tilde{A}}(y) \leq v_{\tilde{A}}(x) = 0,$$

so $y \in A$. This completes the proof.

Lemma 2: A nonempty subset A of an ordered semigroup $\langle S, \cdot, \leq \rangle$ is a bi-ideal of S if and only if $\tilde{A}$ is an intuitionistic fuzzy bi-ideal of S.

Proof: First assume that A is a bi-ideal of an ordered semigroup S, then for any element x, y, z of S.

In the case when $x \in A$ and $z \in A$, then $\mu_{\tilde{A}}(xyz) = 1 \geq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(z)\}$.

In the case when $x \notin A$ and $z \in A$, then $\mu_{\tilde{A}}(xyz) \geq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(z)\} = 0$, and $v_{\tilde{A}}(xyz) \leq \max\{v_{\tilde{A}}(x), v_{\tilde{A}}(z)\} = 1$.

In the case when $x \in A$ and $z \notin A$, then $\mu_{\tilde{A}}(xyz) \geq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(z)\} = 0$, and $v_{\tilde{A}}(xyz) \leq \max\{v_{\tilde{A}}(x), v_{\tilde{A}}(z)\} = 1$.

In the case when $x \notin A$ and $z \notin A$, then $\mu_{\tilde{A}}(xyz) \geq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(z)\} = 0$, and $v_{\tilde{A}}(xyz) \leq \max\{v_{\tilde{A}}(x), v_{\tilde{A}}(z)\} = 1$.

Let $x, y \in S, x \leq y$. Then $\mu_{\tilde{A}}(x) \geq \mu_{\tilde{A}}(y)$ and $v_{\tilde{A}}(x) \leq v_{\tilde{A}}(y)$.Indeed, if $y \in A$, then $\mu_{\tilde{A}}(y) = 1, v_{\tilde{A}}(y) = 0$. Since $Sx \leq y \in A$, we have $x \in A$, then $\mu_{\tilde{A}}(x) = 1, v_{\tilde{A}}(x) = 0$, thus $\mu_{\tilde{A}}(x) \geq \mu_{\tilde{A}}(y)$ and $v_{\tilde{A}}(x) \leq v_{\tilde{A}}(y)$. If $y \notin A$, then $\mu_{\tilde{A}}(y) = 0, v_{\tilde{A}}(y) = 1$. Since $x \in A$, we have

$$\mu_{\tilde{A}}(x) \geq 0 = \mu_{\tilde{A}}(y) \text{ and } v_{\tilde{A}}(x) \leq 1 = v_{\tilde{A}}(y).$$

Thus $\tilde{A}$ is an intuitionistic fuzzy biideal of S.

Conversely, assume $\tilde{A}$ is an intuitionistic fuzzy biideal of S, then for any element $x, z \in A, y \in S$, we have $\mu_{\tilde{A}}(xyz) \geq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(z)\} = 1$,and $v_{\tilde{A}}(xyz) \leq \max\{v_{\tilde{A}}(x), v_{\tilde{A}}(z)\} = 0$. Thus $xyz \in A$, and $ASA \subseteq A$.

Let $x \in A, Sy \leq x$, then $y \in A$. Indeed, since $x \in A$, so$\mu_{\tilde{A}}(x) = 1, v_{\tilde{A}}(x) = 0$. Since $\tilde{A}$ is an intuitionistic fuzzy bi-ideal of $S$and $y \leq x$, we have

$$\mu_{\tilde{A}}(y) \geq \mu_{\tilde{A}}(x) = 1 \text{ and } v_{\tilde{A}}(y) \leq v_{\tilde{A}}(x) = 0,$$

So $y \in A$. This completes the proof.

Lemma 3: An A-IFS A in an ordered semigroup $\langle S, \cdot, \leq \rangle$ is an intuitionistic fuzzy subsemigroup of S if and only if $A \circ A \subseteq A$.

Proof: If $A \circ A \subseteq A$, then for $\forall x, y \in S$, we have $\mu_A(xy) \geq \mu_{A \circ A}(xy) \geq \min\{\mu_A(x), \mu_A(y)\}$, and $v_A(xy) \leq v_{A \circ A}(xy) \leq \max\{v_A(x), v_A(y)\}$. So A is an intuitionistic fuzzy subsemigroup of S.

