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Homogeneity Properties of Hypercubes

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Abstract

The paper reviews properties of hypercubes of arbitrary dimension from the metric geometry point of view. It is shown, in particular, that a hypercube is a homogeneous metric space with respect to a class of partial cubes. This generalizes the ℓ_1-rigidity property of finite partial cubes.

1 Introduction

Hypercubes and their isometric subgraphs known as 'partial cubes' have many applications ranging from chemistry (benzenoid graphs) to preference modeling [14], to computer science (hypercube computers), suggesting that these concepts are ubiquitous. These graphs also provide examples of infinite dimensional discrete geometries with rich metric structures. In the paper we use principles of classical distance geometry (cf. [3]) to investigate homogeneity properties of hypercubes of arbitrary dimension.

Let X and Y be metric spaces. An *isometry* from X onto Y is a distance preserving bijection $X \hookrightarrow Y$. A metric space X is *homogeneous* if for any two points $x, y \in X$ there is an isometry $\alpha : X \hookrightarrow X$ such that $\alpha(x) = y$. Following [4], we say that a metric space X is *fully homogeneous* if, for every two metric subspaces $A, B \subseteq X$ and an isometry $A \hookrightarrow B$, this isometry can be extended to an isometry of the entire space X onto itself. Euclidean, spherical, and hyperbolic spaces are examples of fully homogeneous metric spaces [1]. Another example is a finite metric space X with the discrete metric d (that is, $d(x, y) = 1$ for all distinct $x, y \in X$). Note that such a space is not fully homogeneous if X is an infinite set.

In a more general setting (cf. [3]), let $\mathcal{M}$ be a nonempty class of metric subspaces of a given metric space X. We say that the space X is *homogeneous with respect to* $\mathcal{M}$ (or $\mathcal{M}$-*homogeneous*) if, for every two subspaces $A, B \in \mathcal{M}$ and an isometry $A \hookrightarrow B$, this isometry can be extended to an isometry of the entire space X onto itself. For example, a finite dimensional normed vector space is homogeneous with respect to the family of its vector subspaces (Witt's Theorem [12]). It is shown in [13] that some ℓ_1-spaces are homogeneous with respect to classes of rectangular subspaces.

We begin by introducing basic facts about hypercubes and partial cubes in the next section.

Vertices of partial cubes are well-graded families of sets (see Section 3). On the other hand, well-graded families of subsets of a set X induce partial cubes on the set X. This relationship is crucial for our investigation of homogeneity properties of hypercubes. In Section 3 we introduce geometric properties of well-graded families of sets. The isometry group of a hypercube of arbitrary dimension is characterized in Section 4.

Finally, in Section 5 we show that a hypercube is a homogeneous metric space with respect to a particular class of its partial subcubes.

Some results of the paper are known for finite hypercubes. However, our proofs are different from the standard ones and cover the infinite as well as the finite dimensional hypercubes and partial cubes.

2 Hypercubes and partial cubes

Let X be a set. We denote $\mathcal{P}_f(X)$ the set of all finite subsets of X. A graph $\mathcal{H}(X)$ has the set $\mathcal{P}_f(X)$ as the set of its vertices; a pair of vertices $\{P, Q\}$ is an edge of $\mathcal{H}(X)$ if the symmetric difference $P \bigtriangleup Q$ is a singleton. The graph $\mathcal{H}(X)$ is called the *hypercube on* X [5]. If X is a finite set of cardinality n, then the graph $\mathcal{H}(X)$ is the n*-cube* Q_n.

The shortest path distance $d(P, Q)$ on the hypercube $\mathcal{H}(X)$ is the *Hamming distance* between sets P and Q:

$$d(P, Q) = |P \bigtriangleup Q| \quad \text{for } P, Q \in \mathcal{P}_f.$$

The set $\mathcal{P}_f(X)$ is a metric space with the metric d.

A set $R \in \mathcal{P}_f(X)$ is *between* two sets $P, Q \in \mathcal{P}_f(X)$ if

$$d(P, R) + d(R, Q) = d(P, Q)$$

(metric betweenness relation). It is well-known (see, for instance, [2]) that R lies between P and Q if and only if

$$P \cap Q \subseteq R \subseteq P \cup Q$$

(lattice betweenness relation).

Let $\mathcal{F}$ be a nonempty family of finite subsets of X and let G be a subgraph of $\mathcal{H}(X)$ induced by $\mathcal{F}$. If G is an isometric subgraph of $\mathcal{H}(X)$, then G is called a *partial cube on* X [11]. In general, a graph is a *partial cube* if it can be isometrically embedded into $\mathcal{H}(X)$ for some set X.

Let $G = (V, E)$ be a (simple) connected graph. For an edge $\{u, v\} \in E$, let W_{uv} be the set of vertices of G that are closer to u than to v:

$$W_{uv} = \{w \in V : d_G(w, u) < d_G(w, v)\}.$$

If G is bipartite, then sets W_{uv} and W_{vu} form a bipartition of V. These sets and graphs induced by these sets are called *semicubes* of G [8].

Let $G = (V, E)$ be a connected bipartite graph. We say that two edges of G *stand in the relation* θ if their respective semicubes define the same partition of V. If G is a partial cube, then θ is an equivalence relation on E [5].

The (isometric) *dimension*, $\dim_I G$, of a graph G is the smallest cardinality of a set X such that G can be isometrically embedded into the hypercube $\mathcal{H}(X)$. The following theorem is Theorem 2 in [5].

Theorem 2.1. *Let G be a partial cube. Then $\dim_I G = |E/\theta|$, where E/θ is the set of equivalence classes of the relation θ.*

3 Partial cubes and well-graded families of sets

It is clear that a family $\mathcal{F}$ of finite subsets of a set X induces a partial cube on X if and only if for any two distinct $P, Q \in \mathcal{F}$ there is a sequence of sets $R_0 = P, R_1, \ldots, R_n = Q$ in $\mathcal{F}$ such that

$$d(R_i, R_{i+1}) = 1 \quad \text{for all } 0 \le i < n, \text{ and} \quad d(P, Q) = n. \tag{3.1}$$

The families of sets satisfying these conditions are known as *well-graded families* of sets [7]. We shall call them *wg-families* of sets. Note that the sequence (R_i) is a shortest path from P to Q in $\mathcal{H}(X)$ (and in the subgraph induced by $\mathcal{F}$).

In the rest of the paper we use the same symbol, say $\mathcal{F}$, for a family of finite subsets of X and a subgraph of $\mathcal{H}(X)$ induced by this family. We consider $\mathcal{F}$ as a metric space with the Hamming distance as its metric. To avoid trivialities we assume that $|\mathcal{F}| \ge 2$.

Let $\mathcal{F}$ be a nonempty family of sets in $\mathcal{P}_f(X)$. The set of all $R \in \mathcal{F}$ that lie between $P, Q \in \mathcal{F}$ is the *interval* $\mathcal{I}(P, Q)$ *between P and Q in* $\mathcal{F}$. Note that the set $\mathcal{I}(P, Q)$ is defined relative to the family $\mathcal{F}$. In general, the interval between P and Q in $\mathcal{F}$ might be a proper subset of the usual interval $[P \cap Q, P \cup Q]$ defined by P and Q in $\mathcal{P}_f(X)$.

Two distinct sets $P, Q \in \mathcal{F}$ are *adjacent in* $\mathcal{F}$ if $\mathcal{I}(P, Q) = \{P, Q\}$. If sets P and Q form an edge in the graph induced by $\mathcal{F}$, then P and Q are adjacent in $\mathcal{F}$ but, in general, not vice versa. The following proposition gives a 'local' characterization of wg-families of sets. The proof is by a simple induction argument on the distance between two sets and omitted.