Conversely, for $\forall x \in S$, if there are $y, z \in S$ such that $x = yz$, then

$$\mu_{A \circ A}(x) \leq \min\{\mu_A(y), \mu_A(z)\} \leq \mu_A(yz) = \mu_A(x),$$

$$v_{A \circ A}(x) \geq \max\{v_A(y), v_A(z)\} \geq v_A(yz) = v_A(x);$$

otherwise

$$\mu_{A \circ A}(x) = 0 \leq \mu_A(x), v_{A \circ A}(x) = 1 \geq v_A(x).$$

Thus $A \circ A \subseteq A$, this completes the proof.

Lemma 4[13] :An ordered semigroup $\langle S, \cdot, \leq \rangle$ is completely regular if and only if $A \subseteq \left(A^2 S A^2\right]$ for every $A \subseteq S$. (Equivalently, if $a \in \left(a^2 S a^2\right]$ for every $a \in S$.)

Lemma 5[13] : An ordered semgiroup $\langle S, \cdot, \leq \rangle$ is strongly regular if and only if one of the following equivalent conditions is satisfied:

(1) $\langle S, \cdot, \leq \rangle$ is left regular, right regular, and $(SaS]$ is strongly regular subsemigroup of S, for every $a \in S$.

(2) For each $a \in S$, we have $a \in (Sa] \cap (aS]$, and $(SaS]$ is a strongly regular subsemigroup of S.

3 Main Results

Theorem 1: For an A-IFS A of an ordered semigroup $\langle S, \cdot, \leq \rangle$, the following conditions are equivalent:

(1) A is an intuitionistic fuzzy left ideal of S.

(2) $S \circ A \subseteq A$and

$$x \leq y \Rightarrow \mu_A(x) \geq \mu_A(y) \text{ and } v_A(x) \leq v_A(y)$$

for all $x, y \in S$.

Proof: First assume that (1) holds. Let a be any element of S. In the case when there exist element $x, y \in S$ such that $a = xy$. Then, since A is an intuitionistic fuzzy left ideal of S, we have

$$\begin{aligned}
\mu_{S\circ A}(a) &= \sup_{a=xy}\{\min\{\mu_S(x), \mu_A(y)\}\} \\
&\leq \sup_{a=xy}\{\min\{1, \mu_A(xy)\}\} \\
&= \sup_{a=xy}\{\min\{1, \mu_A(a)\}\} \\
&= \mu_A(a), \\
v_{S\circ A}(a) &= \inf_{a=xy}\{\max\{v_S(x), v_A(y)\}\} \\
&\geq \inf_{a=xy}\{\max\{0, v_A(xy)\}\} \\
&= \inf_{a=xy}\{\max\{0, v_A(a)\}\} \\
&= v_A(a).
\end{aligned}$$

Otherwise $\mu_{S\circ A}(a) = 0 \leq \mu_A(a), v_{S\circ A}(a) = 1 \geq v_A(a)$.

So we have $S \circ A \subseteq A$.By the definition of intuitionistic fuzzy left ideal of S, we have $x \leq y \Rightarrow \mu_A(x) \geq \mu_A(y)$ and $v_A(x) \leq v_A(y)$ for all $x, y \in S$.

Conversely, assume (2) holds. Let x, y be any element of S, put $a = xy$. Since $S \circ A \subseteq A$, we have

$$\begin{aligned}
\mu_A(xy) &\geq \mu_{S\circ A}(a) \\
&= \sup_{a=bc} \{\min\{\mu_S(b), \mu_A(c)\}\} \\
&\geq \min\{\mu_S(x), \mu_A(y)\} \\
&= \min\{1, \mu_A(y)\} \\
&= \mu_A(y), \\
v_{A(xy)} &\leq v_{S\circ A}(a) \\
&= \inf_{a=bc} \{\max\{v_S(b), v_A(c)\}\} \\
&\leq \max\{v_S(x), v_A(y)\} \\
&= \max\{0, \mu_A(y)\} \\
&= v_A(y).
\end{aligned}$$

This means that A is an intuitionistic fuzzy left ideal of S, thus (2) implies (1). This completes the proof.

The left-right dual of *Theorem 1* states as follows:

Theorem 2: For an A-IFS A of an ordered semigroup $\langle S, \cdot, \leq \rangle$, the following conditions are equivalent:

(1) A is an intuitionistic fuzzy right ideal of S.

(2) $A \circ S \subseteq A$ and $x \leq y \Rightarrow \mu_A(x) \geq \mu_A(y)$ and $v_A(x) \leq v_A(y)$ for all $x, y \in S$.

Combining the above two *Theorems* , we can obtain the following result:

Theorem 3: For an A-IFS A of an ordered semigroup $\langle S, \cdot, \leq \rangle$, the following conditions are equivalent:

(1) A is an intuitionistic fuzzy two-sided ideal of S.