Proposition 3.1. *A family $\mathcal{F} \subseteq \mathcal{P}_f(X)$ is well-graded if and only if $d(P, Q) = 1$ for any two sets P and Q that are adjacent in $\mathcal{F}$.*

Let G be a partial cube. This graph admits isometric representations as partial cubes on various sets X. For instance, the complete graph K_2 can be isometrically embedded in different ways into any hypercube $\mathcal{H}(X)$ with $|X| > 2$. It is desirable to 'minimize' the class of hypercubes $\mathcal{H}(X)$ that can be used as target graphs for isometric embeddings of G. We do it by 'reducing the domain' of a wg-family of sets.

Let $\mathcal{F}'$ be a family of subsets of a set X'. We define the *reduction* of $\mathcal{F}'$ as a family $\mathcal{F}$ of subsets of $X = \cup\mathcal{F}' \setminus \cap\mathcal{F}'$ consisting of the intersections of sets in $\mathcal{F}'$ with X. It is clear that $\mathcal{F}$ satisfies the following two conditions

$$\cap\mathcal{F} = \varnothing \quad \text{and} \quad \cup\mathcal{F} = X. \tag{3.2}$$

In other words, the reduction $\mathcal{F}$ of $\mathcal{F}'$ is obtained by eliminating 'inactive' elements from the set X'.

Lemma 3.1. *The partial cubes induced by a wg-family $\mathcal{F}'$ and its reduction $\mathcal{F}$ are isomorphic.*

Proof. It suffices to prove that metric spaces $\mathcal{F}'$ and $\mathcal{F}$ are isometric. Let us define a mapping $\alpha : \mathcal{F}' \to \mathcal{F}$ by $P \mapsto P \cap X$. Clearly, α is surjective. We have

$$(P \cap X) \triangle (Q \cap X) = (P \triangle Q) \cap X = (P \triangle Q) \cap (\cup\mathcal{F}' \setminus \cap\mathcal{F}') = P \triangle Q.$$

Thus, $d(\alpha(P), \alpha(Q)) = d(P, Q)$. Consequently, α is an isometry. □

Let G be a partial cube on some set X induced by a wg-family $\mathcal{F}$ and let $\{P, Q\}$ be an edge of G. Then there is $x \in \cup\mathcal{F} \setminus \cap\mathcal{F}$ such that $P \triangle Q = \{x\}$. It is not difficult to show that two sets

$$\{R \in \mathcal{F} : x \in R\} \quad \text{and} \quad \{R \in \mathcal{F} : x \notin R\}$$

form the same partition of $\mathcal{F}$ as semicubes W_{PQ} and W_{QP}. Thus there is one-to-one correspondence between equivalence classes of the relation θ and those elements $x \in X$ that define edges in G.

Lemma 3.2. *If $\mathcal{F}$ is a wg-family of sets, then for any $x \in (\cup\mathcal{F} \setminus \cap\mathcal{F})$ there are sets $P, Q \in \mathcal{F}$ such that $P \triangle Q = \{x\}$.*

Proof. For a given $x \in \cup\mathcal{F} \setminus \cap\mathcal{F}$ there are sets S and T in $\mathcal{F}$ such that $x \in S$ and $x \notin T$. Let $R_0 = S, R_1, \ldots, R_n = T$ be a sequence of sets in $\mathcal{F}$ satisfying conditions (3.1). It is clear that there is i such that $x \in R_i$ and $x \notin R_{i+1}$. Hence, $R_i \triangle R_{i+1} = \{x\}$, so we can choose $P = R_i$ and $Q = R_{i+1}$. □

It follows that any element of the set $\cup\mathcal{F}\setminus\cap\mathcal{F}$ defines an edge of G. By applying Theorem 2.1, we have the following result.

Theorem 3.1. *Let $\mathcal{F}$ be a wg-family of finite subsets of a set X and G be the partial cube induced by $\mathcal{F}$. Then*

$$\dim_I G = |\cup\mathcal{F} \setminus \cap\mathcal{F}|.$$

By Lemma 3.1, we may assume that the wg-family $\mathcal{F}$ satisfies conditions (3.2). Then we can reformulate Theorem 3.1 as follows:

Theorem 3.2. *Let $\mathcal{F}$ be a wg-family of finite subsets of a set X satisfying conditions (3.2) and let G be the partial cube induced by $\mathcal{F}$. Then*

$$\dim_I G = |X|.$$

The result of Theorem 3.1 suggests the definition of the isometric dimension of an arbitrary family $\mathcal{F}$ of subsets of a given set X as the cardinality of the set $\cup\mathcal{F} \setminus \cap\mathcal{F}$:

$$\dim_I \mathcal{F} = |\cup\mathcal{F} \setminus \cap\mathcal{F}|.$$

4 The isometry group of a hypercube

Let X be a set and let $Iso(\mathcal{H}(X))$ be the *isometry group* of the hypercube $\mathcal{H}(X)$, that is, the group of all isometries of the metric space $\mathcal{P}_f(X)$ onto itself.

For $P \in \mathcal{P}_f(X)$ we define a function α_P from $\mathcal{P}_f(X)$ onto itself by

$$\alpha_P(S) = S \bigtriangleup P \quad \text{for } S \in \mathcal{P}_f(X).$$

We have

$$d(\alpha_P(S), \alpha_P(T)) = |S \bigtriangleup P \bigtriangleup T \bigtriangleup P| = |S \bigtriangleup T| = d(S,T).$$

Thus α_P is an isometry of $\mathcal{P}_f$ onto itself. Clearly, the isometries α_P form a subgroup K of $Iso(\mathcal{H}(X))$ with the identity element $e = \alpha_\varnothing$.

The set $\mathcal{P}_f(X)$ is a commutative group under the operation of symmetric difference of sets. The empty set is the identity of this group. It is clear that the group K is isomorphic to the group $\mathcal{P}_f(X)$. Since $P \bigtriangleup P = \varnothing$, all elements of the group $\mathcal{P}_f(X)$ have order 2. Thus $\mathcal{P}_f(X)$ is an elementary Abelian two-group.

Let $P = \{x_1, \ldots, x_n\}$. Then

$$P = \{x_1\} \bigtriangleup \cdots \bigtriangleup \{x_n\}.$$

Hence, $\alpha_P = \alpha_{\{x_1\}} \circ \cdots \circ \alpha_{\{x_n\}}$. It follows that the family $\{\alpha_{\{x\}}\}_{x\in X}$ is a set of generators of the group K. These generators satisfy relations $\alpha^2_{\{x\}} = e$ for all $x \in X$.

The properties of the group $\mathcal{P}_f(X)$ established in the preceding paragraphs characterize this group (we omit the proof):

Proposition 4.1. *Let G be a group generated by a family of elements $\{g_x\}_{x\in X}$ satisfying the relations (and only these relations):*

(i) $g_x g_y = g_y g_x$,

(ii) $g_x^2 = e$

for all $x, y \in X$. Then G is isomorphic to $\mathcal{P}_f(X)$.

Let π be a permutation on X, that is, a bijection from X onto itself. The permutation π defines an isometry $\hat{\pi} : \mathcal{P}_f(X) \hookrightarrow \mathcal{P}_f(X)$ by

$$\hat{\pi}(S) = \pi(S) = \{\pi(x) : x \in S\} \quad \text{for } S \in \mathcal{P}_f(X).$$

These isometries form a subgroup H of $Iso(\mathcal{H}(X))$. It is clear that the group H is isomorphic to the symmetric group $S(X)$.

A rather standard argument proves the following well-known result.

Theorem 4.1. *The isometry group of the hypercube $\mathcal{H}(X)$ is a semidirect product of the subgroup K by the subgroup H:*

$$Iso(\mathcal{H}(X)) = K \rtimes H.$$

Remark 4.1. The vertices of the n-cube Q_n form the n-dimensional vector space L over the field F_2. The transformations α_P form the translation group K of L and elements of the group H can be regarded as 'orthogonal' transformations of L. In this case, Theorem 4.1 is an analog of the classical result in geometry: The group of motions of the n-dimensional Euclidean space is a semidirect product of its translation group by the orthogonal group $O(n)$.