(2) $S \circ A \subseteq A$ and $A \circ S \subseteq A$ and $x \leq y \Rightarrow \mu_A(x) \geq \mu_A(y)$ and $v_A(x) \leq v_A(y)$ for all $x, y \in S$.

For an intuitionistic fuzzy bi-ideal of an ordered semigroup we have the following:

Theorem 4: For an A-IFS A of an ordered semigroup $\langle S, \cdot, \leq \rangle$, the following conditions are equivalent:

(1) A is an intuitionistic fuzzy bi-ideal of S.

(2) $A \circ A \subseteq A$ and $A \circ S \circ A \subseteq A$and $x \leq y \Rightarrow \mu_A(x) \geq \mu_A(y)$ and $v_A(x) \leq v_A(y)$ for all $x, y \in S$.

Proof: First assume that (1) holds. Since A is an intuitionistic fuzzy bi-ideal of S, then by the definition of intuitionistic fuzzy bi-ideal, it is obvious that

$x \leq y \Rightarrow \mu_A(x) \geq \mu_A(y)$ and $v_A(x) \leq v_A(y)$for all $x, y \in S$.

Since A is an intuitionistic fuzzy semigroup of S, by Lemma 4 that $A \circ A \subseteq A$. Let a be any element of S, in the case when $\mu_{A\circ S\circ A}(a) = 0, v_{A\circ S\circ A}(a) = 1$, it is obvious that $A \circ S \circ A \subseteq A$.

Otherwise, there exist elements $x, y, p, q \in S$ such that $a = xy$ and $x = pq$. Since A is an intuitionistic fuzzy biideal of S, we have

$$\mu_A(pqy) \geq \min\{\mu_A(p), \mu_A(y)\},$$
$$v_A(pqy) \leq \max\{v_A(p), v_A(y)\}.$$

Then we have

$$\begin{aligned}
\mu_{A\circ S\circ A}(a) &= \sup_{a=xy}\{\min\{\mu_{A\circ S}(x), \mu_A(y)\}\} \\
&= \sup_{a=xy}\left\{\min\left\{\sup_{x=pq}\{\min\{\mu_A(p), \mu_S(q)\}\}, \mu_A(y)\right\}\right\} \\
&= \sup_{a=xy}\left\{\min\left\{\sup_{x=pq}\{\min\{\mu_A(p), 1\}\}, \mu_A(y)\right\}\right\} \\
&= \sup_{a=xy}\{\min\{\mu_A(p), \mu_A(y)\}\} \\
&\leq \sup_{a=xy}\{\mu_A(pqy)\} \\
&= \sup_{a=xy}\{\mu_A(xy)\} \\
&= \mu_A(a),
\end{aligned}$$

$$\begin{aligned}
v_{A\circ S\circ A}(a) &= \inf_{a=xy}\{\max\{v_{A\circ S}(x), v_A(y)\}\} \\
&= \inf_{a=xy}\left\{\max\left\{\inf_{x=pq}\{\max\{v_A(p), v_S(q)\}\}, v_A(y)\right\}\right\} \\
&= \inf_{a=xy}\left\{\max\left\{\inf_{x=pq}\{\max\{v_A(p), 0\}\}, v_A(y)\right\}\right\} \\
&= \inf_{a=xy}\{\max\{v_A(p), v_A(y)\}\} \\
&\geq \inf_{a=xy}\{v_A(pqy)\} \\
&= \inf_{a=xy}\{v_A(xy)\} \\
&= v_A(a),
\end{aligned}$$

and so we have $A \circ S \circ A \subseteq A$.

Conversely, assume that (2) holds. Since $A \circ A \subseteq A$, by lemma 4 that A is an intuitionistic fuzzy subsemigroup of S. Let x, y, z be any element of S. Put $a = xyz$, since $A \circ S \circ A \subseteq A$, we have

$$\begin{aligned}
\mu_A(xyz) &= \mu_A(a) \\
&\geq \mu_{A\circ S\circ A}(a) \\
&= \sup_{a=bc}\{\min\{\mu_{A\circ S}(b), \mu_A(c)\}\} \\
&\geq \min\{\mu_{A\circ S}(xy), \mu_A(z)\} \\
&= \min\left\{\sup_{xy=pq}\{\min\{\mu_A(p), \mu_S(q)\}\}, \mu_A(z)\right\} \\
&\geq \min\{\min\{\mu_A(x), \mu_S(y)\}, \mu_A(z)\} \\
&= \min\{\min\{\mu_A(x), 1\}, \mu_A(z)\} \\
&= \min\{\mu_A(x), \mu_A(z)\}
\end{aligned}$$