5 Homogeneity properties of hypercubes

We begin with an example demonstrating that, in general, the metric space $\mathcal{H}(X)$ is not fully homogeneous.

Example 5.1. (cf. Example 4.3.7 in [6]) Let $X = \{a, b, c, d\}$. Consider two families of subsets of X:

$$\mathcal{A} = \{\varnothing, \{a, d\}, \{b, d\}, \{c, d\}\}$$

and

$$\mathcal{B} = \{\varnothing, \{a, b\}, \{a, c\}, \{b, c\}\}.$$

Clearly, $\mathcal{A}$ and $\mathcal{B}$ are isometric. The distance from the set $\{d\}$ to all sets in $\mathcal{A}$ is 1. On the other hand, it is easy to verify that there is no subset of X which is on distance 1 from all sets in $\mathcal{B}$. Thus an isometry from $\mathcal{A}$ onto $\mathcal{B}$ cannot be extended to an isometry from $\mathcal{H}(X)$ onto itself. Note that $\dim_I \mathcal{A} = 4$ but $\dim_I \mathcal{B} = 3$.

This example can be easily modified to show that $\mathcal{H}(X)$ is not fully homogeneous if $|X| > 3$. The metric space $\mathcal{H}(X)$ is fully homogeneous if $|X| \leq 3$ (we omit the proof). Note that any metric space $\mathcal{H}(X)$ is homogeneous. Indeed, for $P, Q \in \mathcal{P}_f(X)$ we have $Q = (\alpha_Q \circ \alpha_P)(P)$.

It is convenient to prove the main result of this section (Theorem 5.1) in terms of wg-families of finite subsets of a given set X.

For $\mathcal{F} \subseteq \mathcal{P}_f(X)$ we denote $D(\mathcal{F}) = \cup \mathcal{F}$ and define

$$\mathcal{M} = \{\mathcal{F} \subseteq \mathcal{P}_f(X) : \mathcal{F} \text{ is well-graded and } D(\mathcal{F}) \in \{X\} \cup \mathcal{P}_f(X)\}.$$

In other words, a wg-family $\mathcal{F}$ belongs to the class $\mathcal{M}$ if $D(\mathcal{F}) = X$ or $D(\mathcal{F})$ is a finite set.

We show that the metric space $\mathcal{P}_f(X)$ is $\mathcal{M}$-homogeneous by proving few lemmas.

A general remark is in order. Let Y be a homogeneous metric space, A and B be two subspaces of Y, and α be an isometry from A onto B. Let c be a fixed point in Y. For a given $a \in A$, let $b = \alpha(a) \in B$. Since Y is homogeneous, there are isometries β and γ from Y onto itself such that $\beta(a) = c$ and $\gamma(b) = c$. Then $\delta = \gamma\alpha\beta^{-1}$ is an isometry from $\beta(A)$ onto $\gamma(B)$ such that $\delta(c) = c$. Clearly, α is extendable to an isometry of Y if and only if δ is extendable. Thus, in the case of the space $\mathcal{P}_f(X)$, we may consider only wg-families of subsets containing the empty set $\varnothing$ and isometries between these families fixing this point.

The following property of an isometry $\alpha : \mathcal{F} \hookrightarrow \mathcal{G}$ between two families of sets is an immediate consequence of equivalence of lattice and metric betweenness relations:

$$P \cap Q \subseteq R \subseteq P \cup Q \quad \Leftrightarrow \quad \alpha(P) \cap \alpha(Q) \subseteq \alpha(R) \subseteq \alpha(P) \cup \alpha(Q) \tag{5.1}$$

for $P, Q, R \in \mathcal{F}$.

In what follows, $\mathcal{F}, \mathcal{G} \in \mathcal{M}$ are two wg-families each containing the empty set and $\alpha : \mathcal{F} \hookrightarrow \mathcal{G}$ is an isometry such that $\alpha(\varnothing) = \varnothing$.

As a special case of (5.1), we have

$$P \subseteq Q \quad \Leftrightarrow \quad \alpha(P) \subseteq \alpha(Q) \tag{5.2}$$

for $P, Q \in \mathcal{F}$, since P lies between $\varnothing$ and Q. We also have

$$|\alpha(P)| = |P| \quad \text{for } P \in \mathcal{F}, \tag{5.3}$$

since $|P| = d(\varnothing, P) = d(\varnothing, \alpha(P)) = |\alpha(P)|$.

We define a function $r_{\mathcal{F}} : D(\mathcal{F}) \to \mathbb{N}$ by

$$r_{\mathcal{F}}(x) = \min\{|R| : x \in R,\ R \in \mathcal{F}\}.$$

For $k \in \mathbb{N}$ a subset $X_k^{\mathcal{F}}$ of X is defined by

$$X_k^{\mathcal{F}} = \{x \in X : r_{\mathcal{F}}(x) = k\}.$$

We have $X_i^{\mathcal{F}} \cap X_j^{\mathcal{F}} = \varnothing$ for $i \neq j$, and $\cup_k X_k^{\mathcal{F}} = D(\mathcal{F})$. Note that some of the sets $X_k^{\mathcal{F}}$ could be empty for $k > 1$.

Example 5.2. Let $X = \{a, b, c\}$ and $\mathcal{F} = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. We have $r_{\mathcal{F}}(a) = r_{\mathcal{F}}(b) = 1,\ r_{\mathcal{F}}(c) = 3$ and

$$X_1^{\mathcal{F}} = \{a, b\},\ X_2^{\mathcal{F}} = \varnothing,\ X_3^{\mathcal{F}} = \{c\}.$$

Lemma 5.1. *The set $X_1^{\mathcal{F}}$ is not empty and for any nonempty set $P \in \mathcal{F}$ there is $x \in P$ such that $P \setminus \{x\} \in \mathcal{F}$.*

Proof. Since $\mathcal{F}$ is well-graded and contains the empty set, there is a nested sequence $\varnothing, R_1, \ldots, R_k = P$ of distinct sets in $\mathcal{F}$ such that $|R_{i+1} \setminus R_i| = 1$. Since R_1 is a singleton, we have $X_1^{\mathcal{F}} \neq \varnothing$. Clearly, $R_{k-1} = P \setminus \{x\}$ for some $x \in P$. Thus $P \setminus \{x\} \in \mathcal{F}$. □

Lemma 5.2. *For $P \in \mathcal{F}$ and $x \in P$ we have*

$$r_{\mathcal{F}}(x) = |P| \quad \Rightarrow \quad P \setminus \{x\} \in \mathcal{F}.$$

Proof. By Lemma 5.1, there is $y \in P$ such that $P \setminus \{y\} \in \mathcal{F}$. Since

$$|P \setminus \{y\}| = |P| - 1 < r_{\mathcal{F}}(x),$$

we have $x \notin P \setminus \{y\}$. Therefore, $y = x$. □

Lemma 5.3. *If $x \in P \in \mathcal{F}$ and $P \setminus \{x\} \in \mathcal{F}$, then there is $y \in \alpha(P)$ such that $\alpha(P) \setminus \{y\} = \alpha(P \setminus \{x\})$.*

Proof. By (5.2), $P \setminus \{x\} \subset P$ implies $\alpha(P \setminus \{x\}) \subset \alpha(P)$. Since $d(P \setminus \{x\}, P) = 1$, we have $d(\alpha(P \setminus \{x\}), \alpha(P)) = 1$. The result follows. □

We define a relation $\pi \subseteq D(\mathcal{F}) \times D(\mathcal{G})$ as follows: $(x, y) \in \pi$ if and only if $x \in D(\mathcal{F})$ and $y \in D(\mathcal{G})$ satisfy conditions of Lemma 5.3 for some $P \in \mathcal{F}$. By Lemmas 5.2 and 5.3, for any $x \in D(\mathcal{F})$ there is $y \in D(\mathcal{G})$ such that $(x, y) \in \pi$. Conversely, for any $y \in D(\mathcal{G})$ there is $x \in D(\mathcal{F})$ such that $(x, y) \in \pi$. Indeed, it suffices to apply the results of Lemmas 5.2 and 5.3 to the family $\mathcal{G}$ and the inverse isometry α^{-1}.