$$\begin{aligned} v_A(xyz) &= v_A(a) \\ &\leq v_{A\circ S\circ A}(a) \\ &= \inf_{a=bc}\{\max\{v_{A\circ S}(b), v_A(c)\}\} \\ &\leq \max\{v_{A\circ S}(xy), v_A(z)\} \\ &= \max\left\{\inf_{xy=pq}\{\max\{v_A(p), v_S(q)\}\}, v_A(z)\right\} \\ &\leq \max\{\max\{v_A(x), v_S(y)\}, v_A(z)\} \\ &= \max\{\max\{v_A(x), 0\}, v_A(z)\} \\ &= \max\{\mu_A(x), \mu_A(z)\} \end{aligned}$$

This means that A is an intuitionistic fuzzy bi-ideal of S. This completes the proof.

Theorem 5: An ordered semigroup $\langle S, \cdot, \leq\rangle$ is completely regular if and only if for each intuitionistic fuzzy bi-ideal A of S, we have

$$\mu_A(a) = \mu_A(a^2) \text{ and } v_A(a) = v_A(a^2) \text{ for every } a \in S.$$

Proof: Let A be an intuitionistic fuzzy bi-ideal of $S$and let $a \in S$, then by lemma 4 , we have $a \in (a^2Sa^2]$, i.e. there exists $x \in S$ such that $a \leq a^2xa^2$. Since A is an intuitionistic fuzzy bi-ideal of S, we have

$$\begin{aligned} \mu_A(a) \geq \mu_A(a^2xa^2) \geq \min\{\mu_A(a^2), \mu_A(a^2)\} = \mu_A(a^2), \\ v_A(a) \leq v_A(a^2xa^2) \leq \max\{v_A(a^2), v_A(a^2)\} = v_A(a^2). \end{aligned}$$

On the other hand, since A is an intuitionistic fuzzy subsemigroup of S, we have

$$\begin{aligned} \mu_A(a^2) = \mu_A(aa) \geq \min\{\mu_A(a), \mu_A(a)\} = \mu_A(a), \\ v_A(a^2) = v_A(aa) \leq \max\{v_A(a), v_A(a)\} = v_A(a). \end{aligned}$$

So we have $\mu_A(a) = \mu_A(a^2)$ and $v_A(a) = v_A(a^2)$ for every $a \in S$.

Conversely, let $a \in S$, we consider the bi-ideal of S generated by a^2, that is , the set $A := (a^2 \cup a^4 \cup a^2Sa^2]$. Since A is a bi-ideal of S,by Lemma 2, $\tilde{A}$ is an intuitionistic fuzzy bi-ideal of S, by hypothesis, we have

$$\mu_{\tilde{A}}(a) = \mu_{\tilde{A}}(a^2) \text{ and } v_{\tilde{A}}(a) = v_{\tilde{A}}(a^2) \text{ for every } a \in S.$$

Since $a^2 \in A$, we have $\mu_{\tilde{A}}(a^2) = 1$ and $v_{\tilde{A}}(a^2) = 0$ .Thus$\mu_{\tilde{A}}(a) = 1$ and $v_{\tilde{A}}(a) = 0$, and $a \in A = (a^2 \cup a^4 \cup a^2Sa^2]$, then $a \leq y$ for some $y \in a^2 \cup a^4 \cup a^2Sa^2$.

In the case of $y = a^2$, then $a \leq y = a^2 = aa \leq a^2a^2 = aaa^2 \leq a^2aa^2 \in a^2Sa^2$, and $a \in (a^2Sa^2]$.

In the case of $y = a^4$, then $a \leq y = a^4 = aaa^2 \leq a^4aa^2 \in a^2Sa^2$, and $a \in (a^2Sa^2]$.

In the case of $y \in a^2Sa^2$, then $a \in (a^2Sa^2]$.

So we have $a \in (a^2Sa^2]$ for every $a \in S$. This completes the proof.

The strongly regular ordered semigroups are completely regular. By *theorem 5* and *lemma 5*, we have the following:

Theorem 6: An ordered semigroup $\langle S, \cdot, \leq \rangle$ is strongly regular if and only if the following two conditions are satisfied:

(1) For each intuitionistic fuzzy bi-ideal A of S, we have $\mu_A(a) = \mu_A(a^2)$ and $v_A(a) = v_A(a^2)$ for every $a \in S$.

(2) $(SaS]$ is a strongly regular subsemigroup of S for every $a \in S$.

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