Lemma 5.4. *If $x \in X_k^{\mathcal{F}}$ and $(x, y) \in \pi$, then $y \in X_k^{\mathcal{G}}$. Conversely, if $y \in X_k^{\mathcal{G}}$ and $(x, y) \in \pi$, then $x \in X_k^{\mathcal{F}}$.*

Proof. Let $P \in \mathcal{F}$ be a set of cardinality k defining $r_{\mathcal{F}}(x) = k$. Then $r_{\mathcal{G}}(y) \leq k$, since $y \in \alpha(P)$ and, by (5.3), $|\alpha(P)| = k$.

Suppose that $m = r_{\mathcal{G}}(y) < k$. Then there is $Q \in \mathcal{G}$ such that $y \in Q$ and $|Q| = m$. By Lemma 5.2, $Q \setminus \{y\} \in \mathcal{G}$. By Lemma 5.3,

$$\alpha(P \setminus \{x\}) \cap Q \subseteq \alpha(P) \subseteq \alpha(P \setminus \{x\}) \cup Q.$$

By (5.1), we have

$$(P \setminus \{x\}) \cap \alpha^{-1}(Q) \subseteq P \subseteq (P \setminus \{x\}) \cup \alpha^{-1}(Q).$$

Thus, $x \in \alpha^{-1}(Q)$, a contradiction, since $r_{\mathcal{F}}(x) = k$ and, by (5.3),

$$|\alpha^{-1}(Q)| = |Q| = m < k.$$

It follows that $r_{\mathcal{G}}(y) = k$, that is, $y \in X_k^{\mathcal{G}}$.

We prove the converse statement by applying the above argument to the inverse isometry α^{-1}. □

We proved that for every $k \geq 1$ the restriction of π to $X_k^{\mathcal{F}}$ is a relation $\pi_k \subseteq X_k^{\mathcal{F}} \times X_k^{\mathcal{G}}$.

Lemma 5.5. *The relation π_k is a bijection for every $k \geq 1$.*

Proof. First we prove that π_k is a function. Suppose that there are $z \neq y$ such that $(x, y) \in \pi_k$ and $(x, z) \in \pi_k$. Then there are two distinct sets $P, Q \in \mathcal{F}$ defining y and z, respectively, such that

$$x \in P \cap Q, \;\; k = r_{\mathcal{F}}(x) = |P| = |Q|, \;\; P \setminus \{x\} \in \mathcal{F}, \; Q \setminus \{x\} \in \mathcal{F}.$$

By Lemma 5.3,

$$\alpha(P) \setminus \{y\} = \alpha(P \setminus \{x\}), \;\; \alpha(Q) \setminus \{z\} = \alpha(Q \setminus \{x\})$$

for some $y \in \alpha(P)$ and $z \in \alpha(Q)$. We have

$$\begin{aligned} d\,(\alpha(P), \alpha(Q)) &= d\,(P, Q) = d\,(P \setminus \{x\}, Q \setminus \{x\}) \\ &= d\,(\alpha(P) \setminus \{y\}, \alpha(Q) \setminus \{z\}). \end{aligned}$$

Thus, $y, z \in \alpha(P) \cap \alpha(Q)$. In particular, $z \in \alpha(P) \setminus \{y\}$, a contradiction, because $|\alpha(P) \setminus \{y\}| = k - 1$ but, by Lemma 5.4, $r_{\mathcal{G}}(z) = k$.

By applying the above argument to α^{-1}, we prove that for any $y \in X_k^{\mathcal{G}}$ there is a unique $x \in X_k^{\mathcal{F}}$ such that $(x, y) \in \pi_k$. Hence, π_k is a bijection. □

Corollary 5.1. *The relation π is a bijection from $D(\mathcal{F})$ onto $D(\mathcal{G})$.*

Recall that the wg-families $\mathcal{F}$ and $\mathcal{G}$ belong to the class $\mathcal{M}$. If $D(\mathcal{F}) = X$, then $D(\mathcal{G}) = X$, so π is a permutation on X. If $D(\mathcal{F})$ is a finite set, then the bijection π can be extended to a permutation on the set X. We denote this permutation by the same symbol π.

Lemma 5.6. *$\alpha(P) = \pi(P)$ for any $P \in \mathcal{F}$.*

Proof. The proof is by induction on $k = |P|$. The case $k = 1$ is trivial, since $\alpha(\{x\}) = \{\pi_1(x)\}$ for $\{x\} \in \mathcal{F}$.

Suppose that $\alpha(R) = \pi(P)$ for all $R \in \mathcal{F}$ such that $|R| < k$. Let P be a set in $\mathcal{F}$ of cardinality k. By Lemma 5.1, $P = R \cup \{x\}$ for some $R \in \mathcal{F}$ and $x \notin R$. It is clear that $m = r_{\mathcal{F}}(x) \leq k$ and $|R| = k - 1$.

If $m = k$, then $\alpha(P) = \alpha(R) \cup \{\pi(x)\} = \pi(P)$, by the definition of π and the induction hypothesis.

Suppose that $m < k$. Since $m = r_{\mathcal{F}}(x)$, there is a set $Q \in \mathcal{F}$ containing x such that $|Q| = m$. By Lemma 5.2, there is $S \in \mathcal{F}$ such that $S = Q \setminus \{x\}$. Since $x \in P$, we have

$$S \cap P \subseteq Q \subseteq S \cup P.$$

By (5.1), we have

$$\alpha(S) \cap \alpha(P) \subseteq \alpha(Q) \subseteq \alpha(S) \cup \alpha(P).$$

Thus, by the induction hypothesis,

$$\pi(S) \cup \{\pi(x)\} = \pi(Q) \subseteq \pi(S) \cup \alpha(P).$$

Since $\pi(x) \notin \pi(S)$, we have $\pi(x) \in \alpha(P)$. Since $\alpha(P) = \pi(R) \cup \{y\}$ for $y \notin \pi(R)$, and $x \notin R$, we have $y = \pi(x)$, that is, $\alpha(P) = \pi(P)$. □

We proved that $\alpha = \hat{\pi}|_{\mathcal{F}}$, that is, $\hat{\pi} \in \mathrm{Iso}(\mathcal{H}(X))$ is an extension of the isometry α.

Theorem 5.1. *The metric space $\mathcal{P}_f(X)$ is $\mathcal{M}$-homogeneous.*

Corollary 5.2. (cf. Theorem 5 in [10]) *Every isometry of a partial cube on a set X onto itself is the restriction of an isometry of the hypercube $\mathcal{H}(X)$.*

Corollary 5.3. (cf. Proposition 19.1.2 in [6]) *Let X be a finite set and $\mathcal{M}$ be the class of all partial cubes on X of dimension $|X|$. The metric space $\mathcal{H}(X)$ is $\mathcal{M}$-homogeneous.*

Example 5.3. Let Y be a proper subset of an infinite set X such that $|Y| = |X|$. It is easy to show that the hypercubes $\mathcal{H}(Y)$ and $\mathcal{H}(X)$ are isomorphic. These hypercubes are partial cubes on X of the same dimension $|X|$. It is clear that an isometry $\mathcal{H}(Y) \hookrightarrow \mathcal{H}(X)$ cannot be extended to an isometry of $\mathcal{H}(X)$ onto itself. Thus Corollary 5.3 does not hold for infinite sets X.

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